

# In search of Complete, Unambiguous Multilateral Orderings of States: Human Wellbeing and Family Background

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***“ . . . for as long as the worst and the best have not been defined in terms of a clearly and concretely conceived ideal, and then the precise margins of possibilities determined, we do not know which is the lesser evil, and consequently we are compelled to accept under this name anything effectively imposed . . . ”***

***Simone Weil (1934) *Oppression and Liberty****

# Introduction.

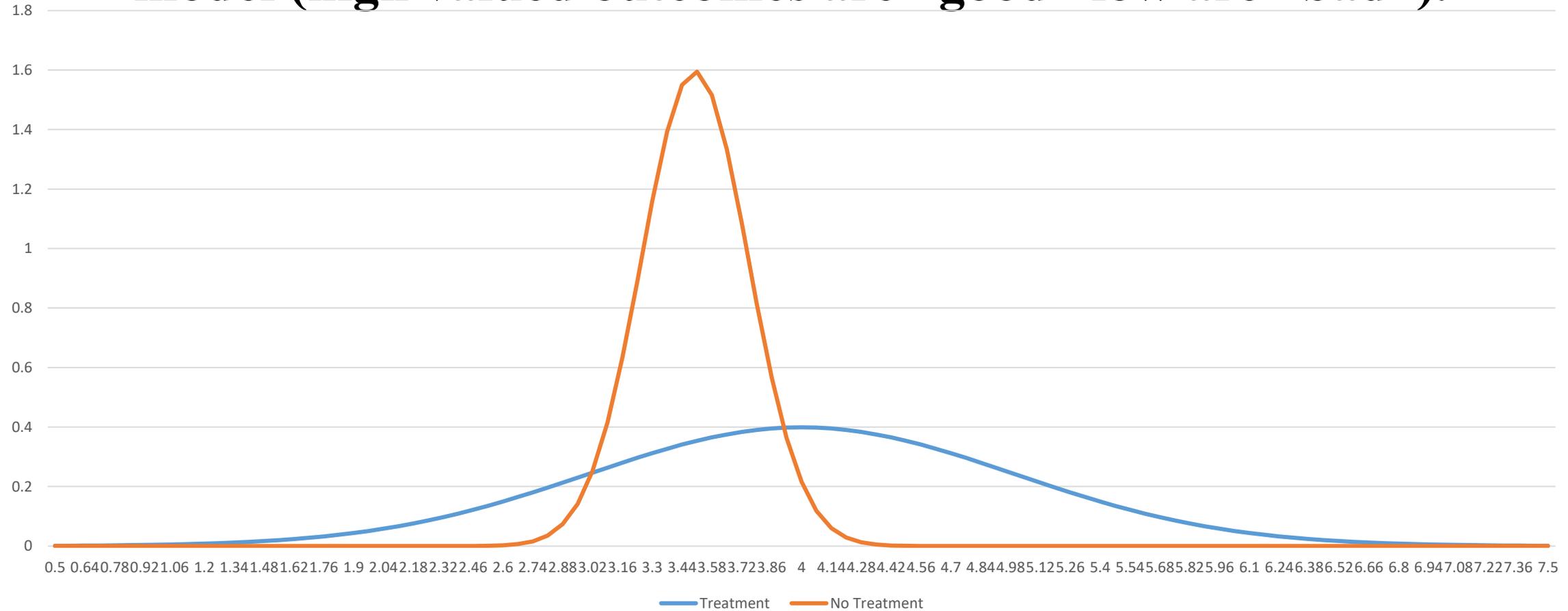
- In many comparison situations, summary statistics may not reveal the full extent of prospect differences engendering some ambiguity (Knights' unmeasurable uncertainty).
- Sometimes the problem is the choosers priority structure or value function is not well defined, but when it is, in continuous domains, Stochastic Dominance techniques provide a complete range of bilateral comparison criteria for making unambiguous orderings in certain circumstances.
- However, the cumbersome and partial nature of the SD ordering process, hampers implementation in multilateral situations and, in ordered categorical domains (e.g. self reported health, happiness ..etc), the technique encounters serious difficulties in interpretation and implementation beyond first order comparisons, here these issues are addressed.
- Using notions of distributional proximity, we provide a normative interpretation of dominance in an ordered categorical contexts together with a family of indices (and their asymptotic distributions) which facilitate ordering and ranking in ordered categorical environments together with measures calibrating the extent of comparative ignorance in a collection of comparisons, in essence measuring the extent of unmeasurable uncertainty.

