

Nearly optimal central limit theorem and bootstrap approximations in high dimensions

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NEARLY OPTIMAL CENTRAL LIMIT THEOREM AND BOOTSTRAP APPROXIMATIONS IN HIGH DIMENSIONS

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ABSTRACT. In this paper, we derive new, nearly optimal bounds for the Gaussian approximation to scaled averages of n independent high-dimensional centered random vectors X_1, \dots, X_n over the class of rectangles in the case when the covariance matrix of the scaled average is non-degenerate. In the case of bounded X_i 's, the implied bound for the Kolmogorov distance between the distribution of the scaled average and the Gaussian vector takes the form

$$C(B_n^2 \log^3 d/n)^{1/2} \log n,$$

where d is the dimension of the vectors and B_n is a uniform envelope constant on components of X_i 's. This bound is sharp in terms of d and B_n , and is nearly (up to $\log n$) sharp in terms of the sample size n . In addition, we show that similar bounds hold for the multiplier and empirical bootstrap approximations. Moreover, we establish bounds that allow for unbounded X_i 's, formulated solely in terms of moments of X_i 's. Finally, we demonstrate that the bounds can be further improved in some special smooth and zero-skewness cases.

1. INTRODUCTION

Let X_1, \dots, X_n be a sequence of centered independent random vectors in \mathbb{R}^d . Denote

$$W := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$$

and let \mathcal{R} be the class of rectangles in \mathbb{R}^d , which we choose to be sets of the form $A = \prod_{j=1}^d (a_j, b_j]$ for some $-\infty \leq a_j \leq b_j \leq \infty$, $j = 1, \dots, d$. In this paper, we are interested in deriving new bounds on

$$\varrho := \sup_{A \in \mathcal{R}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)|, \quad Z \sim N(0, \Sigma_W), \quad (1.1)$$

where $\Sigma_W := \mathbb{E}[WW^T]$. We are particularly interested in the high-dimensional case, where d is potentially much larger than n .

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The problem of bounding ϱ has attracted considerable attention in the literature because the class of rectangles \mathcal{R} strikes an interesting balance: it is large enough so that bounds on ϱ are useful in mathematical statistics, and, at the same time, it is small enough so that, as $n \rightarrow \infty$, under minimal conditions, we have $\varrho = \varrho_n \rightarrow 0$ even if $d = d_n \rightarrow \infty$ much faster than $n \rightarrow \infty$, as was originally shown in [12], making bounds on ϱ particularly useful in high-dimensional statistics and machine learning, e.g. in multiple hypothesis testing with the family-wise error rate control and in selecting penalty parameters for regularized estimators of high-dimensional models; see [6] for details on these and other examples.

Various bounds on ϱ and on closely related quantities were derived in [12, 13, 15, 42, 20, 41, 34, 30, 17, 23, 32, 31, 19, 18] but in our discussion, we only focus on the results that are particularly relevant for comparisons with our results. In addition, for clarity of the introduction, we assume below that components of X_i 's are uniformly bounded by the envelope constant $B_n = B_n(d)$, i.e. $\|X_i\|_\infty := \max_{1 \leq j \leq d} |X_{ij}| \leq B_n$, for all $i = 1, \dots, n$, even though all aforementioned papers, as well as ours, considered the case of unbounded X_i 's as well. It then follows from [17] that

$$\varrho \leq C \left(\frac{B_n^2 (\log(dn))^5}{n} \right)^{1/4}, \quad (1.2)$$

where $C > 0$ is a constant that is independent of n and d . This bound is conjectured to be near-optimal when Σ_W is unrestricted.

Next, [23] demonstrated that if, in addition, we assume that all eigenvalues of Σ_W are bounded below from zero (strongly non-degenerate case, in their terminology), then the bound (1.2) can be substantially improved: they showed that

$$\varrho \leq C \left(\frac{B_n^2 (\log(dn))^4}{n} \right)^{1/3} \quad (1.3)$$

in this case. Moreover, they established that this bound can be further improved to the near-sharp $n^{-1/2} \log n$ and the sharp $(\log d)^3$ dependencies but only for the case of jointly log-concave X_i 's. Their results exploit the implicit smoothing that occurs when Σ_W is strongly non-degenerate. Further, [32] and [31] demonstrated, again in the strongly non-degenerate case, that

$$\varrho \leq C \left(\left(\frac{B_n^6 (\log d)^4 \log(dn)}{n} \right)^{1/2} + \left(\frac{(\log d)^7 \log(dn)}{n} \right)^{1/2} \right) \log n, \quad (1.4)$$

which nearly matches the dependence on n in the classical Berry-Esseen bound for the one-dimensional ($d = 1$) case, e.g. Theorem 2.2.15 in [39], but does not provide optimal dependence on $B_n = B_n(d)$ and $\log d$.

In this paper, our main result is to establish that in the strongly non-degenerate case,

$$\varrho \leq C \left(\frac{B_n^2 (\log d)^3}{n} \right)^{1/2} \log n, \quad (1.5)$$

which we show to be optimal up to the $\log n$ factor, i.e. in general

$$\varrho \geq c \left(\frac{B_n^2 (\log d)^3}{n} \right)^{1/2}. \quad (1.6)$$

In addition, we extend this result to allow for unbounded X_i 's, which yields a bound depending solely on some moments of X_i 's. A critical ingredient in our proofs is an explicit use of smoothing, combined with the previous implicit smoothing ideas, as we further comment on below.

Our result (1.5) strongly improves bounds (1.3) and (1.4), and features the optimal dependence on the ambient dimension d , the optimal dependence on the envelope constant B_n , and a nearly optimal dependence on the sample size n (up to the $\log n$ factor). This result improves over (1.4) by replacing $(\log d)^8$ by the optimal $(\log d)^3$ and replacing B_n^6 by the optimal B_n^2 . The first improvement is particularly important when $\log d$ is growing as some fractional power of the sample size n , in which case our bound (1.5) has much better dependence on n . The second improvement is important when the envelope constant B_n is increasing with the sample size n , in which case our bound (1.5) also has much better dependence on n . This, for example, occurs in the many local means settings arising in nonparametric statistics, discussed in detail in Section 5, where the envelope constant $B_n = B_n(d)$ has dependency on the dimension of problem d of the form $B_n(d) \propto \sqrt{d}$. In particular, the bound (1.4) would then require $d^3 \ll n$ for $\rho \rightarrow 0$, whereas our bound would only require $d \ll n$. Therefore, the improvement is critical for obtaining the sharp dependency on the dimension d in general. Also, as discussed in Section 5, in the many local means setting, our bound tends to be either at least as sharp (up to \log factors) or much sharper than the Gaussian approximation based on the Hungarian coupling.

Moreover, we also consider bootstrap approximations, i.e. we derive bounds on

$$\varrho^* := \sup_{A \in \mathcal{R}} |\mathbb{P}(Z \in A) - \mathbb{P}(W^* \in A \mid X_1, \dots, X_n)|,$$

where W^* denotes a bootstrap version of W . These approximations are important in mathematical statistics because they allow to estimate probabilities $\mathbb{P}(Z \in A)$, $A \in \mathcal{R}$, using random vectors X_1, \dots, X_n , which is useful when Σ_W is unknown so that probabilities can not be calculated directly. For the multiplier and empirical bootstrap approximations, we derive bounds that are generally similar to those for the Gaussian approximation (1.1).

Finally, we show that the $\log n$ factor in (1.5) can be removed if we assume that X_i 's have a Gaussian component, and we also show that if X_i 's have a Gaussian component

and satisfy a zero-skewness condition, then

$$\varrho \leq \frac{C(\log d)^2}{n}.$$

This last bound substantially extends the results of [18], who showed that $\varrho \rightarrow 0$ if $(\log d)^2/n \rightarrow 0$ in the case when X_i 's have independent components with vanishing odd moments.

Our results are built using the exchangeable pair approach coupled with the Slepian-Stein method and employ ideas of many authors, e.g. [7, 28, 3, 8, 11, 36, 12, 15, 23, 32, 31], but the key technical tool behind our results is a set of new smoothing inequalities, which we call mixed smoothing inequalities. Specifically, for any rectangle $A \in \mathcal{R}$, we first approximate the indicator of A by a Lipschitz-smooth function and then approximate it further via convolution with a centered Gaussian distribution. Building on the results of [7, 3, 23], we then prove several bounds for sums of absolute values of partial derivatives of the resulting function, which play a crucial role in our derivations.

This mixed smoothing turns out important for two reasons. First, smoothing via convolutions allows to obtain nearly optimal dependence on n , as demonstrated by [28] in the moderate-dimensional case and then by [23, 32, 31] in the high-dimensional case. Second, smoothing via Lipschitz-smooth functions allows to obtain optimal dependence on d , as follows from our results. Our approach here is inspired by [8], who used related but different mixed smoothing to derive a Berry-Esseen bound with good dependence on d for convex sets in the moderate-dimensional case.

The rest of the paper is organized as follows. In the next section, we consider Gaussian approximations and derive various bounds on ϱ . In Section 3, we derive bounds for bootstrap approximations. In Section 4, we discuss special cases, where bounds for the Gaussian approximations can be improved. In Section 5, we demonstrate usefulness of our results in a particular problem of nonparametric statistics: many local means problem. In Section 6, we develop our new smoothing inequalities. In Sections 7–10, we give all the proofs. Finally, in Section 11, we collect several known lemmas, which are used in our derivations.

1.1. Notation. In the following, we assume $n \geq 3$ and $d \geq 3$ so that $\log n > 1$ and $\log d > 1$. Also, for any $\eta > 0$, we use $R(0, \eta)$ to denote the centered ℓ_∞ -ball with radius η , namely $R(0, \eta) := \{y \in \mathbb{R}^d : \|y\|_\infty \leq \eta\}$. For any $A = \prod_{j=1}^d (a_j, b_j] \in \mathcal{R}$ and $t \in \mathbb{R}$, we denote $A^t := \prod_{j=1}^d (a_j - t, b_j + t]$ and $(\partial A)^t := A^t \setminus A^{-t}$. For any $r = (r_1, \dots, r_d)^T \in \mathbb{R}^d$ and $t \in \mathbb{R}$, we denote $r + t := (r_1 + t, \dots, r_d + t)^T \in \mathbb{R}^d$. In addition, for any $r \in \mathbb{R}^d$, we define the function $1_r : \mathbb{R}^d \rightarrow \mathbb{R}$ by $1_r(y) := 1\{y \leq r\}$ for all $y \in \mathbb{R}^d$. Moreover, for any matrix $S = (S_{jk})_{j,k=1}^{J,K}$, we use $\|S\|_\infty$ to denote its ℓ_∞ -norm, i.e. $\|S\|_\infty := \max_{1 \leq j \leq J} \max_{1 \leq k \leq K} |S_{ij}|$. For any matrices $S = (S_{jk})_{j,k=1}^{J,K}$ and $Q = (Q_{jk})_{j,k=1}^{J,K}$, we denote $\langle S, Q \rangle := \sum_{j=1}^J \sum_{k=1}^K S_{jk} Q_{jk}$.

Furthermore, we write $a \lesssim b$ if there exists a universal constant $C > 0$ such that $a \leq Cb$. Finally, for any random variable X and $q \geq 1$, we write $\|X\|_{L_q}$ and $\|X\|_{\psi_q}$ to denote the L_q - and ψ_q -norms of X , respectively, i.e. $\|X\|_{L_q} := (\mathbb{E}|X|^q)^{1/q}$ and $\|X\|_{\psi_q} := \inf\{C > 0: \mathbb{E}\psi_q(|X|/C) \leq 1\}$, where $\psi_q(x) := \exp(x^q) - 1$ for all $x > 0$. We formally define $\|X\|_{\psi_q}$ in the same way even when $q \in (0, 1]$, although it is not a norm but a quasi-norm.

2. GAUSSIAN APPROXIMATIONS

Let Σ be any $d \times d$ positive definite symmetric matrix with unit diagonal entries and let $\sigma_* > 0$ be the square root of its smallest eigenvalue. Define

$$\varrho_\Sigma = \sup_{A \in \mathcal{R}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)|, \quad Z \sim N(0, \Sigma),$$

so that $\varrho = \varrho_{\Sigma_W}$. In this subsection, we will derive bounds on ϱ_Σ . By substituting $\Sigma = \Sigma_W$, we are then able to derive direct bounds on ϱ . In addition, it will sometimes be possible to obtain better bounds on ϱ using the triangle inequality, namely $\varrho \leq |\varrho - \varrho_\Sigma| + \varrho_\Sigma$. The latter is possible when Σ_W is degenerate but can be well approximated by a non-degenerate Σ in the $\|\cdot\|_\infty$ -norm; see Remark 2.3 below for details.

Denote

$$\Delta_0 := \frac{\log d}{\sigma_*^2} \|\Sigma - \Sigma_W\|_\infty \quad \text{and} \quad \Delta_1 := \frac{(\log d)^2}{n^2 \sigma_*^4} \max_{1 \leq j \leq d} \sum_{i=1}^n \mathbb{E} X_{ij}^4.$$

First, we derive a bound on ϱ_Σ in the case of bounded X_i 's:

Theorem 2.1 (Gaussian Approximation, Bounded Case). *Suppose that there is a constant $\delta > 0$ such that $\|X_i\|_\infty/\sqrt{n} \leq \delta$ for every $i = 1, \dots, n$ almost surely. Then*

$$\varrho_\Sigma \leq C \left\{ \left(1 \vee \left| \log \left(\frac{\Delta_1}{\log d} + \frac{\delta^2 \log d}{\sigma_*^2} \right) \right| \right) \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\delta \log d)^2}{\sigma_*^2} \right) + \frac{\delta (\log d)^{3/2}}{\sigma_*} \right\},$$

where $C > 0$ is a universal constant.