# Unmeasurable Uncertainty Example 1: Choosing Preferred Nation Location on the basis of income levels and differences

- Consider Adult equivalent PPP adjusted Eurozone nation household income levels in 2015 (Anderson and Thomas 2018).
- Using average log incomes: Latvia (9.28) > Greece (9.16) and Portugal (11.89) > Slovakia (11.41)
- Using median log incomes: Latvia (7.27) < Greece (8.10) and Portugal (9.98) < Slovakia (10.61).
- Similarly, with respect to relative inequality: Belgium is more equal than Slovenia using the coefficient of variation whereas Slovenia is more equal than Belgium using Atkinson's measure ( $\alpha = 0.25$ )

Figure 3.1: A Two-Group Comparison

**Example 2. The veil of Ignorance Problem (Carniero, Hansen, Heckman 2003) Suppose a random trial in a Treatment effects model (high valued outcomes are “good” low are “bad”).**



# Seeking Clarity: The Choosers value function and the stochastic dominance rules with continuous $x$ .

- Suppose  $K$  treatments with outcome pdf's of  $x$ :  $f_k(x)$  for  $k=1,\dots,K$  defined on  $R^+$ .
- Consider a family of value functions  $U(x)$  reflecting the quality  $x$  yields in terms of the choosers' priorities. Here "i" is specified such that  $(-1)^{i+1}(dU(x)/dx) > 0$  for  $i = 1, 2, \dots$  {Alternative conditions are derivable for different  $U(x)$  types}.
- Standard SD analysis: For two distributions  $f_k(x)$  and  $f_j(x)$  given  $U(x)$  of type "i", let  $F^i(x) = \int_0^x F^{i-1}(z) dz$  with  $F^0(x) = f(x)$ , if  $F_k^i(x) \leq F_j^i(x) \forall x$  with  $F_k^i(x) < F_j^i(x)$  somewhere  $\Rightarrow$   $k$  is unambiguously preferred to  $j$  for all possible  $U(x)$  of type "i".
- Similar rules requiring one surface to be unambiguously below the other apply when  $x$  is a vector (Atkinson and Bourguignon 1982). when they do not prevail alternative instruments in the same class could yield contradictory rankings.

# A Stochastic Dominance Based Ordering Function: The Utopia-Dystopia (Hull) Index for the $i$ 'th comparison level (Anderson Leo 2016, Anderson, Post and Whang JBES 2020).

- Consider upper and lower hulls  $M^i(x)$  and  $L^i(x)$  where:
- $M^i(x) = \max_{F_k^i \in \check{F}} \{F_1^i(x), F_2^i(x), \dots, F_K^i(x)\}$ ,  $L^i(x) = \min_{F_k^i \in \check{F}} \{F_1^i(x), F_2^i(x), \dots, F_K^i(x)\}$  [1]
- $M^i(x)$  is the synthetic worst case “Dystopian” scenario if the worst bits of the distributions are combined over the support of  $x$ ,  $L^i(x)$  is the synthetic best case “Utopian” scenario if the best bits are combined over the support of  $x$ .
- A relative distance measure  $UD \left( F_k^i(x), M^i(x), L^i(x) \right) = \frac{\int_0^M (M^i(x) - F_k^i(x)) dx}{\int_0^M (M^i(x) - L^i(x)) dx}$  [2]
- $UD \left( F_k^i(x), M^i(x), L^i(x) \right)$  satisfies the Continuity, Function and Scale Independence, Coherence, Normalization, IIA axioms of choice and a newly introduced Non Ambiguity Axiom  $\{UD(F_k) = 1 (0) \Rightarrow F_k \text{ unambiguously best (worst)}\}$