Remark 2.1 (Optimality of Theorem 2.1). The most important feature of Theorem 2.1 is that it implies a nearly optimal bound on ϱ . Indeed, assuming that (i) $n^{-1} \sum_{i=1}^n \mathbb{E} X_{ij}^2 = 1$ for all $j = 1, \dots, d$, (ii) $|X_{ij}| \leq B_n$ almost surely for all $i = 1, \dots, n$ and $j = 1, \dots, d$ and some constant $B_n > 0$, possibly depending on n , and (iii) $\sigma_{*,W} \geq b$ for some constant $b > 0$, where $\sigma_{*,W}$ is the square root of the smallest eigenvalue of the correlation matrix of W , it follows from Theorem 2.1 that

$$\varrho \leq \frac{CB_n (\log d)^{3/2}}{\sqrt{n}} \log n, \tag{2.1}$$

where $C > 0$ is a constant depending only on b ; see Corollary 2.1 below for details. On the other hand, we will show in Proposition 2.1 below that under mild conditions on B_n and

d , there exists a distribution of X_i 's such that

$$\varrho \geq \frac{cB_n(\log d)^{3/2}}{\sqrt{n}}, \quad (2.2)$$

where $c > 0$ is a constant that is independent of (n, d, B_n) . Comparing (2.1) and (2.2), we conclude that the bound in Theorem 2.1 is optimal up to the $\log n$ factor. In addition, we will be able to get rid of the excessive $\log n$ factor in the case when X_i 's have an additive Gaussian component; see Theorem 4.1 below. \blacksquare

Remark 2.2 (Relation to Previous Work). The bound in (2.1) is as sharp as the bound obtained by [23] for the log-concave X_i 's, which is the first (nearly) sharp result in the non-degenerate case using implicit smoothing ideas and Stein's method. Subsequent work of [32] and [31], using the same implicit smoothing ideas combined with Lindeberg's method, obtained the following bound for more general non-degenerate cases:

$$\varrho \leq C' \left(\left(\frac{B_n^6(\log d)^4 \log(dn)}{n} \right)^{1/2} + \left(\frac{(\log d)^7 \log(dn)}{n} \right)^{1/2} \right) \log n, \quad (2.3)$$

under the same conditions as those aforementioned in Remark 2.1 and assuming also that $\mathbb{E}X_{ij}^2 = 1$ for all $i = 1, \dots, n$ and $j = 1, \dots, d$, where $C' > 0$ is a constant depending only on b . Our bound (2.1) is considerably sharper. First, it has much better dependence on the dimension d . For example, with B_n being independent of n and $d \geq n$, (2.1) depends on d via $(\log d)^{3/2}$ whereas (2.3) depends on d via $(\log d)^4$, which is a large improvement in the high-dimensional case, where $\log d$ is growing as some fractional power of the sample size n . Second, (2.1) has much better dependence on the envelope constant B_n : (2.1) depends on B_n linearly whereas (2.3) depends on B_n via B_n^3 . This second improvement is particularly important in classical applications to nonparametric statistics, where the intrinsic dimensionality of the problem often shows up not only via d but also via B_n . We illustrate this point in Section 5 through an example. \blacksquare

In Remark 2.3 below, we discuss how the bound on ϱ_Σ in Theorem 2.1 can be used to obtain bounds on ϱ_{Σ_W} when Σ_W is degenerate. To this end, we need the following Gaussian comparison lemma, which is a special case of Theorem 1.1 in [23] and is similar to Theorem 2.2 in [32].

Lemma 2.1 (Gaussian Comparison; [23], Theorem 1.1). *Let $Z \sim N(0, \Sigma)$, where Σ has unit entries on the diagonal, and $Z' \sim N(0, \Sigma')$, then*

$$\sup_{A \in \mathcal{R}} |\mathbb{P}(Z \in A) - \mathbb{P}(Z' \in A)| \leq C \frac{D}{\sigma_*^2} \log d \left(1 \vee \left| \log \frac{D}{\sigma_*^2} \right| \right),$$

where σ_*^2 is the smallest eigenvalue of Σ and $D = \|\Sigma - \Sigma'\|_\infty$.

Remark 2.3 (On Degenerate Cases). As we briefly mentioned above, having bounds on ϱ_Σ for general Σ in Theorem 2.1 rather than for $\Sigma = \Sigma_W$ is useful when Σ_W is degenerate.

Indeed, the direct application of Theorem 2.1 with $\Sigma = \Sigma_W$ gives a trivial bound as $\sigma_* = 0$ in this case. Instead, by the triangle inequality and Lemma 2.1, we have

$$\varrho \leq \varrho_\Sigma + C\Delta_0 \left(1 \vee \left| \log \left(\frac{\Delta_0}{\log d} \right) \right| \right), \quad (2.4)$$

where $C > 0$ is a universal constant. This bound can be combined with Theorem 2.1 to obtain useful bounds on ϱ whenever there exists Σ such that the square root of its smallest eigenvalue σ_* is strictly positive and $\|\Sigma - \Sigma_W\|_\infty$ is small. We illustrate this point in Section 5 through an example. \blacksquare

Next, we extend the result in Theorem 2.1 to allow for unbounded X_i 's. Denote

$$\mathcal{M} := \left(\mathbb{E} \left[\max_{1 \leq j \leq d} \max_{1 \leq i \leq n} |X_{ij}|^4 \right] \right)^{1/4}.$$

Also, denote

$$\Lambda_1 := (\log d)^2 (\log n) \log(dn).$$

Finally, for all $\psi > 0$, denote

$$M(\psi) := \max_{1 \leq i \leq n} \mathbb{E} \left[\|X_i\|_\infty^4 \mathbf{1}\{\|X_i\|_\infty > \psi\} \right].$$

We then have the following result:

Theorem 2.2 (Gaussian Approximation, Unbounded Case). *For all $\psi > 0$,*

$$\varrho_\Sigma \leq C \left\{ (\log n) \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\mathcal{M} \log d)^2}{n\sigma_*^2} \right) + \sqrt{\frac{\Lambda_1 M(\psi)}{n\sigma_*^4} + \frac{\psi (\log d)^{3/2}}{\sigma_* \sqrt{n}}} \right\},$$

where $C > 0$ is a universal constant.

We now apply Theorems 2.1 and 2.2 to derive bounds on $\varrho = \varrho_{\Sigma_W}$ under easily interpretable conditions. Let $q \geq 4$ be a constant and let $\{B_n\}_{n \geq 1}$ be a sequence of positive constants, possibly growing to infinity. Also, let $\sigma_{*,W}$ be the square root of the smallest eigenvalue of the correlation matrix of W and for all $j = 1, \dots, d$, denote $\sigma_j := (\mathbb{E}[W_j^2])^{1/2}$. Consider the following conditions:

- (E.1) $|X_{ij}/\sigma_j| \leq B_n$ for all $i = 1, \dots, n$ and $j = 1, \dots, d$ almost surely;
- (E.2) $\|X_{ij}/\sigma_j\|_{\psi_2} \leq B_n$ for all $i = 1, \dots, n$ and $j = 1, \dots, d$;
- (E.3) $\|\max_{1 \leq j \leq d} |X_{ij}/\sigma_j|\|_{L_q} \leq B_n$ for all $i = 1, \dots, n$;

and also consider condition

$$(M) \quad n^{-1} \sum_{i=1}^n \mathbb{E} |X_{ij}/\sigma_j|^4 \leq B_n^2 \text{ for all } j = 1, \dots, d;$$

Similar conditions were previously used and motivated by applications in [12, 15, 17, 20, 23, 19].

Corollary 2.1 (Gaussian Approximation under Simple Conditions). *Under condition (E.1), we have*

$$\varrho \leq \frac{CB_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2}, \quad (2.5)$$

where $C > 0$ is a universal constant; under conditions (M) and (E.2), we have

$$\varrho \leq C \left(\frac{B_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n(\log d)^2}{\sqrt{n}\sigma_{*,W}} \right),$$

where $C > 0$ is a universal constant; under conditions (M) and (E.3), we have

$$\varrho \leq C \left\{ \frac{B_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n^2(\log d)^2 \log n}{n^{1-2/q}\sigma_{*,W}^2} + \left(\frac{B_n^q(\log d)^{3q/2-4}(\log n) \log(dn)}{n^{q/2-1}\sigma_{*,W}^q} \right)^{\frac{1}{q-2}} \right\},$$

where $C > 0$ is a constant depending only on q .

Remark 2.4 (Dropping Condition (M)). Like in the case of condition (E.1), meaningful bounds on ϱ can be obtained without imposing condition (M) in the cases of (E.2) and (E.3) as well. This is so because both (E.2) and (E.3) imply bounds on the left-hand side of the inequality in condition (M). Indeed, it is straightforward to check that, for all $j = 1, \dots, d$, under (E.2), we have $n^{-1} \sum_{i=1}^n \mathbb{E}|X_{ij}/\sigma_j|^4 \lesssim B_n^2 \log n + B_n^4/n^2$ whereas under (E.3), we have $n^{-1} \sum_{i=1}^n \mathbb{E}|X_{ij}/\sigma_j|^4 \lesssim B_n^{2q/(q-2)}$. We do not present the implied bounds on ϱ for brevity of the paper. \blacksquare

We conclude this section with the proposition that provides a lower bound on the convergence rate of ϱ and demonstrates that the convergence rate in (2.1) is sharp up to the $\log n$ factor:

Proposition 2.1 (Lower Bound on ϱ). *Let $\{B_n\}_{n \geq 1}$ be a sequence of positive constants such that $B_n \geq 2$ for all n . Suppose that d depends on n so that*

$$\frac{B_n(\log d)^{3/2}}{\sqrt{n}} \rightarrow 0, \quad \frac{B_n^4}{\sqrt{\log d}} \rightarrow 0, \quad \frac{\sqrt{n}}{dB_n(\log d)^{3/2}} \rightarrow 0$$

as $n \rightarrow \infty$. Then, we can construct i.i.d. random vectors $X_{n,1}, \dots, X_{n,n}$ in \mathbb{R}^d for every n such that

$$\mathbb{E}[X_{n,ij}] = 0, \quad \mathbb{E}[X_{n,ij}^2] = 1, \quad |X_{n,ij}| \leq B_n, \quad \text{for all } n \geq 1, \quad i = 1, \dots, n, \quad j = 1, \dots, d;$$

and

$$\liminf_{n \rightarrow \infty} \frac{\sqrt{n}}{B_n(\log d)^{3/2}} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left(\max_{1 \leq j \leq d} \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{n,ij} \leq x \right) - \mathbb{P} \left(\max_{1 \leq j \leq d} Z_j \leq x \right) \right| > 0,$$

where Z_1, Z_2, \dots are independent standard normal variables.

This proposition extends Proposition 1.1 in [23] to allow for the n -dependent envelope constant B_n .

3. BOOTSTRAP APPROXIMATIONS

Since Σ_W is in practice typically unknown, the Gaussian approximations obtained in the previous section are typically infeasible in the sense that we are unable to calculate probabilities $\mathbb{P}(Z \in A)$, $A \in \mathcal{R}$ and $Z \sim N(0, \Sigma_W)$, which is needed in statistical applications. In this section, we therefore consider bootstrap approximations. These approximations allow to estimate $\mathbb{P}(Z \in A)$ from the sample X_1, \dots, X_n . We focus on the multiplier and empirical bootstrap approximations.

Throughout this section, let Σ be any $d \times d$ positive definite symmetric matrix with unit diagonal entries and let $\sigma_* > 0$ be the square root of its smallest eigenvalue. This is the same convention as that in the previous section.

3.1. Multiplier Bootstrap Approximation. Let ξ_1, \dots, ξ_n be i.i.d. $N(0, 1)$ random variables that are independent of $X := (X_1, \dots, X_n)$. Denote $\bar{X} := (\bar{X}_1, \dots, \bar{X}_d)^T := n^{-1} \sum_{i=1}^n X_i$ and consider the (Gaussian) multiplier bootstrap version of W :

$$W^\xi := \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i (X_i - \bar{X}).$$

In this subsection, we are interested in bounding

$$\varrho_\Sigma^\xi := \sup_{A \in \mathcal{R}} |\mathbb{P}(W^\xi \in A \mid X) - \mathbb{P}(Z \in A)|, \quad Z \sim N(0, \Sigma),$$

and, in particular, $\varrho^\xi := \varrho_{\Sigma_W}^\xi$. Denote

$$\Delta'_0 := \frac{\log d}{\sigma_*^2} \left\| \Sigma - \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})^T \right\|_\infty.$$

The following result is as an easy consequence of Lemma 2.1.

Theorem 3.1 (Multiplier Bootstrap). *We have*

$$\varrho_\Sigma^\xi \leq C \Delta'_0 \left(1 \vee \left| \log \left(\frac{\Delta'_0}{\log d} \right) \right| \right),$$

where $C > 0$ is a universal constant.

Applying Theorem 3.1 under easily interpretable conditions (M) and (E), we obtain the following analog of Corollary 2.1.

Corollary 3.1 (Multiplier Bootstrap under Simple Conditions). *Under condition (E.1), we have with probability at least $1 - \alpha$ that*

$$\varrho^\xi \leq \frac{CB_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2},$$

where $C > 0$ is a universal constant; under conditions (M) and (E.2), we have with probability at least $1 - \alpha$ that

$$\varrho^\xi \leq \frac{CB_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2},$$

where $C > 0$ is a universal constant; under conditions (M) and (E.3), we have with probability at least $1 - \alpha$ that

$$\varrho^\xi \leq \frac{C(\log d)(\log n)}{\sigma_{*,W}^2} \left(\frac{B_n\sqrt{\log(d/\alpha)}}{\sqrt{n}} + \frac{B_n^2(\log d + \alpha^{-2/q})}{n^{1-2/q}} \right),$$

where $C > 0$ is a constant depending only on q .

Remark 3.1 (Main Features of Corollary 3.1). The bounds in Corollary 3.1 are generally comparable with the corresponding bounds in Corollary 2.1. For example, under condition (E.1), combining Corollaries 2.1 and 3.1, we have that for some universal constant $C > 0$, with probability at least $1 - 1/d$,

$$\sup_{A \in \mathcal{R}} \left| \mathbb{P}(W \in A) - \mathbb{P}(W^\xi \in A \mid X) \right| \leq \frac{CB_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2}, \quad (3.1)$$

which has the same right-hand side as that of (2.5). Thus, we are able to obtain a feasible bootstrap approximation bound to probabilities $\mathbb{P}(W \in A)$ with the same convergence rate as that of the infeasible Gaussian approximation. Note also that under the assumption that $\sigma_{*,W}$ is bounded below from zero (strongly non-degenerate case in the terminology of [23]), (3.1) is much better than the general bound (which does not require $\sigma_{*,W} > 0$) following from the results in [17]. ■

Remark 3.2 (Other Types of Multipliers). In Theorem 3.1 and Corollary 3.1, we focused on Gaussian multipliers but we note that similar results can be obtained for other multipliers, e.g. Rademacher or Mammen multipliers; see [33] and [17] for definitions. To do so, we can apply Theorem 2.1 conditional on X_i 's to bound

$$\sup_{A \in \mathcal{R}} |\mathbb{P}(W^\xi \in A \mid X) - \mathbb{P}(W^\zeta \in A \mid X)|,$$

where W^ζ is defined by analogy with W^ξ with multipliers represented by random variables ζ_1, \dots, ζ_n instead of ξ_1, \dots, ξ_n . ■

3.2. Empirical Bootstrap Approximation. Let X_1^*, \dots, X_n^* be i.i.d. draws from the empirical distribution of X_1, \dots, X_n and consider the empirical bootstrap version of W :

$$W^* := \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i^* - \bar{X}).$$

In this subsection, we are interested in bounding

$$\varrho_{\Sigma}^* := \sup_{A \in \mathcal{R}} |\mathbb{P}(W^* \in A \mid X) - \mathbb{P}(Z \in A)|, \quad Z \sim N(0, \Sigma),$$

and, in particular, $\varrho^* := \varrho_{\Sigma_W}^*$. To do so, denote

$$\mathcal{M}^* := \max_{1 \leq j \leq d} \max_{1 \leq i \leq n} |X_{ij} - \bar{X}_j|$$

and, for all $\psi > 0$,

$$M^*(\psi) := \frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}\|_{\infty}^4 \mathbf{1}\{\|X_i - \bar{X}\|_{\infty} > \psi\}.$$

Also, denote

$$\Delta'_1 := \frac{(\log d)^2}{n^2 \sigma_*^4} \max_{1 \leq j \leq d} \sum_{i=1}^n (X_{ij} - \bar{X}_j)^4.$$

The following result is an easy consequence of Theorem 2.2.

Theorem 3.2 (Empirical Bootstrap). *For all $\psi > 0$,*

$$\varrho_{\Sigma}^* \leq C \left\{ (\log n) \left(\Delta'_0 + \sqrt{\Delta'_1 \log d} + \frac{(\mathcal{M}^* \log d)^2}{n \sigma_*^2} \right) + \sqrt{\frac{\Lambda_1 M^*(\psi)}{n \sigma_*^4} + \frac{\psi (\log d)^{3/2}}{\sigma_* \sqrt{n}}} \right\},$$

where $C > 0$ is a universal constant.