# Ordinal Variable Considerations.

- Typically, when  $x$  is an ordered categorical variate, categories are arbitrarily assigned a numerical value or Cantril scale in accord with the ordering which, when standard summary statistics are employed or higher orders of “ $i$ ” are to be entertained, poses scale dependency problems (Bond and Lang 2019, Schroder and Yitzhaki 2017).
- In the following we suppose  $J$  ordered categories,  $j=1, \dots, J$  such that  $x_{j,g}$ , the latent value associated with category  $j$  in distribution  $g$ , accords with the preference ordering  $x_{1,g} < x_{2,g} < \dots < x_{j,g} < \dots < x_{J,g}$  and denote the probability density  $f(x_{j,g}) = f_{j,g}$  and cumulative density  $F_{j,g} = \sum_{h=1}^j f_{h,g}$ .
- Then, noting that  $f_{j,g}$  does not depend upon  $x_{j,g}$ , when:
  - $F_{j,g} \leq F_{j,g'}$  for all  $j$  and  $F_{j,g} < F_{j,g'}$  for some  $j$ ,  $j = 1, \dots, J$  [3]
- $g$  stochastically dominates  $g'$  at the first order and, if the latent values had true cardinal value, the ordering would be robust to any specified  $U(x)$  in the  $i=1$  class. However in the absence of cardinality the linkage is lost at higher orders  $i > 1$ .

# Higher order Ordinal Dominance and an Ordinal Utopia-Dystopia (Hull) Index.

- Let the situation when [3] is met be known as First Order Ordinal Dominance (denoted  $\succ_0$ ).
- A parallel to the higher orders of dominance in the continuous case can be achieved by considering successive sums of cdf's  $F_{j,g}^s$ ,  $s=1,..$  where  $F_{j,g}^s = \sum_{h=1}^j F_{j,g}^{s-1}$  and  $F_{j,g}^0 = f_{j,g}$ , it can be shown that:
- $F_{j,g}^s = \sum_{h=1}^j \frac{[j-h+(s-1)]!}{(j-h)!(s-1)!} f_{j,g} = \sum_{h=1}^j \binom{[j-i+(s-1)]}{s-1} f_{j,g}$  where the term in the bracket is the binomial coefficient.
- Defining  $M_j^s$  and  $L_j^s$   $j = 1, \dots, J$  as the worst and best case scenarios over a collection of  $K$  distributions analogous to [1] facilitates  $H(F_k^s, M_j^s, L_j^s)$ , a Ordinal Hull index equivalent to [2] where for  $T^s = \sum_{j=1}^J (F_j^s - L_{j,k}^s)$  and  $S^s = \sum_{j=1}^J (M_j^s - L_j^s)$ :
- $$H(F_k^s, M_j^s, L_j^s) = \frac{\sum_{j=1}^J (M_j^s - F_{j,k}^s)}{\sum_{j=1}^J (M_j^s - L_j^s)} = 1 - \frac{T^s}{S^s}$$
- Note these measures do not depend upon values of  $x$  or any such scale.

# Distributional Properties and the Ambiguity issue.

- Following the asymptotic normality of estimates of the ordered categorical density functions (Rao 2009), asymptotic normality of estimates of the  $F_{j,g}^S$  and Ordinal Utopia – Dystopia indices are readily established.
- The notion of an unambiguous difference between two distributions transcends statistically significant differences between respective location parameter estimates. Satisfaction of a dominance condition between the two distributions at some order, ensures there will be no such ambiguity at that order.
- A measure of the extent to which dominance does or does not prevail provides a measure of the absence of or propensity for such ambiguity which is complementary to the sense in which the distributions are statistically significantly different.
- $AMB_{g,g'}^S = \frac{|\sum_{j=1}^J (F_{j,g}^S - F_{j,g'}^S)|}{\sum_{j=1}^J |F_{j,g}^S - F_{j,g'}^S|}$  (a weighted average over all possible pairs provides an index of ambiguity in the collection)