Like in the previous subsection, applying Theorem 3.2 under easily interpretable conditions (M) and (E), we obtain the following analog of Corollary 2.1.

Corollary 3.2 (Empirical Bootstrap under Simple Conditions). *Under condition (E.1), we have with probability at least $1 - \alpha$ that*

$$\varrho^* \leq \frac{CB_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2},$$

where $C > 0$ is a universal constant; under conditions (M) and (E.2), we have with probability at least $1 - \alpha$ that

$$\varrho^* \leq C \left(\frac{B_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n(\log(dn))^2\sqrt{\log(1/\alpha)}}{\sqrt{n}\sigma_{*,W}} \right),$$

where $C > 0$ is a universal constant; under conditions (M) and (E.3), we have with probability at least $1 - \alpha$ that

$$\varrho^* \leq C \left(\frac{B_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n\sqrt{\log(dn)}\log d}{n^{1/2-1/q}\alpha^{1/q}\sigma_{*,W}} \right),$$

where $C > 0$ is a constant depending only on q .

Remark 3.3 (Main Features of Corollary 3.2). Bounds for the empirical bootstrap approximation in this corollary are comparable but slightly worse than the corresponding bounds in Corollary 3.1 for the multiplier bootstrap approximation. However, since we only have upper bounds on the approximation error, this does not imply that the multiplier bootstrap is necessarily more precise than the empirical bootstrap. In fact, simulations in [20, 17, 19] suggest the opposite may be true, with approximation errors being rather similar for most practical purposes. Note also that, like in the case of Corollary 3.1, under the assumption that $\sigma_{*,W}$ is bounded below from zero, bounds in Corollary 3.2 are much better than the general bound (which does not require $\sigma_{*,W} > 0$) following from the results in [17]. ■

4. GAUSSIAN APPROXIMATION FOR SPECIAL CASES: SMOOTH AND ZERO SKEWNESS

In some special cases, the bounds in Theorems 2.1 and 2.2 can be improved. In this section, we consider two such cases and derive an improved version of Theorem 2.1. For brevity, we do not provide an improved version of Theorem 2.2.

An interesting practical case occurs when X_i 's are generated with additive Gaussian noise (for example, due to measurement error or injection of noise for data privacy). As we demonstrate here, we can improve the bound in Theorem 2.1 by removing a logarithmic pre-factor in this case. The proof of this result is relatively simple, so that it may be useful to review it before reading the more complicated proofs of Theorems 2.1 and 2.2.

As before, let X_1, \dots, X_n be independent centered random vectors in \mathbb{R}^d but now suppose that we observe only their noisy versions, say $\tilde{X}_1, \dots, \tilde{X}_n$, where $\tilde{X}_i = X_i + g_i$ for some centered Gaussian g_i and $G = \sum_{i=1}^n g_i/\sqrt{n} \sim N(0, \Sigma_0)$, such that G is independent of X_i 's. Assume that Σ_0 is non-degenerate and let $\sigma_{*,0} > 0$ be the square root of its smallest eigenvalue. Assume also that Σ_0 has unit diagonal entries (this assumption is not essential and is made to simplify the results below; by rescaling, similar results can be obtained as long as all diagonal entries of Σ_0 are of the same order). In addition, let Σ be any $d \times d$ non-negative definite symmetric matrix and let $\tilde{\Sigma} := \Sigma + \Sigma_0$. Denote $\tilde{W} := \sum_{i=1}^n \tilde{X}_i/\sqrt{n}$ and $W := \sum_{i=1}^n X_i/\sqrt{n}$, so that $\Sigma_{\tilde{W}} = \Sigma_W + \Sigma_0$, where $\Sigma_{\tilde{W}} := \mathbb{E}\tilde{W}\tilde{W}^T$ and $\Sigma_W := \mathbb{E}W W^T$. Also, denote

$$\tilde{\varrho}_\Sigma := \sup_{A \in \mathcal{R}} |\mathbb{P}(\tilde{W} \in A) - \mathbb{P}(\tilde{Z} \in A)|, \quad \tilde{Z} \sim N(0, \tilde{\Sigma}).$$

Below, we derive a bound on $\tilde{\varrho}_\Sigma$. Since the distribution of \tilde{X}_i 's is smooth because of the presence of the additive Gaussian components g_i , we refer to the results below as the Gaussian approximation in the smooth case. Following the literature, e.g. [43], we also sometimes refer to the distribution of \tilde{X}_i 's as quasi-Gaussian.

Theorem 4.1 (Gaussian Approximation, Smooth Case). *Suppose that there are constants $\delta, c > 0$ such that $\|X_i\|_\infty/\sqrt{n} \leq \delta$ for every $i = 1, \dots, n$ almost surely and $\delta\sqrt{\log d} \leq c\sigma_{*,0}$. Then*

$$\tilde{\varrho}_\Sigma \leq C \left(\tilde{\Delta}_0 + \tilde{\Delta}_1 \right),$$

where $C > 0$ is a constant depending only on c and

$$\tilde{\Delta}_0 := \frac{\log d}{\sigma_{*,0}^2} \|\Sigma - \Sigma_W\|_\infty, \quad \tilde{\Delta}_1 := \frac{(\log d)^{3/2}}{n^{3/2}\sigma_{*,0}^3} \max_{1 \leq j \leq d} \sum_{i=1}^n \mathbb{E}X_{ij}^3.$$

Remark 4.1 (Optimality of Theorem 4.1). In comparison with Theorem 2.1 and Corollary 2.1, Theorem 4.1 does not contain the logarithmic pre-factor. Assuming that (i) $n^{-1} \sum_{i=1}^n \mathbb{E}X_{ij}^2 \leq 1$ for all $j = 1, \dots, d$, (ii) $|X_{ij}| \leq B_n$ almost surely for all $i = 1, \dots, n$ and $j = 1, \dots, d$ and some constant $B_n > 0$, possibly depending on n , and (iii) $\sigma_{*,0}^3 \geq b$ for some constant $b > 0$, it follows from Theorem 4.1 that

$$\tilde{\varrho}_{\Sigma_W} \leq \frac{CB_n(\log d)^{3/2}}{\sqrt{n}}, \quad (4.1)$$

where $C > 0$ is a constant depending only on b . This bound is optimal in the quasi-Gaussian case with respect to both the sample size n and the dimension d , as follows from Proposition 1.1 in [23], which yields a lower bound and allows the lower bound to be achieved by the quasi-Gaussian distributions. Hence, it is not possible to obtain a better bound without imposing further conditions, such as zero skewness or symmetry of the distribution of X_i 's. \blacksquare

Our second example in this section demonstrates that, with a bit more structure, namely assuming the zero skewness condition, we can further improve the bounds. Most notably, the theorem below implies dependence on n via $1/n$ instead of $1/\sqrt{n}$ for uniformly bounded X_i 's.

Theorem 4.2 (Gaussian Approximation, Smooth and Zero Skewness Case). *Under the assumptions of Theorem 4.1, assume additionally that*

$$\mathbb{E}[X_{ij}X_{ik}X_{il}] = 0 \quad \text{for all } i = 1, \dots, n \text{ and } j, k, l = 1, \dots, d. \quad (4.2)$$

Then

$$\tilde{\varrho}_\Sigma \leq C \left(\tilde{\Delta}_0 + \tilde{\Delta}_2 \right),$$

where $C > 0$ is a constant depending only on c and

$$\tilde{\Delta}_0 := \frac{\log d}{\sigma_{*,0}^2} \|\Sigma - \Sigma_W\|_\infty, \quad \tilde{\Delta}_2 := \frac{(\log d)^2}{n^2\sigma_{*,0}^4} \max_{1 \leq j \leq d} \sum_{i=1}^n \mathbb{E}X_{ij}^4.$$

Remark 4.2 (Optimality of Theorem 4.2). Assuming that (i) $n^{-1} \sum_{i=1}^n \mathbb{E} X_{ij}^2 \leq 1$ for all $j = 1, \dots, d$, (ii) $|X_{ij}| \leq B_n$ almost surely for all $i = 1, \dots, n$ and $j = 1, \dots, d$ and some constant $B_n > 0$, possibly depending on n , and (iii) $\sigma_{*,0}^3 \geq b$ for some constant $b > 0$, it follows from Theorem 4.1 that

$$\tilde{\varrho}_{\Sigma_W} \leq \frac{CB_n^2(\log d)^2}{n}, \quad (4.3)$$

where $C > 0$ is a constant depending only on b . This bound is optimal in the quasi-Gaussian case with zero skewness with respect to both the sample size n and the dimension d , as we prove in Proposition 4.1 below. Moreover, neither the zero skewness nor quasi-Gaussian conditions can be dropped in general to get such dependences. In fact, if the former is not satisfied, we can at best get (4.1), as discussed in Remark 4.1 above. Similarly, it is well-known that the dependence on n should be $1/\sqrt{n}$ in the normal approximation rate for sums of independent Rademacher variables (see e.g. page 112 of [35]), so we cannot drop the quasi-Gaussian assumption in general to get a bound proportional to $1/n$. Finally, note that (4.3) is substantially better than (4.1), meaning that imposing the zero skewness condition is rather helpful in the quasi-Gaussian case. ■

We conclude this section with the proposition that provides a lower bound on the convergence rate of $\tilde{\varrho}_{\Sigma_W}$ under the quasi-Gaussian and zero skewness conditions and demonstrates that the convergence rate in (4.3) is sharp:

Proposition 4.1. *Let $X = (X_{ij})_{i,j=1}^\infty$ be an array of i.i.d. random variables such that $\|X_{ij}\|_{\psi_1} < \infty$, $\mathbb{E}[X_{ij}] = 0$, $\mathbb{E}[X_{ij}^2] = 1$, $\mathbb{E}[X_{ij}^3] = 0$ and $\gamma := \mathbb{E}[X_{ij}^4] - 3 \neq 0$. Let $W = n^{-1/2} \sum_{i=1}^n X_i$ with $X_i := (X_{i1}, \dots, X_{id})^T$. Suppose that d depends on n so that $(\log d)^2/n \rightarrow 0$ and $(\log d)^3/n \rightarrow \infty$ as $n \rightarrow \infty$. Also, let $Z \sim N(0, I_d)$. Then*

$$\limsup_{n \rightarrow \infty} \frac{n}{(\log d)^2} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left(\max_{1 \leq j \leq d} W_j \leq x \right) - \mathbb{P} \left(\max_{1 \leq j \leq d} Z_j \leq x \right) \right| > 0.$$

Remark 4.3 (Relation to Previous Work). This proposition complements Theorem 3 in [18], who showed that the Gaussian approximation with vanishing error is not possible if $(\log d)^2/n^{1+\delta} \not\rightarrow 0$ for some $\delta > 0$ and X_{ij} 's are Rademacher random variables. ■

5. APPLICATION TO MANY LOCAL MEANS PROBLEM

An interesting setting that illustrates the value of our new bounds is the problem of many local means, which plays a fundamental role in nonparametric statistics. In this problem, the dimensionality of the problem actually shows up in the envelope and moments of X_i 's and not just via $\log d$. We illustrate this point with the following simple example. Consider i.i.d. random vectors V_1, \dots, V_n in \mathbb{R}^k and non-overlapping regions $(R_j)_{j=1}^d$ that partition the support of V_i 's such that $p_j := \mathbb{P}\{V_i \in R_j\} = p$ for all $j = 1, \dots, d$ and $d = 1/p$. Define

components of X_i via:

$$X_{ij} = \frac{1\{V_i \in R_j\} - p}{\sqrt{p(1-p)}}, \quad j = 1, \dots, d,$$

and set $W := n^{-1/2} \sum_{i=1}^n X_i$. The distribution of W over the class of rectangles \mathcal{R} is of interest in testing hypotheses about the means of X_{ij} 's.

To apply our results in this setting, observe that

$$\Sigma_W = \mathbb{E}WW^T = \frac{p}{p(1-p)}I_d - \frac{p^2}{p(1-p)}1_d1_d',$$

where $1_d := (1, \dots, 1)^T \in \mathbb{R}^d$. The smallest eigenvalue of Σ_W is

$$\frac{p}{p(1-p)} - \frac{p^2}{p(1-p)}d = 0,$$

so this is actually a degenerate case. On the other hand, we have for $\Sigma := \frac{p}{p(1-p)}I_d$ that

$$\|\Sigma_W - \Sigma\|_\infty \leq p/(1-p) = 1/(d-1)$$

and all eigenvalues of Σ are bounded below from zero. Thus, applying (2.4) and Theorem 2.1 with $\Sigma = \frac{p}{p(1-p)}I_d$ and $\delta := \sqrt{d/n}$ to bound ϱ_Σ , we have that

$$\varrho \lesssim \frac{(\log d)^2}{d} + \sqrt{\frac{d(\log d)^3}{n}} \log n. \quad (5.1)$$

This bound may be rather poor if $d \rightarrow \infty$ slowly. Fortunately, we can combine (5.1) with the bound we previously derived in [17] to obtain

$$\varrho \lesssim \left(\frac{(\log d)^2}{d} + \sqrt{\frac{d(\log d)^3}{n}} \log n \right) \wedge \left(\frac{d(\log n)^5}{n} \right)^{1/4}, \quad (5.2)$$

which is much better than (5.1) when $d \rightarrow \infty$ slowly. Specifically, (5.2) gives

$$\varrho \rightarrow 0 \quad \text{if} \quad \frac{d(\log n)^5}{n} \rightarrow 0. \quad (5.3)$$

Turning now to the alternative bounds in the literature, we note that the direct application of results in [32] and [31] give an infinite bound on ϱ because Σ_W is degenerate. This is of course an unfair comparison, so it is possible to modify the arguments in [32] and [31] to have the dependencies in their bounds via $\|\Sigma - \Sigma_W\|_\infty$, as we did in Remark 2.3, and obtain

$$\varrho \lesssim \frac{(\log d)^2}{d} + \left(\sqrt{\frac{d^3(\log d)^4 \log n}{n}} + \sqrt{\frac{(\log d)^7 \log(dn)}{n}} \right) \log n \quad (5.4)$$

This bound gives, when $d \rightarrow \infty$:

$$\varrho \rightarrow 0 \quad \text{if} \quad \frac{d^3(\log n)^7}{n} \rightarrow 0. \quad (5.5)$$

Comparing (5.3) with (5.5), we conclude that (5.2) is substantially better than (5.4).

In addition, it is possible to obtain a bound on ϱ via the Hungarian coupling. In particular, results in [37] and [27] imply that one can construct a centered Gaussian random vector G in \mathbb{R}^d such that

$$\|W - G\|_\infty \lesssim \sqrt{\frac{\log n}{(np)^{1/\kappa}}} + \sqrt{\frac{\log^2 n}{np}}$$

almost surely. Moreover, [5] showed that that bound is sharp up to possible log factors when $\kappa \geq 2$. Combining this bound with the anti-concentration inequality in Lemma 11.3 implies

$$\varrho \lesssim \sqrt{\log d} \left(\sqrt{\frac{\log n}{(np)^{1/\kappa}}} + \sqrt{\frac{\log^2 n}{np}} \right) \quad (5.6)$$

When $d \geq n^{1/3}$, which is the most relevant case, this bound is better than that in (5.2) by a $(\log n)^2$ factor for $\kappa = 1$ but worse for $\kappa \geq 2$. For $\kappa \geq 3$, this bound is much worse than that in (5.2) by a polynomial-in- n factor regardless of d .

6. MIXED SMOOTHING INEQUALITIES

Let $\phi > 0$, $\epsilon \in [0, 1]$, and $r \in \mathbb{R}^d$. Also, let Σ be a $d \times d$ symmetric positive definite matrix with unit diagonal entries, and let $\sigma_* > 0$ be the square root of the smallest eigenvalue of Σ . Consider functions $g^\phi: \mathbb{R} \rightarrow \mathbb{R}$, $m^{r,\phi}: \mathbb{R}^d \rightarrow \mathbb{R}$, and $\rho^{r,\phi,\epsilon,\Sigma}: \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$g^\phi(t) := \begin{cases} 1 & \text{if } t \leq 0, \\ 1 - \phi t & \text{if } 0 < t < 1/\phi, \\ 0 & \text{if } t \geq 1/\phi, \end{cases}$$

$$m^{r,\phi}(w) := g^\phi \left(\max_{1 \leq j \leq d} (w_j - r_j) \right), \quad w \in \mathbb{R}^d, \quad (6.1)$$

and

$$\rho^{r,\phi,\epsilon,\Sigma}(w) := \mathbb{E} m^{r,\phi}(w + \epsilon Z), \quad w \in \mathbb{R}^d, \quad (6.2)$$

where Z is a centered normal random vector in \mathbb{R}^d with covariance matrix Σ . For large ϕ and small ϵ , the function $\rho^{r,\phi,\epsilon,\Sigma}(\cdot)$ provides a good approximation to the indicator function $1_r(\cdot) = 1\{\cdot \leq r\}$ but, in contrast to the indicator function, is smooth. In particular, we will prove the following inequalities, which play a key role in obtain sharp bounds for the Gaussian approximation.

Lemma 6.1. *Let $v \in \mathbb{Z}$ and $K \in \mathbb{R}$ be such that $v \geq 1$ and $K > 0$. Set $\eta = \eta_d = K/\sqrt{\log d}$. Then*

$$\sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \sigma_* \eta)} |\partial_{j_1, \dots, j_v} \rho^{r,\phi,\epsilon,\Sigma}(w + y)| \leq C \frac{\phi(\log d)^{(v-1)/2}}{(\epsilon \sigma_*)^{v-1}}, \quad (6.3)$$

where $C > 0$ is a constant depending only on v and K .

Lemma 6.2. *Let $v \in \mathbb{Z}$ and $K \in \mathbb{R}$ be such that $v \geq 1$ and $K > 0$. Set $\eta = \eta_d = K/\sqrt{\log d}$. Then*

$$\sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \sigma_* \eta)} |\partial_{j_1, \dots, j_v} \rho^{r, \phi, \epsilon, \Sigma}(w + y)| \leq C \frac{(\log d)^{v/2}}{(\epsilon \sigma_*)^v},$$

where $C > 0$ is a constant depending only on v and K .

Lemma 6.3. *Let $v \in \mathbb{Z}$ be such that $v \geq 1$. Then for all $\kappa, \eta > 0$ with $\kappa > \eta$,*

$$\sup_{w \in (A_2 \setminus A_1)^c} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \sigma_* \eta)} |\partial_{j_1, \dots, j_v} \rho^{r, \phi, \epsilon, \Sigma}(w + y)| \leq C \frac{d^v}{(\epsilon \sigma_*)^v} e^{-(\kappa - \eta)^2/4},$$

where $C > 0$ is a constant depending only on v and

$$A_1 = \{w \in \mathbb{R}^d : w \leq r - 2\epsilon\kappa\}, \quad A_2 = \{w \in \mathbb{R}^d : w \leq r + 2\epsilon\kappa + \phi^{-1}\}.$$

Remark 6.1 (Relation to Previous Work). All three lemmas here are new. Their proofs are inspired by the original ideas of [7], who derived Lemma 6.2 without smoothing (with $\phi = \infty$) and $\eta = 0$. See also [23] who extended the result of [7] to allow for $\eta > 0$ in Lemma 6.2 using related methods of [3]. Having $\eta \neq 0$ is important for establishing the optimal dependence on the envelopes. \blacksquare

7. PROOFS FOR SECTION 2

Proof of Theorem 2.1. For all $i = 1, \dots, n$, we denote $\xi_i := X_i/\sqrt{n}$, so that $W = \sum_{i=1}^n \xi_i$ and $\|\xi_i\|_\infty \leq \delta$. Working with ξ_i 's is a little more convenient than working with X_i 's. Also, we assume, without loss of generality, that W and Z are independent. In addition, since $\varrho_\Sigma \leq 1$, we assume, again without loss of generality, that

$$\frac{\Delta_1}{\log d} + \frac{\delta^2 \log d}{\sigma_*^2} \leq \frac{1}{3}. \quad (7.1)$$

Further, observe that for any $w \in \mathbb{R}^d$ and $A = \prod_{j=1}^d (a_j, b_j] \in \mathcal{R}$, $\mathbb{P}(W \in A) = \mathbb{P}(W \leq b) - \mathbb{P}(W \leq a)$, where $a = (a_1, \dots, a_d)^T \in \mathbb{R}^d$ and $b = (b_1, \dots, b_d)^T \in \mathbb{R}^d$. Therefore, it suffices to prove the asserted claim with ϱ_Σ replaced by

$$\varrho'_\Sigma := \sup_{r \in \mathbb{R}^d} |\mathbb{P}(W \leq r) - \mathbb{P}(Z \leq r)|, \quad Z \sim N(0, \Sigma),$$

which is what we do below.

Now, for any bounded measurable function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ and $t \in [0, 1]$, define $T_t h : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$T_t h(w) = \mathbb{E}h(\sqrt{1-t}w + \sqrt{t}Z) - \mathbb{E}h(Z), \quad w \in \mathbb{R}^d.$$

Then, by Lemmas 11.2 and 11.3 and the fact that $\mathbb{P}(\|Z\|_\infty > \sqrt{4\log d}) \leq 1/2$, we have

$$\varrho' \lesssim \sup_{r \in \mathbb{R}^d} |\mathbb{E}T_t 1_r(W)| + \sqrt{\frac{t}{1-t}} \log d. \quad (7.2)$$

By taking the value of t appropriately, we will deduce a recursive inequality for ϱ' ; see (7.23) below. In particular, we set

$$t := \frac{\Delta_1}{\log d} + \frac{\delta^2 \log d}{\sigma_*^2}. \quad (7.3)$$

Note here that because of (7.1), $|\log t| \geq 1$ and $1/\sqrt{1-t} \leq 2$.

Further, fix $\phi > 0$, to be chosen below in (7.24), and for any $r \in \mathbb{R}^d$, consider the smoothed indicator function $m^{r,\phi}: \mathbb{R}^d \rightarrow \mathbb{R}$ as in (6.1) of Section 6. Denoting $\tilde{W} = \sqrt{1-t}W + \sqrt{t}Z$, we have by Lemma 11.3 that

$$\begin{aligned} \mathbb{P}(\tilde{W} \leq t) &\leq \mathbb{E}m^{r,\phi}(\tilde{W}) = \mathbb{E}m^{r,\phi}(Z) + \mathbb{E}m^{r,\phi}(\tilde{W}) - \mathbb{E}m^{r,\phi}(Z) \\ &\leq \mathbb{P}(Z \leq r + 1/\phi) + \mathbb{E}m^{r,\phi}(\tilde{W}) - \mathbb{E}m^{r,\phi}(Z) \\ &\leq \mathbb{P}(Z \leq r) + C\sqrt{\log d}/\phi + \mathbb{E}m^{r,\phi}(\tilde{W}) - \mathbb{E}m^{r,\phi}(Z) \end{aligned}$$

and, similarly,

$$\begin{aligned} \mathbb{P}(Z \leq r) &\leq \mathbb{P}(Z + 1/\phi \leq r) + C\sqrt{\log d}/\phi \leq \mathbb{E}m^{r,\phi}(Z + 1/\phi) + C\sqrt{\log d}/\phi \\ &= \mathbb{E}m^{r,\phi}(\tilde{W} + 1/\phi) + \mathbb{E}m^{r,\phi}(Z + 1/\phi) - \mathbb{E}m^{r,\phi}(\tilde{W} + 1/\phi) + C\sqrt{\log d}/\phi \\ &\leq \mathbb{P}(\tilde{W} \leq r) + \mathbb{E}m^{r,\phi}(Z + 1/\phi) - \mathbb{E}m^{r,\phi}(\tilde{W} + 1/\phi) + C\sqrt{\log d}/\phi, \end{aligned}$$

where $C > 0$ is a universal constants. Hence,

$$\sup_{r \in \mathbb{R}^d} \left| \mathbb{P}(\tilde{W} \leq r) - \mathbb{P}(Z \leq r) \right| \lesssim \sup_{r \in \mathbb{R}^d} \left| \mathbb{E}m^{r,\phi}(\tilde{W}) - \mathbb{E}m^{r,\phi}(Z) \right| + \sqrt{\log d}/\phi,$$

and so,

$$\sup_{r \in \mathbb{R}^d} |\mathbb{E}T_t 1_r(W)| \lesssim \sup_{r \in \mathbb{R}^d} |\mathbb{E}T_t m^{r,\phi}(W)| + \sqrt{\log d}/\phi. \quad (7.4)$$

Given (7.2) and (7.4), we need to bound $\sup_{r \in \mathbb{R}^d} |\mathbb{E}T_t m^{r,\phi}(W)|$.

To do so, fix any $r \in \mathbb{R}$ (we will take the supremum in (7.23)), write $h := m^{r,\phi}$, and proceed to bound $|\mathbb{E}T_t h(W)|$. By the fundamental theorem of calculus and the fact that $\mathbb{E}T_1 h(W) = 0$,

$$\mathbb{E}T_t h(W) = -\frac{1}{2} \int_t^1 \mathbb{E} \left\langle \nabla h(\sqrt{1-s}W + \sqrt{s}Z), \frac{Z}{\sqrt{s}} - \frac{W}{\sqrt{1-s}} \right\rangle ds,$$

and so, using Lemma 11.1,

$$\mathbb{E}T_t h(W) = -\frac{1}{2} \int_t^1 \mathbb{E} \left[\langle \Sigma, \nabla^2 h_s(\sqrt{1-s}W) \rangle - \left\langle \frac{W}{\sqrt{1-s}}, \nabla h_s(\sqrt{1-s}W) \right\rangle \right] ds, \quad (7.5)$$

where for all $s \in [t, 1]$, the function $h_s: \mathbb{R}^d \rightarrow \mathbb{R}$ is given by

$$h_s(w) = \mathbb{E}h(w + \sqrt{s}Z), \quad w \in \mathbb{R}^d.$$

Here, it is useful to note that $h_s = \rho^{r, \phi, \sqrt{s}, \Sigma}$, where $\rho^{r, \phi, \epsilon, \Sigma}$ with $\epsilon = \sqrt{s}$ is the function appearing in (6.2) of Section 6. In particular, h_s is infinitely differentiable, with derivatives satisfying bounds in Lemmas 6.1, 6.2, and 6.3. These bounds will be used below.

To bound the integral in (7.5), we employ the exchangeable pair approach in Stein's method for multivariate normal approximation by [11] and [36] along with a symmetry argument by [23, 24] (cf. (7.13)–(7.14) below). Define $\xi = (\xi_i)_{i=1}^n$ and let $\xi' = (\xi'_i)_{i=1}^n$ be an independent copy of ξ . Also, let I be a random index uniformly chosen from $\{1, \dots, n\}$ and independent of ξ and ξ' . In addition, define $Y_i := \xi'_i - \xi_i$ and $W' := W + Y_I$. It is then easy to verify that (W, W') has the same distribution as (W', W) (exchangeability) and

$$\mathbb{E}[W' - W \mid W'] = \frac{W'}{n}, \quad \mathbb{E}[W' - W \mid W] = -\frac{W}{n}. \quad (7.6)$$

Therefore, denoting $D := W' - W$, we have

$$\begin{aligned} \mathbb{E} \left\langle \frac{W}{\sqrt{1-s}}, \nabla h_s(\sqrt{1-s}W) \right\rangle &= \mathbb{E} \left\langle \frac{W'}{\sqrt{1-s}}, \nabla h_s(\sqrt{1-s}W') \right\rangle \\ &= \mathbb{E} \left\langle \frac{nD}{\sqrt{1-s}}, \nabla h_s(\sqrt{1-s}W') \right\rangle. \end{aligned} \quad (7.7)$$

Express the right-hand side of this chain of identities, using Taylor's expansion around W with exact integral remainder, as:

$$\begin{aligned} \mathbb{E} \left\langle \frac{nD}{\sqrt{1-s}}, \nabla h_s(\sqrt{1-s}W) \right\rangle &+ \mathbb{E} \langle nDD^T, \nabla^2 h_s(\sqrt{1-s}W) \rangle \\ &+ n \sum_{j,k,l=1}^d \sqrt{1-s} \mathbb{E} \left[UD_j D_k D_l \partial_{jkl} h_s \left(\sqrt{1-s}(W + (1-U)D) \right) \right], \end{aligned} \quad (7.8)$$

where U is a uniform random variable on $[0, 1]$ independent of everything else, and note also that by (7.6),

$$\mathbb{E} \left\langle \frac{nD}{\sqrt{1-s}}, \nabla h_s(\sqrt{1-s}W) \right\rangle = -\mathbb{E} \left\langle \frac{W}{\sqrt{1-s}}, \nabla h_s(\sqrt{1-s}W) \right\rangle. \quad (7.9)$$

Therefore, substituting (7.8) and (7.9) into (7.7) and rearranging terms, we obtain

$$\begin{aligned} \mathbb{E} \left\langle \frac{W}{\sqrt{1-s}}, \nabla h_s(\sqrt{1-s}W) \right\rangle &= \frac{n}{2} \mathbb{E} \langle DD^T, \nabla^2 h_s(\sqrt{1-s}W) \rangle \\ &+ \frac{n}{2} \sum_{j,k,l=1}^d \sqrt{1-s} \mathbb{E} \left[UD_j D_k D_l \partial_{jkl} h_s \left(\sqrt{1-s}(W + (1-U)D) \right) \right], \end{aligned}$$

and so, by (7.5),

$$\mathbb{E}T_t h(W) = \mathbb{E}T_t m^{r,\phi}(W) = -\frac{1}{2} \int_t^1 (R_1(s) - R_2(s)) ds, \quad (7.10)$$

where

$$\begin{aligned} R_1(s) &:= \sum_{j,k=1}^d \mathbb{E} \left[\mathbb{E} \left[\Sigma_{jk} - \frac{n}{2} D_j D_k \mid \xi \right] \partial_{jk} h_s(\sqrt{1-s}W) \right], \\ R_2(s) &:= \frac{n}{2} \sum_{j,k,l=1}^d \sqrt{1-s} \mathbb{E} \left[U D_j D_k D_l \partial_{jkl} h_s \left(\sqrt{1-s}(W + (1-U)D) \right) \right]. \end{aligned}$$