# An Example: Adult Quality of Life and Child Circumstances.

- This example follows a substantial Equal Opportunity literature in exploring the dependency of an adult's health, education and income status outcomes on their childhood experiences augmenting health gradient (Allison and Foster 2004; Case et al. 2002, 2005; Currie and Stabile 2003; Currie 2009; Cutler et al. 2011) and educational and income status literatures (Anand and Sen 1997; Atkinson 2003; Grusky and Kanbur 2006; Sen 1995; Stiglitz et al. 2011).
- it seeks to order childhood circumstance types predicated upon the nature of a multidimensional value function class defined over a child's self-reported health, education, and income status when adult, and identifies the worst and best childhood circumstance scenarios.
- Four principle typologies of a child's circumstance are the focus; their parent's income quintile, their parent's educational status, and the sports and exercise engagement that was fostered in the child.
- Panel data from waves I and IV of the Inter-University Consortium for Political and Social Research (ICPSR), National Longitudinal Study of Adolescent Health 1994-2008 are used to relate an adult's self-reported health, education and income status to their parental circumstance, and childhood behaviors.

Table A.4.4:  $\tilde{\mathcal{T}}^{(s)}$  &  $\tilde{\mathcal{H}}^{(s)}$  Ranking of Welfare (Joint CDF of SRH, Income, & Educational Attainment) in Adulthood, Conditional on Parental Income Quintile

	1 <sup>st</sup> Order Comparison					$\widetilde{\mathcal{AM}}^{(1)}$
	5 <sup>th</sup> Quintile	4 <sup>th</sup> Quintile	3 <sup>rd</sup> Quintile	2 <sup>nd</sup> Quintile	1 <sup>st</sup> Quintile	
$\tilde{\mathcal{T}}^{(1)}$	0.0676	21.5417	32.4054	50.5704	65.8194	0.0224
$\tilde{\mathcal{H}}^{(1)}$	0.9990	0.6728	0.5078	0.2319	0.0003	
Std. Error ( $\tilde{\mathcal{H}}^{(1)}$ )	(0.0566)	(0.0718)	(0.0751)	(0.0809)	(0.0015)	
$P - Value$ ( $\tilde{\mathcal{H}}^{(1)} = 1$ )	1.0000	1.0000	1.0000	0.9979	0.5698	
$P - Value$ ( $\tilde{\mathcal{H}}^{(1)} = 0$ )	0.4928	0.0000	0.0000	0.0000	0.0000	
$\tilde{\mathcal{R}}^{(1)}$	0.9937	0.9687	0.9504	0.9773	0.9977	
Observation	686	705	679	662	587	
	2 <sup>nd</sup> Order Comparison					
	5 <sup>th</sup> Quintile	4 <sup>th</sup> Quintile	3 <sup>rd</sup> Quintile	2 <sup>nd</sup> Quintile	1 <sup>st</sup> Quintile	
$\tilde{\mathcal{T}}^{(2)}$	2.6215	4383.8605	8445.9807	12665.8014	18178.2072	0.0147
$\tilde{\mathcal{H}}^{(2)}$	0.9999	0.7589	0.5355	0.3035	0.0003	
Std. Error ( $\tilde{\mathcal{H}}^{(2)}$ )	(0.0320)	(0.0500)	(0.0626)	(0.0736)	(0.0022)	
$P - Value$ ( $\tilde{\mathcal{H}}^{(2)} = 1$ )	1.0000	1.0000	1.0000	1.0000	0.5572	
$P - Value$ ( $\tilde{\mathcal{H}}^{(2)} = 0$ )	0.4982	0.0000	0.0000	0.0000	0.0000	
$\tilde{\mathcal{R}}^{(2)}$	0.9988	0.9871	0.9657	0.9770	0.9979	
Observation	686	705	679	662	587	
	3 <sup>rd</sup> Order Comparison					
	5 <sup>th</sup> Quintile	4 <sup>th</sup> Quintile	3 <sup>rd</sup> Quintile	2 <sup>nd</sup> Quintile	1 <sup>st</sup> Quintile	
$\tilde{\mathcal{T}}^{(3)}$	0.0000	166680.0836	355860.0558	529909.8433	804264.1477	0.0146
$\tilde{\mathcal{H}}^{(3)}$	1.0000	0.7929	0.5578	0.3415	0.0005	
Std. Error ( $\tilde{\mathcal{H}}^{(3)}$ )	(0.0287)	(0.0569)	(0.0755)	(0.0939)	(0.0061)	
$P - Value$ ( $\tilde{\mathcal{H}}^{(3)} = 1$ )	1.0000	1.0000	1.0000	0.9999	0.5358	
$P - Value$ ( $\tilde{\mathcal{H}}^{(3)} = 1$ )	0.500	0.0001	0.0000	0.0000	0.0000	
$\tilde{\mathcal{R}}^{(3)}$	1.0000	0.9908	0.9659	0.9735	0.9967	
Observation	686	705	679	662	587	

# Conclusions

- We have introduced a notion of ordinal dominance which facilitates multilateral ordering of groups based upon their respective multidimensional ordered categorical outcomes.
- It is possible to order groups based upon ordered categorical data and quantify the extent to which such orderings are unambiguous.
- As an example the multidimensional childhood circumstances of children were ordered on the basis of their health, education and income outcomes when adult and the extent to which the orderings were ambiguous measured.

Table 2:  $\tilde{\mathcal{T}}_s$  &  $\tilde{\mathcal{H}}_s$  Ranking of Welfare (Joint CDF of SRH, Income, & Educational Attainment) in Adulthood

		1 <sup>st</sup> Order Comparison							$\tilde{\mathcal{AM}}^{(1)}$	
		Top Income	College/Univ.	Top Sports	Top Exercise	2nd Income	2nd Sports	2nd Exercise	High School	
I	$\tilde{\mathcal{T}}^{(1)}$	0.5827	2.9618	11.2143	19.9415	21.3884	26.4257	30.0320	43.8718	0.3504
	$\tilde{\mathcal{H}}^{(1)}$	0.9868	0.9329	0.7459	0.5481	0.5153	0.4012	0.3194	0.0058	
	Std. Error ( $\tilde{\mathcal{H}}^{(1)}$ )	(0.0804)	(0.0877)	(0.0947)	(0.0919)	(0.0909)	(0.1011)	(0.0930)	(0.0177)	
	$P - Value$ ( $\tilde{\mathcal{H}}^{(1)} = 1$ )	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.6292	
	$P - Value$ ( $\tilde{\mathcal{H}}^{(1)} = 0$ )	0.4348	0.2221	0.0036	0.0000	0.0000	0.0000	0.0000	0.0000	
	No. Observations	950	1386	843	901	852	691	854	1665	
	$\bar{\mathcal{R}}^1$	0.5222	0.6477	0.8455	0.5194	0.3335	0.5790	0.7872	0.9625	
		2 <sup>nd</sup> Order Comparison							$\tilde{\mathcal{AM}}^{(2)}$	
		Top Income	College/Univ.	Top Sports	2nd Income	Top Exercise	2nd Sports	2nd Exercise	High School	
II	$\tilde{\mathcal{T}}^{(2)}$	82.3540	488.0466	2891.8879	4702.6699	5403.7636	7065.8047	7259.3971	11130.6287	0.2021
	$\tilde{\mathcal{H}}^{(2)}$	0.9927	0.9565	0.7422	0.5808	0.5183	0.3702	0.3529	0.0078	
	Std. Error ( $\tilde{\mathcal{H}}^{(2)}$ )	(0.0520)	(0.0600)	(0.0710)	(0.0766)	(0.0772)	(0.0928)	(0.0868)	(0.0244)	
	$P - Value$ ( $\tilde{\mathcal{H}}^{(2)} = 1$ )	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.6257	
	$P - Value$ ( $\tilde{\mathcal{H}}^{(2)} = 0$ )	0.4439	0.2344	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	
	No. Observations	950	1386	843	852	901	691	854	1665	
	$\bar{\mathcal{R}}^{(2)}$	0.6323	0.7693	0.9475	0.8829	0.8775	0.6643	0.6533	0.9560	
		3 <sup>rd</sup> Order Comparison							$\tilde{\mathcal{AM}}^{(3)}$	
		College/Univ.	Top Income	Top Sports	2nd Income	Top Exercise	2nd Sports	2nd Exercise	High School	
III	$\tilde{\mathcal{T}}^{(3)}$	7586.0918	9297.4851	1.17E+05	1.84E+05	2.28E+05	3.11E+05	3.12E+05	4.82E+05	0.3011
	$\tilde{\mathcal{H}}^{(3)}$	0.9845	0.9810	0.7610	0.6225	0.5339	0.3629	0.3617	0.0126	
	Std. Error ( $\tilde{\mathcal{H}}^{(3)}$ )	(0.0462)	(0.0650)	(0.0775)	(0.0909)	(0.0906)	(0.1220)	(0.1135)	(0.0370)	
	$P - Value$ ( $\tilde{\mathcal{H}}^{(3)} = 1$ )	1.0000	1.0000	1.0000	1.0000	1.0000	0.9985	0.9993	0.6328	
	$P - Value$ ( $\tilde{\mathcal{H}}^{(3)} = 0$ )	0.3683	0.3848	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000	
	No. Observations	1386	950	843	852	901	691	854	1665	
	$\bar{\mathcal{R}}^{(3)}$	0.1031	0.5168	0.9541	0.8880	0.8911	0.6664	0.6402	0.9316	