We bound $R_1(s)$ and $R_2(s)$ in turn. Regarding $R_1(s)$, we have by Lemma 6.2 that

$$\sup_{w \in \mathbb{R}^d} \sum_{j,k=1}^d |\partial_{jk} h_s(w)| \lesssim \frac{\log d}{\sigma_*^2 s}.$$

Hence,

$$\int_t^1 |R_1(s)| ds \lesssim \frac{(\log d) |\log t|}{\sigma_*^2} \mathbb{E} \left\| \Sigma - \frac{n}{2} \mathbb{E}[DD^T \mid \xi] \right\|_\infty.$$

Here, recalling that $D = W' - W = Y_I$, one can deduce

$$\Sigma - \frac{n}{2} \mathbb{E}[DD^T \mid \xi] = (\Sigma - \Sigma_W) - \frac{1}{2} \sum_{i=1}^n (\xi_i \xi_i^T - \mathbb{E}[\xi_i \xi_i^T]),$$

and so

$$\mathbb{E} \left\| \Sigma - \frac{n}{2} \mathbb{E}[DD^T \mid \xi] \right\|_\infty \leq \|\Sigma - \Sigma_W\|_\infty + \frac{1}{2} \mathbb{E} \left\| \sum_{i=1}^n (\xi_i \xi_i^T - \mathbb{E}[\xi_i \xi_i^T]) \right\|_\infty.$$

Also, by Lemma 11.4, the second term is bounded, up to an absolute constant, by

$$\begin{aligned} \max_{1 \leq j,k \leq d} \sqrt{\sum_{i=1}^n \mathbb{E}[\xi_{ij}^2 \xi_{ik}^2]} \sqrt{\log d} + \sqrt{\mathbb{E} \left[\max_{1 \leq i \leq n} \max_{1 \leq j,k \leq d} \xi_{ij}^2 \xi_{ik}^2 \right] \log d} \\ \lesssim \max_{1 \leq j \leq d} \sqrt{\sum_{i=1}^n \mathbb{E}[\xi_{ij}^4]} \sqrt{\log d} + \delta^2 \log d. \end{aligned} \quad (7.11)$$

Consequently, we obtain

$$\frac{1}{2} \int_t^1 |R_1(s)| ds \lesssim |\log t| \times \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\delta \log d)^2}{\sigma_*^2} \right). \quad (7.12)$$

Next, we bound $R_2(s)$. We have

$$\mathbb{E} \left[U D_j D_k D_l \partial_{jkl} h_s \left(\sqrt{1-s}(W + (1-U)D) \right) \right]$$

$$\begin{aligned}
&= -\mathbb{E}\left[UD_jD_kD_l\partial_{jkl}h_s\left(\sqrt{1-s}(W'-(1-U)D)\right)\right] \\
&= -\mathbb{E}\left[UD_jD_kD_l\partial_{jkl}h_s\left(\sqrt{1-s}(W+UD)\right)\right], \tag{7.13}
\end{aligned}$$

where the first equality holds by exchangeability and the second by $W' = W + D$. Hence, using Taylor's expansion one more time, we obtain

$$\begin{aligned}
\frac{|R_2(s)|}{\sqrt{1-s}} &= \left| \frac{n}{4} \sum_{j,k,l=1}^d \mathbb{E}\left[UD_jD_kD_l\partial_{jkl}h_s\left(\sqrt{1-s}(W+(1-U)D)\right)\right] \right. \\
&\quad \left. - \frac{n}{4} \sum_{j,k,l=1}^d \mathbb{E}\left[UD_jD_kD_l\partial_{jkl}h_s\left(\sqrt{1-s}(W+UD)\right)\right] \right| \\
&\leq \frac{n\sqrt{1-s}}{4} \sum_{j,k,l,m=1}^d \mathbb{E}\left[\left|D_jD_kD_lD_m\partial_{jklm}h_s\left(\sqrt{1-s}(W+UD+U'(1-2U)D)\right)\right|\right] \\
&= \frac{\sqrt{1-s}}{4} \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E}\left[\left|Y_{ij}Y_{ik}Y_{il}Y_{im}\partial_{jklm}h_s\left(\sqrt{1-s}(W^{(i)}+\tilde{\xi}_i)\right)\right|\right], \tag{7.14}
\end{aligned}$$

where U' is a uniform random variable on $[0, 1]$ independent of everything else, $W^{(i)} := W - \xi_i$, and $\tilde{\xi}_i := \xi_i + UY_i + U'(1-2U)Y_i$. Here, note that $|U+U'(1-2U)| \leq U \vee (1-U) \leq 1$ and thus $\|\tilde{\xi}_i\|_\infty \leq \|\xi_i\|_\infty + \|Y_i\|_\infty \leq 3\delta$. Therefore, given that $\|\xi_i\|_\infty \leq \delta$ and that Y_i is independent of $W^{(i)}$, we have that $4|R_2(s)|/(1-s)$ is bounded by:

$$\begin{aligned}
&\sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E}\left[\left|Y_{ij}Y_{ik}Y_{il}Y_{im} \sup_{\|y\|_\infty \leq 3\delta} \left|\partial_{jklm}h_s\left(\sqrt{1-s}(W^{(i)}+y)\right)\right|\right|\right] \\
&= \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E}[|Y_{ij}Y_{ik}Y_{il}Y_{im}|] \mathbb{E}\left[\sup_{\|y\|_\infty \leq 3\delta} \left|\partial_{jklm}h_s\left(\sqrt{1-s}(W^{(i)}+y)\right)\right|\right] \\
&\leq \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E}[|Y_{ij}Y_{ik}Y_{il}Y_{im}|] \mathbb{E}\left[\sup_{\|y\|_\infty \leq 4\delta} \left|\partial_{jklm}h_s\left(\sqrt{1-s}(W+y)\right)\right|\right] \\
&= \sum_{j,k,l,m=1}^d \left(\sum_{i=1}^n \mathbb{E}[|Y_{ij}Y_{ik}Y_{il}Y_{im}|]\right) \mathbb{E}\left[\sup_{\|y\|_\infty \leq 4\delta} \left|\partial_{jklm}h_s\left(\sqrt{1-s}(W+y)\right)\right|\right] \\
&\lesssim \max_{1 \leq j \leq d} \sum_{i=1}^n \mathbb{E}\xi_{ij}^4 \mathbb{E}\left[\sum_{j,k,l,m=1}^d \sup_{\|y\|_\infty \leq 4\delta} \left|\partial_{jklm}h_s\left(\sqrt{1-s}(W+y)\right)\right|\right].
\end{aligned}$$

Further, let

$$\eta := 4/\sqrt{\log d}, \quad \kappa := \sqrt{16 \log d - 2 \log(1 - \sqrt{1-t})} + \eta, \tag{7.15}$$

so that for any $t \in (0, 1)$,

$$e^{-(\kappa-\eta)^2/4} = \frac{\sqrt{1-\sqrt{1-t}}}{d^4} \leq \frac{\sqrt{t}}{d^4}. \quad (7.16)$$

Also, for any $s \in [t, 1]$,

$$4\delta\sqrt{1-s} \leq 4\delta \leq \eta\sigma_*\sqrt{s}, \quad (7.17)$$

where the second inequality follows from (7.3). Then using (7.17) and denoting

$$\varepsilon_s := 2\sqrt{s}\kappa + 1/\phi, \quad (7.18)$$

we have

$$\begin{aligned} & \sum_{j,k,l,m=1}^d \sup_{\|y\|_\infty \leq 4\delta} \left| \partial_{jklm} h_s(\sqrt{1-s}(W+y)) \right| \\ & \lesssim \frac{\phi(\log d)^{3/2} \mathbb{1}_{\{\sqrt{1-s}W \in (\partial A)^{\varepsilon_s}\}}}{(\sigma_*\sqrt{s})^3} + \frac{\sqrt{t}}{d^4} \frac{d^4}{(\sigma_*\sqrt{s})^4}, \end{aligned} \quad (7.19)$$

where the first term on the right-hand side appears from bounding the left-hand side by Lemma 6.1 and the second term appears from bounding the left-hand side by Lemma 6.3 and using (7.16) (here, we denote $A := \{y \in \mathbb{R}^d : y \leq r\}$ and for any $x \in \mathbb{R}$, we write $A^x := \{y \in \mathbb{R}^d : y \leq r+x\}$ and $(\partial A)^x := A^x \setminus A^{-x}$). Hence,

$$|R_2(s)| \lesssim (1-s) \frac{\Delta_1 \sigma_*^4}{(\log d)^2} \mathbb{E} \left[\frac{\phi(\log d)^{3/2} \mathbb{1}_{\{\sqrt{1-s}W \in (\partial A)^{\varepsilon_s}\}}}{(\sigma_*\sqrt{s})^3} + \frac{\sqrt{t}}{(\sigma_*\sqrt{s})^4} \right]. \quad (7.20)$$

Next,

$$\mathbb{P}(\sqrt{1-s}W \in (\partial A)^{\varepsilon_s}) \leq \mathbb{P}(\sqrt{1-s}Z \in (\partial A)^{\varepsilon_s}) + 2\varrho' \lesssim \frac{\varepsilon_s \sqrt{\log d}}{\sqrt{1-s}} + \varrho' \quad (7.21)$$

by the definition of ϱ' and Lemma 11.3, using that Σ has unit diagonal entries.

Inserting this bound into (7.20), we deduce

$$|R_2(s)| \lesssim \frac{\Delta_1 \sigma_*^4}{(\log d)^2} \left(\frac{\phi(\log d)^{3/2}}{(\sigma_*\sqrt{s})^3} (\varepsilon_s \sqrt{\log d} + \varrho') + \frac{\sqrt{t}}{(\sigma_*\sqrt{s})^4} \right).$$

Thus, using (7.18), we have

$$\begin{aligned} \int_t^1 |R_2(s)| ds & \lesssim \frac{\Delta_1 \sigma_*^4}{(\log d)^2} \left(\frac{\kappa\phi(\log d)^2 |\log t|}{\sigma_*^3} + \frac{(\log d)^2}{\sigma_*^3 \sqrt{t}} + \frac{\varrho' \phi(\log d)^{3/2}}{\sigma_*^3 \sqrt{t}} + \frac{1}{\sigma_*^4 \sqrt{t}} \right) \\ & \lesssim \Delta_1 \left(\kappa\phi |\log t| + \frac{1}{\sqrt{t}} + \frac{\varrho' \phi}{\sqrt{t \log d}} \right) \end{aligned} \quad (7.22)$$

since $\sigma_* \leq 1$ (recall that all diagonal entries of Σ are equal to one). From (7.2), (7.4), (7.10), (7.12) and (7.22), we obtain

$$\varrho' \leq c \left[|\log t| \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\delta \log d)^2}{\sigma_*^2} \right) + \Delta_1 \left(\kappa \phi |\log t| + \frac{1}{\sqrt{t}} + \frac{\varrho' \phi}{\sqrt{t \log d}} \right) + \sqrt{t} \log d + \frac{\sqrt{\log d}}{\phi} \right], \quad (7.23)$$

where $c > 0$ is a universal constant (recall that $1/\sqrt{1-t} \leq 2$). Now we are ready to specify the value of ϕ :

$$\phi := \frac{1}{2c\sqrt{\Delta_1 |\log t|}}. \quad (7.24)$$

Combining this choice with the choice of t in (7.3) yields

$$c\Delta_1 \frac{\varrho' \phi}{\sqrt{t \log d}} \leq \frac{\varrho'}{2}. \quad (7.25)$$

Also, since $t \leq 1/3$ by (7.1), κ defined in (7.15) satisfies

$$\kappa \lesssim \sqrt{\log d} + \sqrt{|\log t|}. \quad (7.26)$$

The asserted claim now follows by substituting (7.3), (7.24), (7.25), and (7.26) into (7.23). \blacksquare

Proof of Theorem 2.2. The proof is a modification of the proof of Theorem 2.1. Here, we describe the changes, keeping all unmentioned notations the same as those in the proof of Theorem 2.1. In a nutshell, we only need to change the values of t and ϕ and use a truncation argument in the bound for $R_2(s)$.

First, we now set

$$t := \frac{\Delta_1}{\log d} + \frac{M(\psi) \log d}{n\sigma_*^4} + \frac{\psi^2 \log d}{n\sigma_*^2} \quad (7.27)$$

instead of using t in (7.3). As in the proof of Theorem 2.1, it is without loss of generality to assume here that $t \leq 1/3$; compare with (7.1) and (7.3). Moreover, since all diagonal entries of Σ are equal to one, it follows that $\sigma_* \leq 1$. In addition, for a while, we assume $n^{-1} \sum_{i=1}^n \mathbb{E} X_{ij}^2 \geq 1/2$ for all $j = 1, \dots, d$. This implies via Jensen's inequality that $\Delta_1 / \log d \geq 1/(4n)$, and so $|\log t| \lesssim \log n$.

Next, note that (7.2), (7.4), and (7.10) hold under our current assumptions by the same arguments as those in the proof of Theorem 2.1. Thus, we only need to bound

$$\int_t^1 |R_1(s)| ds \quad \text{and} \quad \int_t^1 |R_2(s)| ds.$$

Regarding the former, we proceed as in the proof of Theorem 2.1 but we change the second line in (7.11) by

$$\max_{1 \leq j \leq d} \sqrt{\sum_{i=1}^n \mathbb{E}[\xi_{ij}^4]} \sqrt{\log d} + \frac{\mathcal{M}^2 \log d}{n}$$

so that similarly to (7.12), we obtain

$$\frac{1}{2} \int_t^1 |R_1(s)| ds \lesssim |\log t| \times \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\mathcal{M} \log d)^2}{n\sigma_*^2} \right). \quad (7.28)$$

Further, by (7.14), we have

$$R_2(s) \lesssim (1-s) \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E} \left[\left| Y_{ij} Y_{ik} Y_{il} Y_{im} \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + \tilde{\xi}_i) \right) \right| \right] \leq \mathcal{I}_1(s) + \mathcal{I}_2(s),$$

where

$$\mathcal{I}_1(s) := (1-s) \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E} \left[\left| \varsigma_i Y_{ij} Y_{ik} Y_{il} Y_{im} \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + \tilde{\xi}_i) \right) \right| \right],$$

$$\mathcal{I}_2(s) := (1-s) \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E} \left[\left| (1-\varsigma_i) Y_{ij} Y_{ik} Y_{il} Y_{im} \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + \tilde{\xi}_i) \right) \right| \right],$$

and $\varsigma_i := 1\{\|\xi_i\|_\infty \vee \|\xi'_i\|_\infty \leq 2\psi/\sqrt{n}\}$ for all $i = 1, \dots, n$. We first focus on $\mathcal{I}_1(s)$. Given that $\|\tilde{\xi}_i\|_\infty \leq \|\xi_i\|_\infty + \|Y_i\|_\infty \leq 2\|\xi_i\|_\infty + \|\xi'_i\|_\infty$ and that Y_i is independent of $W^{(i)}$, we have that $|\mathcal{I}_1(s)|$ is bounded by:

$$\begin{aligned} & (1-s) \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E} \left[\left| \varsigma_i Y_{ij} Y_{ik} Y_{il} Y_{im} \sup_{\|y\|_\infty \leq 6\psi/\sqrt{n}} \left| \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + y) \right) \right| \right| \right] \\ &= (1-s) \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E}[\varsigma_i |Y_{ij} Y_{ik} Y_{il} Y_{im}|] \mathbb{E} \left[\sup_{\|y\|_\infty \leq 6\psi/\sqrt{n}} \left| \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + y) \right) \right| \right] \\ &\leq \mathcal{I}_{11}(s) + \mathcal{I}_{12}(s), \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_{11}(s) &:= (1-s) \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E}[|Y_{ij} Y_{ik} Y_{il} Y_{im}|] \mathbb{E} \left[\left| \varsigma_i \sup_{\|y\|_\infty \leq 6\psi/\sqrt{n}} \left| \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + y) \right) \right| \right| \right], \\ \mathcal{I}_{12}(s) &:= (1-s) \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E}[|Y_{ij} Y_{ik} Y_{il} Y_{im}|] \mathbb{E} \left[\left| (1-\varsigma_i) \sup_{\|y\|_\infty \leq 6\psi/\sqrt{n}} \left| \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + y) \right) \right| \right| \right]. \end{aligned}$$