Table 1.A: Pairwise Ordinal Dominance of Welfare (Joint CDF of SRH, Income, & Educational Attainment) in Adulthood

	Top Income	2nd Income	Top Income	College/Univ.	Top Income	High School	Top Income	Top Sports
Difference	18.2105	0.3316	4.69E+07	9.44E+07	39.3195	0.0000	12.5260	0.8865
Std. Error	(3.2988)	(3.2988)	(1.28E+08)	(1.28E+08)	(3.2186)	(3.2186)	(3.2757)	(3.2757)
<i>P</i> – Value	1.0000	0.5400	0.6431	0.7698	1.0000	0.5000	0.9999	0.6067
Observations	950	852	950	1386	950	1665	950	843
	Top Income $\succ_O^2$ 2nd Income		$\not\succeq_O^{10}$ & $\not\succeq_O^{10}$		Top Income $\succ_O^2$ High School		Top Income $\succ_O^2$ Top Sports	
			2nd Income	College/Univ.	2nd Income	High School	2nd Income	Top Sports
Difference			0.2778	16.3387	22.6640	0.0873	0.0023	6.8275
Std. Error			(2.9646)	(2.9646)	(3.4857)	(3.4857)	(3.5613)	(3.5613)
<i>P</i> – Value			0.5373	1.0000	1.0000	0.5100	0.5003	0.9724
Observations			852	1386	852	1665	852	843
			College/Univ. $\succ_O^2$ 2nd Income		2nd Income $\succ_O^2$ High School		Top Sports $\succ_O^2$ 2nd Income	
					College/Univ.	High School	College/Univ.	Top Sports
Difference					37.4534	0.2778	11.1495	1.0080
Std. Error					(2.8202)	(2.8202)	(2.9437)	(2.9437)
<i>P</i> – Value					1.0000	0.5392	0.9999	0.6340
Observations					1386	1665	1386	843
					College/Univ. $\succ_O^2$ High School		College/Univ. $\succ_O^2$ Top Sports	
							High School	Top Sports
Difference							3.80E+07	5.04E+06
Std. Error							(5.41E+07)	(5.41E+07)
<i>P</i> – Value							0.7585	0.5371
Observations							1351	843
							Top Sports $\succ_O^2$ High School	

Table 1.B: Pairwise Ordinal Dominance of Welfare (Joint CDF of SRH, Income, & Educational Attainment) in Adult Cont'd