At this step, $\int_t^1 \mathcal{I}_{11}(s) ds$ is bounded in the same way as $\int_t^1 |R_2(s)| ds$ in the proof of Theorem 2.1; namely,

$$\int_t^1 \mathcal{I}_{11}(s) ds \lesssim \Delta_1 \left(\kappa\phi |\log t| + \frac{1}{\sqrt{t}} + \frac{\phi' \phi}{\sqrt{t} \log d} \right);$$

compare with (7.22). Meanwhile, using the independence between $W^{(i)}$ and Y_i , we obtain

$$\begin{aligned} \mathcal{I}_{12}(s) &\leq (1-s) \sum_{i=1}^n \sum_{j,k,l,m=1}^d \mathbb{E} \|Y_i\|_\infty^4 \mathbb{E}[1 - \varsigma_i] \mathbb{E} \left[\sup_{\|y\|_\infty \leq 6\psi/\sqrt{n}} \left| \partial_{jklm} h_s \left(\sqrt{1-s}(W^{(i)} + y) \right) \right| \right] \\ &\lesssim (1-s) \sum_{i=1}^n \mathbb{E}(1 - \varsigma_i) (\|\xi_i\|_\infty \vee \|\xi'_i\|_\infty)^4 \mathbb{E} \left[\sum_{j,k,l,m=1}^d \sup_{\|y\|_\infty \leq 6\psi/\sqrt{n}} \left| \partial_{jklm} h_s \left(\sqrt{1-s}(W^{(i)} + y) \right) \right| \right], \end{aligned}$$

where the last inequality follows from Chebyshev's association inequality; see Theorem 2.14 in [10]. Then, applying (7.19) with replacing W and 4δ by $W^{(i)}$ and $6\psi/\sqrt{n}$ respectively, we deduce

$$\mathcal{I}_{12}(s) \lesssim (1-s) \sum_{i=1}^n \mathbb{E}(1 - \varsigma_i) (\|\xi_i\|_\infty \vee \|\xi'_i\|_\infty)^4 \mathbb{E} \left[\frac{\phi(\log d)^{3/2} \mathbf{1}_{\{\sqrt{1-s}W^{(i)} \in (\partial A)^{\varepsilon_s}\}}}{(\sigma_* \sqrt{s})^3} + \frac{\sqrt{t}}{(\sigma_* \sqrt{s})^4} \right].$$

Now, as in the proof of (7.22), we obtain

$$\mathcal{I}_{12}(s) \lesssim \sum_{i=1}^n \mathbb{E}(1 - \varsigma_i) (\|\xi_i\|_\infty \vee \|\xi'_i\|_\infty)^4 \left(\frac{\phi(\log d)^{3/2}}{(\sigma_* \sqrt{s})^3} (\varepsilon_s \sqrt{\log d} + \varrho^{\{i\}}) + \frac{\sqrt{t}}{(\sigma_* \sqrt{s})^4} \right),$$

where, for any subset $\mathcal{I} \subset \{1, \dots, n\}$,

$$\varrho^{\mathcal{I}} := \sup_{r \in \mathbb{R}^d} \left| \mathbb{P} \left(\frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}^c} X_i \leq r \right) - \mathbb{P}(Z \leq r) \right|, \quad Z \sim N(0, \Sigma), \quad i = 1, \dots, n.$$

In addition, for all $i = 1, \dots, n$,

$$\begin{aligned} \mathbb{E}(1 - \varsigma_i) (\|\xi_i\|_\infty \vee \|\xi'_i\|_\infty)^4 &\leq \mathbb{E}(1 - \varsigma_i) (\|\xi_i\|_\infty^4 + \|\xi'_i\|_\infty^4) \\ &\lesssim \mathbb{E}(\mathbf{1}_{\{\|\xi_i\|_\infty > \psi/\sqrt{n}\}} + \mathbf{1}_{\{\|\xi'_i\|_\infty > \psi/\sqrt{n}\}}) (\|\xi_i\|_\infty^4 + \|\xi'_i\|_\infty^4) \\ &\lesssim \mathbb{E} \mathbf{1}_{\{\|\xi_i\|_\infty > \psi/\sqrt{n}\}} \|\xi_i\|_\infty^4 \leq M(\psi)/n^2, \end{aligned} \quad (7.29)$$

where the penultimate inequality holds by Chebyshev's association inequality. Thus,

$$\begin{aligned} \mathcal{I}_{12}(s) &\lesssim \frac{M(\psi)}{n} \left(\frac{\phi(\log d)^{3/2}}{(\sigma_* \sqrt{s})^3} ((\sqrt{s}\kappa + 1/\phi)\sqrt{\log d} + \bar{\varrho}) + \frac{\sqrt{t}}{(\sigma_* \sqrt{s})^4} \right) \\ &\lesssim \frac{M(\psi)}{n} \left(\frac{\kappa\phi(\log d)^2}{\sigma_*^3 s} + \frac{(\log d)^2}{\sigma_*^3 s^{3/2}} + \frac{\bar{\varrho}\phi(\log d)^{3/2}}{\sigma_*^3 s^{3/2}} + \frac{\sqrt{t}}{\sigma_*^4 s^2} \right), \end{aligned}$$

where $\bar{\varrho} := n^{-1} \sum_{i=1}^n \varrho^{\{i\}}$. Therefore, we conclude

$$\int_t^1 \mathcal{I}_{12}(s) ds \lesssim \frac{M(\psi)(\log d)^2}{n\sigma_*^4} \left(\kappa\phi |\log t| + \frac{1}{\sqrt{t}} + \frac{\bar{\varrho}\phi}{\sqrt{t \log d}} \right).$$

Turning to $\mathcal{I}_2(s)$, we have by the law of iterated expectations,

$$\mathcal{I}_2(s) \lesssim (1-s) \sum_{i=1}^n \mathbb{E} \left[(1-\varsigma_i) \|\xi_i - \xi'_i\|_\infty^4 \sum_{j,k,l,m=1}^d \mathbb{E} \left[\left| \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + \tilde{\xi}_i) \right) \right| \mid \tilde{\xi}_i \right] \right].$$

Here, we bound the internal sum as in (7.15) with $\eta = 0$, (7.18), (7.19), and (7.21); namely, by Lemmas 6.1 and 6.3,

$$\begin{aligned} (1-s) \sum_{j,k,l,m=1}^d \mathbb{E} \left[\left| \partial_{jklm} h_s \left(\sqrt{1-s} (W^{(i)} + \tilde{\xi}_i) \right) \right| \mid \tilde{\xi}_i \right] \\ \lesssim \frac{\phi(\log d)^{3/2}}{(\sigma_* \sqrt{s})^3} \left((\sqrt{s} \kappa + 1/\phi) \sqrt{\log d} + \varrho^{(i)} \right) + \frac{\sqrt{t}}{(\sigma_* \sqrt{s})^4} \\ \lesssim \frac{\kappa \phi (\log d)^2}{\sigma_*^3 s} + \frac{(\log d)^2}{\sigma_*^3 s^{3/2}} + \frac{\varrho^{(i)} \phi (\log d)^{3/2}}{\sigma_*^3 s^{3/2}} + \frac{\sqrt{t}}{\sigma_*^4 s^2}. \end{aligned}$$

Thus, by (7.29),

$$\int_t^1 \mathcal{I}_2(s) ds \lesssim \frac{M(\psi)(\log d)^2}{n\sigma_*^4} \left(\kappa \phi |\log t| + \frac{1}{\sqrt{t}} + \frac{\bar{\varrho} \phi}{\sqrt{t \log d}} \right).$$

Combining all terms and recalling that $|\log t| \lesssim \log n$, we now have

$$\begin{aligned} \varrho' \leq c \left[(\log n) \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\mathcal{M} \log d)^2}{n\sigma_*^2} \right) \right. \\ \left. + \left(\Delta_1 + \frac{M(\psi)(\log d)^2}{n\sigma_*^4} \right) \left(\kappa \phi \log n + \frac{1}{\sqrt{t}} + \frac{(\varrho' + \bar{\varrho}) \phi}{\sqrt{t \log d}} \right) + \sqrt{t} \log d + \frac{\sqrt{\log d}}{\phi} \right], \end{aligned} \quad (7.30)$$

where $c > 0$ is a universal constant; compare with (7.23). Here, we set

$$\phi := \frac{1}{2ec \sqrt{\left(\Delta_1 + \frac{M(\psi)(\log d)^2}{n\sigma_*^4} \right) \log n}}$$

and use t defined in (7.27) to obtain

$$\begin{aligned} \varrho' \leq C' \left[(\log n) \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\mathcal{M} \log d)^2}{n\sigma_*^2} \right) \right. \\ \left. + \sqrt{\left(\Delta_1 + \frac{M(\psi)(\log d)^2}{n\sigma_*^4} \right) (\log n) \log(dn) + \frac{\psi(\log d)^{3/2}}{\sigma_* \sqrt{n}}} \right] + \frac{\varrho' + \bar{\varrho}}{2e} \end{aligned} \quad (7.31)$$

since $\kappa \lesssim \sqrt{\log(dn)}$ by (7.26), where C' is a universal constant. Hence, rearranging the terms and substituting the definition of Λ_1 ,

$$\varrho' \leq 2C' \left[(\log n) \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\mathcal{M} \log d)^2}{n\sigma_*^2} \right) + \sqrt{\frac{\Lambda_1 M(\psi)}{n\sigma_*^4} + \frac{\psi(\log d)^{3/2}}{\sigma_* \sqrt{n}}} \right] + \frac{\bar{\varrho}}{e}.$$

Now we note that this bound is valid even when $n^{-1} \sum_{i=1}^n \mathbb{E}X_{ij}^2 < 1/2$ for some j ; in fact, we have $\Delta_0 \geq (\log d/\sigma_*^2)|\Sigma_{jj} - n^{-1} \sum_{i=1}^n \mathbb{E}X_{ij}^2| > 1/2$ in this case because $\Sigma_{jj} = 1$ and $\sigma_* \leq 1$. As a result, this bound holds without the restriction $n^{-1} \sum_{i=1}^n \mathbb{E}X_{ij}^2 \geq 1/2$ for all j imposed at the beginning of the proof. We now iterate this bound to obtain inequalities for each $\varrho^{\{i\}}$ and repeat the procedure $\lceil \log n \rceil + 1$ times, dropping one observation at a time. Here, given a proper subset $\mathcal{I} \subset \{1, \dots, n\}$, we apply this bound to $\varrho^{\mathcal{I}}$ with replacing $(X_i)_{i=1}^n$ and ψ by $(\sqrt{\frac{n-|\mathcal{I}|}{n}}X_i)_{i \in \mathcal{I}^c}$ and $\sqrt{\frac{n-|\mathcal{I}|}{n}}\psi$, respectively. Then, denoting the corresponding $\Delta_0, \Delta_1, \mathcal{M}, M(\psi)$ by $\Delta_0^{\mathcal{I}}, \Delta_1^{\mathcal{I}}, \mathcal{M}^{\mathcal{I}}, M^{\mathcal{I}}(\sqrt{\frac{n-|\mathcal{I}|}{n}}\psi)$ respectively, we have

$$\begin{aligned} \Delta_0^{\mathcal{I}} &= \frac{\log d}{\sigma_*^2} \max_{1 \leq j, k \leq d} \left| \Sigma_{jk} - \frac{1}{n-|\mathcal{I}|} \sum_{i \in \mathcal{I}^c} \frac{n-|\mathcal{I}|}{n} \mathbb{E}X_{ij}X_{ik} \right| \leq \Delta_0 + \frac{\log d}{n\sigma_*^2} \max_{1 \leq j \leq d} \sum_{i \in \mathcal{I}} \mathbb{E}X_{ij}^2 \\ &\leq \Delta_0 + \frac{\log d}{n\sigma_*^2} \max_{1 \leq j \leq d} \sqrt{|\mathcal{I}| \sum_{i \in \mathcal{I}} \mathbb{E}X_{ij}^4} \leq \Delta_0 + \sqrt{|\mathcal{I}| \Delta_1 \log d}, \\ \Delta_1^{\mathcal{I}} &= \frac{(\log d)^2}{(n-|\mathcal{I}|)^2 \sigma_*^4} \max_{1 \leq j \leq d} \sum_{i \in \mathcal{I}^c} \frac{(n-|\mathcal{I}|)^2}{n^2} \mathbb{E}X_{ij}^4 \leq \Delta_1, \end{aligned}$$

and

$$\frac{(\mathcal{M}^{\mathcal{I}})^2}{n-|\mathcal{I}|} \leq \frac{\mathcal{M}^2}{n}, \quad \frac{M^{\mathcal{I}}(\sqrt{\frac{n-|\mathcal{I}|}{n}}\psi)}{n-|\mathcal{I}|} \leq \frac{M(\psi)}{n}.$$

Note that $\sum_{i=1}^{\infty} e^{-i\sqrt{i}} < \infty$. We thus obtain

$$\varrho' \lesssim (\log n) \left(\Delta_0 + \sqrt{\Delta_1 \log d} + \frac{(\mathcal{M} \log d)^2}{n\sigma_*^2} \right) + \sqrt{\frac{\Lambda_1 M(\psi)}{n\sigma_*^4}} + \frac{\psi(\log d)^{3/2}}{\sigma_* \sqrt{n}} + \frac{\widehat{\varrho}}{e^{\log n}},$$

where

$$\widehat{\varrho} := \frac{1}{|\widehat{\mathcal{N}}|} \sum_{N \in \widehat{\mathcal{N}}} \sup_{r \in \mathbb{R}^d} \left| \mathbb{P} \left(\frac{1}{\sqrt{n}} \sum_{i \in N} X_i \leq r \right) - \mathbb{P}(Z \leq r) \right|, \quad Z \sim N(0, \Sigma),$$

and $\widehat{\mathcal{N}} := \{N \subset \{1, \dots, n\} : |N| = n - \lceil \log n \rceil - 2\}$. Since $\widehat{\varrho} \leq 1$ and $\Delta_1/\log d \geq 1/(4n)$, as discussed above, the asserted claim follows. \blacksquare

Proof of Corollary 2.1. Since

$$\sup_{A \in \mathcal{R}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| = \sup_{A \in \mathcal{R}} |\mathbb{P}(SW \in A) - \mathbb{P}(SZ \in A)|$$

for any $d \times d$ diagonal matrix S , we assume, without loss of generality, that $\sigma_j = 1$ for all $j = 1, \dots, d$. Then $\sigma_{*,W}$ is the square root of the smallest eigenvalue of $\Sigma_W = \mathbb{E}WW^T$. To prove the asserted claims, we will apply Theorems 2.1 and 2.2 with $\Sigma = \Sigma_W$, so that $\sigma_* = \sigma_{*,W}$.