	Top Income	2nd Sports	Top Income	Top Exercise	Top Income	2nd Exercise
Difference	24.6895	0.0000	19.9952	0.3316	24.9734	0.0000
Std. Error	(3.7683)	(3.7683)	(3.3949)	(3.3949)	(3.5473)	(3.5473)
<i>P</i> – Value	1.0000	0.5000	1.0000	0.5389	1.0000	0.5000
Observations	950	691	950	901	950	854
	Top Income $\succ_O^2$	2nd Sports	Top Income $\succ_O^2$	Top Exercise	Top Income $\succ_O^2$	2nd Exercise
	2nd Income	2nd Sports	2nd Income	Top Exercise	2nd Income	2nd Exercise
Difference	9.1949	0.0000	5.7940	1.4034	9.3943	0.0000
Std. Error	(4.0512)	(4.0512)	(3.6741)	(3.6741)	(3.8252)	(3.8252)
<i>P</i> – Value	0.9884	0.5000	0.9426	0.6488	0.9930	0.5000
Observations	852	691	852	901	852	854
	2nd Income $\succ_O^2$	2nd Sports	2nd Income $\succ_O^2$	Top Exercise	2nd Income $\succ_O^2$	2nd Exercise
	College/Univ.	2nd Sports	College/Univ.	Top Exercise	College/Univ.	2nd Exercise
Difference	0.1459	0.0000	18.1281	0.2778	0.1557	0.0000
Std. Error	(0.0000)	(0.0000)	(3.0499)	(3.0499)	(0.0000)	(0.0000)
<i>P</i> – Value	1.0000	0.5000	1.0000	0.5363	1.0000	0.5000
Observations	1386	691	1386	901	1386	854
	College/Univ. $\succ_O^1$	2nd Sports	College/Univ. $\succ_O^2$	Top Exercise	College/Univ. $\succ_O^1$	2nd Exercise
	High School	2nd Sports	High School	Top Exercise	High School	2nd Exercise
Difference	1.0576	15.0954	0.0000	19.6278	0.9886	14.3489
Std. Error	(3.9555)	(3.9555)	(3.5311)	(3.5311)	(3.6739)	(3.6739)
<i>P</i> – Value	0.6054	0.9999	0.5000	1.0000	0.6061	1.0000
Observations	1665	691	1665	901	1665	854
	2nd Sports $\succ_O^2$	High School	Top Exercise $\succ_O^2$	High School	2nd Exercise $\succ_O^2$	High School

Table 1.C: Pairwise Ordinal Dominance of Welfare (Joint CDF of SRH, Income, & Educational Attainment) in Adulthood, Cont'd

	Top Sports	2nd Sports	Top Sports	Top Exercise	Top Sports	2nd Exercise
Difference	13.6814	0.0000	7.9789	0.0000	13.9391	0.0000
Std. Error	(4.0298)	(4.0298)	(3.6543)	(3.6543)	(3.8060)	(3.8060)
<i>P</i> – Value	0.9997	0.5000	0.9855	0.5000	0.9999	0.5000
Observations	843	691	843	901	843	854
	Top Sports $\succ_O^2$ 2nd Sports		Top Sports $\succ_O^2$ Top Exercise		Top Sports $\succ_O^2$ 2nd Exercise	
			2nd Sports	Top Exercise	2nd Sports	2nd Exercise
Difference			1.1166	5.7024	3.50E+07	1.06E+08
Std. Error			(4.1403)	(4.1403)	(5.80E+08)	(5.80E+08)
<i>P</i> – Value			0.6063	0.9158	0.5240	0.5722
Observations			691	901	691	854
			Top Exercise $\succ_O^2$ 2nd Sports		$\not\succeq_O^{10}$ & $\not\succeq_O^{10}$	
					Top Exercise	2nd Exercise
Difference					6.4908	0.8209
Std. Error					(3.9040)	(3.9040)
<i>P</i> – Value					0.9518	0.5833
Observations					901	854
					Top Exercise $\succ_O^2$ 2nd Exercise	