Consider first the case when (E.1) holds. By Jensen's inequality, $\Delta_1 \geq (\log d)^2/(n\sigma_{*,W}^4) \geq (\log d)/n$, and so

$$\left| \log \left(\frac{\Delta_1}{\log d} + \frac{\delta^2 \log d}{\sigma_*^2} \right) \right| \leq \log n.$$

Also, since (E.1) implies (M),

$$(\log n)\sqrt{\Delta_1 \log d} \leq \frac{B_n(\log d)^{3/2}(\log n)}{\sqrt{n}\sigma_{*,W}^2}. \quad (7.32)$$

Combining these inequalities and using Theorem 2.1 with $\delta = B_n/\sqrt{n}$ and $\Delta_0 = 0$ gives the asserted claim under condition (E.1).

Next, consider the case when (M) and (E.2) hold. Without loss of generality, we assume that

$$\frac{B_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}} \leq 1 \quad (7.33)$$

since otherwise the asserted claims are trivial. (7.32) holds by condition (M). In addition, $\mathcal{M} \lesssim B_n\sqrt{\log(dn)}$ by Lemma 2.2.2 and discussion on page 95 of [40], and so

$$\frac{(\log n)(\mathcal{M} \log d)^2}{n\sigma_{*,W}^2} \lesssim \frac{B_n^2(\log d)^2(\log n) \log(dn)}{n\sigma_{*,W}^2} \lesssim \frac{B_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2} \quad (7.34)$$

by (7.33). Moreover, since for any $i = 1, \dots, n$ and $\psi > 0$,

$$\mathbb{E}\|X_i\|_\infty^4 \mathbf{1}\{\|X_i\|_\infty > \psi\} \leq (\mathbb{E}\|X_i\|_\infty^8) \mathbb{P}(\|X_i\|_\infty > \psi)^{1/2} \lesssim B_n^4 \sqrt{\mathbb{P}(\|X_i\|_\infty > \psi)},$$

setting $\psi = CB_n\sqrt{\log(dn)}$ for a sufficiently large but universal constant C , we have

$$\begin{aligned} \sqrt{\frac{\Lambda_1 M(\psi)}{n\sigma_{*,W}^4} + \frac{\psi(\log d)^{3/2}}{\sqrt{n}\sigma_{*,W}}} &\lesssim \frac{B_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n(\log d)^{3/2} \sqrt{\log(dn)}}{\sqrt{n}\sigma_{*,W}} \\ &\lesssim \frac{B_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n(\log d)^2}{\sqrt{n}\sigma_{*,W}}, \end{aligned} \quad (7.35)$$

where the last inequality follows from $\sigma_{*,W} \leq 1$. Combining $\Delta_0 = 0$, (7.32), (7.34), and (7.35) and applying Theorem 2.2 gives the asserted claim under conditions (M) and (E.2).

Now consider the case when (M) and (E.3) hold. In this case,

$$\mathcal{M} \leq \left(\mathbb{E} \left[\sum_{i=1}^n \max_{1 \leq j \leq d} |X_{ij}|^q \right] \right)^{1/q} \leq n^{1/q} B_n,$$

and so

$$\frac{(\log n)(\mathcal{M} \log d)^2}{n\sigma_{*,W}^2} \leq \frac{B_n^2(\log d)^2 \log n}{n^{1-2/q}\sigma_{*,W}^2}. \quad (7.36)$$

Also, (7.32) holds by the same arguments as those in the previous case. In addition, since for any $i = 1, \dots, n$ and $\psi > 0$,

$$\mathbb{E}\|X_i\|_\infty^4 1_{\{\|X_i\|_\infty > \psi\}} \leq \mathbb{E}\|X_i\|_\infty^q / \psi^{q-4} \leq B_n^q / \psi^{q-4},$$

setting

$$\psi = \left(\frac{B_n^q (\log n) \log(dn)}{\sigma_{*,W}^2 \log d} \right)^{1/(q-2)},$$

we obtain

$$\sqrt{\frac{\Lambda_1 M(\psi)}{n\sigma_{*,W}^4}} + \frac{\psi (\log d)^{3/2}}{\sqrt{n}\sigma_{*,W}} \lesssim \left(\frac{B_n^q (\log d)^{3q/2-4} (\log n) \log(dn)}{n^{q/2-1} \sigma_{*,W}^q} \right)^{\frac{1}{q-2}}. \quad (7.37)$$

Combining $\Delta_0 = 0$, (7.32), (7.36), and (7.37) and applying Theorem 2.2 gives the asserted claim under conditions (M) and (E.3) and completes the proof of the theorem. \blacksquare

Proof of Proposition 2.1. For every $n \geq 1$, let $(X_{n,ij})_{i,j=1}^\infty$ be an array of i.i.d. variables such that

$$\mathbb{P}(X_{n,ij} = a_n) = 1 - \mathbb{P}(X_{n,ij} = b_n) = p_n,$$

where

$$p_n := \frac{1}{B_n^2}, \quad a_n := \sqrt{\frac{1-p_n}{p_n}}, \quad b_n := -\sqrt{\frac{p_n}{1-p_n}}.$$

Since $B_n \geq 2$, we have $|X_{n,ij}| \leq \max\{B_n, 1/\sqrt{3}\} \leq B_n$. Also, it is straightforward to check that $\mathbb{E}[X_{n,ij}] = 0$ and $\mathbb{E}[X_{n,ij}^2] = 1$. Therefore, we complete the proof once we show that there is a sequence $(x_n)_{n=1}^\infty$ of real numbers such that

$$\rho := \liminf_{n \rightarrow \infty} \frac{\sqrt{n}}{B_n \log^{3/2} d} \left| \mathbb{P} \left(\max_{1 \leq j \leq d} W_{n,j} \leq x_n \right) - \mathbb{P} \left(\max_{1 \leq j \leq d} Z_j \leq x_n \right) \right| > 0,$$

where

$$W_{n,j} := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{n,ij}.$$

For every n , we define $x_n \in \mathbb{R}$ as the solution of the equation $\Phi_1(x)^d = e^{-1}$, i.e. $x_n := \Phi_1^{-1}(e^{-1/d})$. Then we have $x_n/\sqrt{2 \log d} \rightarrow 1$ as $n \rightarrow \infty$ (cf. the proof of Proposition 2.1 in [30]). We also have

$$d(1 - \Phi_1(x_n)) = d(1 - e^{-1/d}) = 1 + O(d^{-1}) \quad \text{as } n \rightarrow \infty. \quad (7.38)$$

Now, applying Theorem 1 in [4] with $I = \{1, \dots, d\}$, $B_\alpha = \{\alpha\}$ and $X_\alpha = 1_{\{W_{n,\alpha} > x_n\}}$ in their notation, we obtain

$$\left| \mathbb{P} \left(\max_{1 \leq j \leq d} W_{n,j} \leq x_n \right) - e^{-\lambda_n} \right| \leq d \mathbb{P}(W_{n,1} > x_n)^2,$$

where $\lambda_n := d\mathbb{P}(W_{n,1} > x_n)$. Meanwhile, we have by definition

$$\mathbb{P}\left(\max_{1 \leq j \leq d} Z_j \leq x_n\right) = \Phi_1(x_n)^d = e^{-1}.$$

Hence we obtain

$$\left| \mathbb{P}\left(\max_{1 \leq j \leq d} W_{n,j} \leq x_n\right) - \mathbb{P}\left(\max_{1 \leq j \leq d} Z_j \leq x_n\right) \right| \geq |e^{-\lambda_n} - e^{-1}| - \frac{\lambda_n^2}{d}. \quad (7.39)$$

To evaluate λ_n , we apply Theorem 2.1 in [25] with $m = n$, $X_\alpha = X_{n,\alpha 1}$, $\mathcal{I}_i = \{i\}$, $\xi_i = X_{n,i1}/\sqrt{n}$, $\delta = \delta_n := B_n/\sqrt{n}$ and $s = d = 1$ in their notation. We have by assumption

$$n^2 \delta_n^5 x_n^2 = O\left(\frac{B_n^5 \log d}{\sqrt{n}}\right) = o\left(\frac{B_n \log^{3/2} d}{\sqrt{n}}\right) = o(1). \quad (7.40)$$

Hence there is a constant $C_0 > 0$ such that $\sqrt{n^2 \delta_n^5} x_n \leq C_0$ for all n . Therefore, Theorem 2.1 in [25] yields

$$\left| \frac{\mathbb{P}(W_{n,1} > x_n)}{(1 - \Phi_1(x_n))e^{\gamma_n x_n^3/6}} - 1 \right| \leq C n^2 \delta_n^5 (1 + x_n^2) \quad \text{for all } n, \quad (7.41)$$

where $\gamma_n := \mathbb{E}[W_{n,1}^3]$ and C is a constant depending only on C_0 . Since

$$\gamma_n = \frac{\mathbb{E}[X_{n,11}^3]}{\sqrt{n}} = \frac{1 - 2p_n}{\sqrt{np_n(1-p_n)}},$$

we have

$$\liminf_{n \rightarrow \infty} \frac{\gamma_n x_n^3}{B_n (\log d)^{3/2} / \sqrt{n}} \geq \liminf_{n \rightarrow \infty} \frac{1}{2B_n \sqrt{p_n}} \left(\frac{x_n}{\sqrt{\log d}}\right)^3 = \sqrt{2}. \quad (7.42)$$

In particular, $\gamma_n x_n^3 = O(B_n (\log d)^{3/2} / \sqrt{n}) = o(1)$ by assumption. Combining this estimate with (7.38) and (7.40), we deduce from (7.41)

$$\begin{aligned} \lambda_n &= \frac{\mathbb{P}(W_{n,1} > x_n)}{(1 - \Phi_1(x_n))e^{\gamma_n x_n^3/6}} \cdot d(1 - \Phi_1(x_n))e^{\gamma_n x_n^3/6} \\ &= d(1 - \Phi_1(x_n))e^{\gamma_n x_n^3/6} + o\left(\frac{B_n \log^{3/2} d}{\sqrt{n}}\right) \\ &= e^{\gamma_n x_n^3/6} + O\left(\frac{1}{d}\right) + o\left(\frac{B_n \log^{3/2} d}{\sqrt{n}}\right). \end{aligned}$$

Using the Maclaurin expansion of the exponential function and (7.42), we obtain

$$\lambda_n = 1 + \frac{\gamma_n x_n^3}{6} + O\left(\frac{1}{d}\right) + o\left(\frac{B_n \log^{3/2} d}{\sqrt{n}}\right)$$

and thus

$$e^{-\lambda_n + 1} = 1 - \frac{\gamma_n x_n^3}{6} + O\left(\frac{1}{d}\right) + o\left(\frac{B_n \log^{3/2} d}{\sqrt{n}}\right).$$

Note that we particularly have $\lambda_n = O(1)$. Thus, (7.39) yields

$$\rho \geq \liminf_{n \rightarrow \infty} \frac{\sqrt{n}}{B_n \log^{3/2} d} |e^{-\lambda_n} - e^{-1}| = \liminf_{n \rightarrow \infty} e^{-1} \frac{\sqrt{n}}{B_n \log^{3/2} d} \frac{\gamma_n x_n^3}{6},$$

where we used the assumption $d^{-1} = o(B_n \log^{3/2} d / \sqrt{n})$. Hence we obtain by (7.42)

$$\rho \geq e^{-1} \frac{\sqrt{2}}{6} > 0.$$

This completes the proof. \blacksquare

Remark 7.1. In the above proof, it seems impossible to use a more traditional moderate deviation result instead of Theorem 2.1 in [25]. This is because such a one requires that the moment generating function of $X_{n,11}$ is bounded uniformly in n on a neighborhood of the origin (see Lemma 4.1 in [29] for instance). \blacksquare

8. PROOFS FOR SECTION 3

Proof of Theorem 3.1. The asserted claim is an immediate consequence of Theorem 1.1 in [23] (restated as Lemma 2.1 in this paper) once we note that (i) conditional on X , the random vector $W^\xi = n^{-1} \sum_{i=1}^n \xi_i (X_i - \bar{X})(X_i - \bar{X})^T$ is centered Gaussian with covariance matrix $n^{-1} \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})^T$ and (ii) for centered Gaussian random vectors, the Stein kernel is equal to the covariance matrix. \blacksquare

Proof of Corollary 3.1. Like in the proof of Corollary 2.1, we assume, without loss of generality, that $\sigma_j = 1$ for all $j = 1, \dots, d$, so that $\sigma_{*,W}$ is the square root of the smallest eigenvalue of $\Sigma_W = \mathbb{E}W W^T$. To prove the asserted claims, we will apply Theorem 3.1 with $\Sigma = \Sigma_W$, so that $\sigma_* = \sigma_{*,W}$. This requires bounding Δ'_0 with $\Sigma = \Sigma_W$. We do so separately for each case.

Consider first the case when (E.1) holds. We assume, without loss of generality, that

$$\frac{B_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} \leq 1, \quad (8.1)$$

since otherwise the asserted claim is trivial. Using this assumption, we will now prove that there exists a universal constant $C' \geq 1$ such that

$$\mathbb{P}\left(\Delta'_0 > \frac{C' B_n(\log d)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2}\right) \leq \alpha. \quad (8.2)$$

This derivation is similar to the proof of Proposition 4.1 in [15].

Note that

$$\sigma_{*,W}^2 \Delta'_0 / \log d \leq \Delta^{(1)} + \Delta^{(2)}, \quad (8.3)$$

where

$$\Delta^{(1)} := \left\| \frac{1}{n} \sum_{i=1}^n (X_i X_i^T - \mathbb{E} X_i X_i^T) \right\|_{\infty}, \quad \Delta^{(2)} := \|\bar{X} \bar{X}^T\|_{\infty} = \|\bar{X}\|_{\infty}^2. \quad (8.4)$$

We first bound $\Delta^{(1)}$. To do so, since (E.1) implies (M),

$$\sigma_n^2 := \max_{1 \leq j, k \leq d} \sum_{i=1}^n \mathbb{E} (X_{ij} X_{ik} - \mathbb{E}[X_{ij} X_{ik}])^2 \leq \max_{1 \leq j, k \leq d} \sum_{i=1}^n \mathbb{E} (X_{ij} X_{ik})^2 \leq n B_n^2.$$

Also,

$$\left\| \max_{1 \leq i \leq n} \max_{1 \leq j, k \leq d} |X_{ij} X_{ik}| \right\|_{\psi_1} \lesssim B_n^2 \log(dn),$$

so that $M_n := \max_{1 \leq i \leq n} \max_{1 \leq j, k \leq d} |X_{ij} X_{ik} - \mathbb{E}[X_{ij} X_{ik}]|$ satisfies

$$\sqrt{\mathbb{E} M_n^2} \lesssim \|M_n\|_{\psi_1} \lesssim B_n^2 \log(dn).$$

Hence, by Lemma 11.4,

$$\begin{aligned} \mathbb{E} \Delta^{(1)} &\lesssim n^{-1} \left(\sqrt{\sigma_n^2 \log d} + \sqrt{\mathbb{E} M_n^2 \log d} \right) \\ &\lesssim \sqrt{n^{-1} B_n^2 \log d} + n^{-1} B_n^2 (\log d) \log(dn) \lesssim \sqrt{n^{-1} B_n^2 \log d}, \end{aligned}$$

where the last inequality follows from (8.1). Thus, applying Lemma 11.5(i) with $\beta = \eta = 1$, we have for all $t > 0$ that $\Delta^{(1)} \lesssim \sqrt{n^{-1} B_n^2 \log d} + t$ with probability at least

$$1 - \exp\left(-\frac{nt^2}{3B_n^2}\right) - 3 \exp\left(-\frac{cnt}{B_n^2 \log(dn)}\right),$$

where $c > 0$ is a universal constant. Setting here

$$t = \frac{B_n \sqrt{3 \log(4/\alpha)}}{\sqrt{n}} + \frac{B_n^2 \log(dn) \log(12/\alpha)}{cn}$$

and recalling (8.1), we conclude that

$$\Delta^{(1)} \lesssim \sqrt{\frac{B_n^2 \log(d/\alpha)}{n}}$$

with probability at least $1 - \alpha/2$, and, by the same argument, we can also find that

$$\Delta^{(2)} \lesssim \sqrt{\frac{B_n^2 \log(d/\alpha)}{n}}$$

again with probability at least $1 - \alpha/2$. Combining these inequalities and recalling (8.3), we obtain (8.2).

Now, observe that the function $f: (0, 1) \rightarrow \mathbb{R}$ defined by $f(x) := x(1 \vee |\log x|)$ for all $x \in (0, 1)$ is increasing. Also, note that by (8.1),

$$\left| \log \left(\frac{C' B_n \sqrt{\log(d/\alpha)}}{\sqrt{n} \sigma_{*,W}^2} \right) \right| \lesssim \log n.$$

Therefore, the asserted claim under condition (E.1) follows from combining Theorem 3.1 and (8.2).

Further, the asserted claim in the case when (M) and (E.2) hold can be proven using the exactly same calculations as those in the case of (E.1).

Now consider the case when (M) and (E.3) hold. In this case, we assume, again without loss of generality, that

$$\frac{(\log d)(\log n)}{\sigma_{*,W}^2} \left(\frac{B_n \sqrt{\log(d/\alpha)}}{\sqrt{n}} + \frac{B_n^2(\log d + \alpha^{-2/q})}{n^{1-2/q}} \right) \leq 1. \quad (8.5)$$

Then, defining σ_n^2 and M_n as above, we have $\sigma_n^2 \leq nB_n^2$ and $\sqrt{\mathbb{E}M_n^2} \lesssim \|M_n\|_{L_{q/2}} \lesssim n^{2/q}B_n^2$. Hence, by Lemma 11.4, $\Delta^{(1)}$ defined in (8.4) satisfies

$$\mathbb{E}\Delta^{(1)} \lesssim n^{-1} \left(\sqrt{\sigma_n^2 \log d} + \sqrt{\mathbb{E}M_n^2 \log d} \right) \lesssim \sqrt{n^{-1}B_n^2 \log d} + n^{-1+2/q}B_n^2 \log d.$$

Thus, applying Lemma 11.5(ii) with $\eta = 1$ and $s = q/2$, we have for all $t > 0$ that

$$\Delta^{(1)} \lesssim \sqrt{n^{-1}B_n^2 \log d} + n^{-1+2/q}B_n^2 \log d + t$$

with probability at least

$$1 - \exp\left(-\frac{nt^2}{3B_n^2}\right) - \frac{cnB_n^q}{n^{q/2}t^{q/2}},$$

where $c > 0$ is a universal constant. Setting here

$$t = \frac{B_n \sqrt{3 \log(4/\alpha)}}{\sqrt{n}} + \frac{(4c/\alpha)^{2/q} B_n^2}{n^{1-2/q}},$$

we conclude that

$$\Delta^{(1)} \lesssim \frac{B_n \sqrt{\log(d/\alpha)}}{\sqrt{n}} + \frac{B_n^2(\log d + \alpha^{-2/q})}{n^{1-2/q}}$$

with probability at least $1 - \alpha/2$, and, by the same argument, we can also find, for $\Delta^{(2)}$ defined in (8.4), that

$$\Delta^{(2)} \lesssim \frac{B_n \sqrt{\log(d/\alpha)}}{\sqrt{n}} + \frac{B_n^2(\log d + \alpha^{-2/q})}{n^{1-2/q}}$$

again with probability at least $1 - \alpha/2$. Combining these inequalities and using (8.3), we obtain

$$\mathbb{P} \left(\Delta'_0 > \frac{C' \log d}{\sigma_{*,W}^2} \left(\frac{B_n \sqrt{\log(d/\alpha)}}{\sqrt{n}} + \frac{B_n^2(\log d + \alpha^{-2/q})}{n^{1-2/q}} \right) \right) \leq 1 - \alpha,$$

where $C' \geq 1$ is a universal constant. Here,

$$\left| \log \left(\frac{C'}{\sigma_{*,W}^2} \left(\frac{B_n \sqrt{\log(d/\alpha)}}{\sqrt{n}} + \frac{B_n^2(\log d + \alpha^{-2/q})}{n^{1-2/q}} \right) \right) \right| \lesssim \log n$$

by (8.5). The proof can now be completed by applying Theorem 3.1 as we did in the case of (E.1). \blacksquare

Proof of Theorem 3.2. The asserted claim follows as a direct application of Theorem 2.2 once we note that Δ_0 , Δ_1 , and $M(\psi)$ are equal to Δ'_0 , Δ'_1 , and $M^*(\psi)$ and \mathcal{M} is bounded from above by \mathcal{M}^* if we substitute $X_1^* - \bar{X}, \dots, X_n^* - \bar{X}$ instead of X_1, \dots, X_n in Theorem 2.2. \blacksquare

Proof of Corollary 3.2. Like in the proof of Corollary 2.1, we assume, without loss of generality, that $\sigma_j = 1$ for all $j = 1, \dots, d$, so that $\sigma_{*,W}$ is the square root of the smallest eigenvalue of $\Sigma_W = \mathbb{E}WW^T$. To prove the asserted claims, we will apply Theorem 3.2 with $\Sigma = \Sigma_W$, so that $\sigma_* = \sigma_{*,W}$. This requires bounding all terms appearing in Theorem 3.2. We do so separately for each case.

Consider first the case when (E.1) holds. We assume, without loss of generality, that

$$\frac{B_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} \leq 1, \quad (8.6)$$

since otherwise the asserted claim is trivial. Then by the proof of Corollary 3.1,

$$\Delta'_0 \log n \lesssim \frac{B_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} \quad (8.7)$$

with probability at least $1 - \alpha/2$. Also, by Lemma 11.6 and (8.6),

$$\mathbb{E} \max_{1 \leq j \leq d} \sum_{i=1}^n |X_{ij}|^4 \lesssim nB_n^2 + B_n^4 \log d \lesssim nB_n^2.$$

Thus, by Lemma 11.7(i) with $\eta = \beta = 1$, we have for any $t > 0$ that $\max_{1 \leq j \leq d} \sum_{i=1}^n |X_{ij}|^4 \lesssim nB_n^2 + t$ with probability at least

$$1 - 3 \exp\left(-\frac{t}{cB_n^4}\right),$$

where $c > 0$ is a universal constant. Setting here $t = cB_n^4 \log(6/\alpha)$, using Jensen's inequality, and recalling (8.6), we have that

$$\max_{1 \leq j \leq d} \sum_{i=1}^n (X_{ij} - \bar{X}_j)^4 \lesssim \max_{1 \leq j \leq d} \sum_{i=1}^n |X_{ij}|^4 \lesssim nB_n^2 + B_n^4 \log(6/\alpha) \lesssim nB_n^2$$

with probability at least $1 - \alpha/2$, and so

$$(\log n)\sqrt{\Delta'_1 \log d} \lesssim \frac{B_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2}$$

with the same probability. Further, $\mathcal{M}^* \lesssim B_n$, and so

$$\frac{(\mathcal{M}^* \log d)^2 \log n}{n\sigma_{*,W}^2} \lesssim \frac{B_n^2 (\log d)^2 \log n}{n\sigma_{*,W}^2} \lesssim \frac{B_n (\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2}$$

by (8.6). Finally, setting $\psi = 2B_n$ gives $M^*(\psi) = 0$ and

$$\frac{\psi (\log d)^{3/2}}{\sqrt{n}\sigma_{*,W}} \lesssim \frac{B_n (\log d)^{3/2}}{\sqrt{n}\sigma_{*,W}}.$$

Combining presented inequalities and applying Theorem 3.2 gives the asserted claim under condition (E.1).

Next, consider the case when (M) and (E.2) hold. We assume, without loss of generality, that

$$\frac{B_n (\log d) (\log n) \sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n (\log(dn))^2 \sqrt{\log(1/\alpha)}}{\sqrt{n}\sigma_{*,W}} \leq 1 \quad (8.8)$$

since otherwise the asserted claim is trivial. Then, again by the proof of Corollary 3.1, Δ'_0 satisfies (8.7) with probability at least $1 - \alpha/4$. Further, by Jensen's inequality, (8.8), and Lemmas 11.6 and 11.7(i) with $\eta = 1$ and $\beta = 1/2$,

$$\begin{aligned} \max_{1 \leq j \leq d} \sum_{i=1}^n (X_{ij} - \bar{X}_j)^4 &\lesssim nB_n^2 + B_n^4 (\log(dn))^2 \log d + B_n^4 (\log(dn))^2 (\log(1/\alpha))^2 \\ &\lesssim nB_n^2 + B_n^4 (\log(dn))^2 (\log(1/\alpha))^2, \end{aligned}$$

with probability at least $1 - \alpha/4$, and so

$$\begin{aligned} (\log n) \sqrt{\Delta'_1 \log d} &\lesssim \frac{B_n (\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n^2 (\log d)^{3/2} \log(dn) \log(1/\alpha) \log n}{n\sigma_{*,W}^2} \\ &\lesssim \frac{B_n (\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n (\log(dn))^2 \sqrt{\log(1/\alpha)}}{\sqrt{n}\sigma_{*,W}} \end{aligned}$$

with the same probability by (8.8). In addition, $\mathcal{M}^* \lesssim B_n \sqrt{\log(dn)} \sqrt{\log(1/\alpha)}$ with probability at least $1 - \alpha/4$, and so

$$\frac{(\mathcal{M}^* \log d)^2 \log n}{n\sigma_{*,W}^2} \lesssim \frac{B_n^2 (\log(dn))^4 \log(1/\alpha)}{n\sigma_{*,W}^2} \lesssim \frac{B_n (\log(dn))^2 \sqrt{\log(1/\alpha)}}{\sqrt{n}\sigma_{*,W}}$$

with the same probability by (8.8). Finally, setting $\psi = C' B_n \sqrt{\log(dn)} \sqrt{\log(1/\alpha)}$ with a sufficiently large but universal constant $C' > 0$, we have $M^*(\psi) = 0$ with probability at least $1 - \alpha/4$, and

$$\frac{\psi (\log d)^{3/2}}{\sqrt{n}\sigma_{*,W}} \lesssim \frac{B_n (\log(dn))^2 \sqrt{\log(1/\alpha)}}{\sqrt{n}\sigma_{*,W}}.$$

Combining presented inequalities and applying Theorem 3.2 gives the asserted claim under conditions (M) and (E.2).

Now consider the case when (M) and (E.3) hold. We assume, without loss of generality, that

$$\frac{B_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n\sqrt{\log(dn)}\log d}{n^{1/2-1/q}\alpha^{1/q}\sigma_{*,W}} \leq 1 \quad (8.9)$$

since otherwise the asserted claim is trivial. Using this assumption, by the proof of Corollary 3.1,

$$\begin{aligned} \Delta'_0 \log n &\lesssim \frac{(\log d)(\log n)}{\sigma_{*,W}^2} \left(\frac{B_n\sqrt{\log(d/\alpha)}}{\sqrt{n}} + \frac{B_n^2(\log d + \alpha^{-2/q})}{n^{1-2/q}} \right) \\ &\lesssim \frac{B_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n\sqrt{\log(dn)}\log d}{n^{1/2-1/q}\alpha^{1/q}\sigma_{*,W}} \end{aligned}$$

with probability at least $1 - \alpha/4$. Also, given that

$$\mathbb{E} \max_{1 \leq i \leq n} \max_{1 \leq j \leq d} |X_{ij}|^4 \leq \left(\mathbb{E} \max_{1 \leq i \leq n} \max_{1 \leq j \leq d} |X_{ij}|^q \right)^{4/q} \leq n^{4/q} B_n^4,$$

we have by Lemma 11.6 that

$$\mathbb{E} \max_{1 \leq j \leq d} \sum_{i=1}^n |X_{ij}|^4 \lesssim n B_n^2 + n^{4/q} B_n^4 \log d.$$

Thus, by Lemma 11.7(ii) with $\eta = 1$ and $s = q/4$, we have for any $t > 0$ that

$$\max_{1 \leq j \leq d} \sum_{i=1}^n |X_{ij}|^4 \lesssim n B_n^2 + n^{4/q} B_n^4 \log d + t$$

with probability at least $1 - (cn B_n^q)/t^{q/4}$, where $c > 0$ is a universal constant. Setting here $t = (4cn/\alpha)^{4/q} B_n^4$ and using Jensen's inequality, we have that

$$\max_{1 \leq j \leq d} \sum_{i=1}^n (X_{ij} - \bar{X}_j)^4 \lesssim \max_{1 \leq j \leq d} \sum_{i=1}^n |X_{ij}|^4 \lesssim n B_n^2 + n^{4/q} B_n^4 (\log d + \alpha^{-4/q})$$

with probability at least $1 - \alpha/4$, and so

$$\begin{aligned} (\log n) \sqrt{\Delta'_1 \log d} &\lesssim \frac{B_n(\log d)^{3/2} \log n}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n^2(\log d)^{3/2}(\log n)(\sqrt{\log d} + \alpha^{-2/q})}{n^{1-2/q}\sigma_{*,W}^2} \\ &\lesssim \frac{B_n(\log d)(\log n)\sqrt{\log(d/\alpha)}}{\sqrt{n}\sigma_{*,W}^2} + \frac{B_n\sqrt{\log(dn)}\log d}{n^{1/2-1/q}\alpha^{1/q}\sigma_{*,W}} \end{aligned}$$

with the same probability by (8.9). In addition, by Markov's inequality, for any $t > 0$,

$$\mathbb{P} \left(\max_{1 \leq j \leq d} \max_{1 \leq i \leq n} |X_{ij}| > t \right) \leq t^{-q} \mathbb{E} \max_{1 \leq j \leq d} \max_{1 \leq i \leq n} |X_{ij}|^q \leq n B_n^q / t^q,$$

and so $\mathcal{M}^* \lesssim n^{1/q} B_n / \alpha^{1/q}$ with probability at least $1 - \alpha/4$, so that

$$\frac{(\mathcal{M}^* \log d)^2 \log n}{n \sigma_{*,W}^2} \lesssim \frac{B_n^2 (\log d)^2 \log n}{n^{1-2/q} \alpha^{2/q} \sigma_{*,W}^2} \lesssim \frac{B_n \sqrt{\log(dn)} \log d}{n^{1/2-1/q} \alpha^{1/q} \sigma_{*,W}}$$

with the same probability by (8.9). Finally, setting $\psi = C' n^{1/q} B_n / \alpha^{1/q}$ for a sufficiently large but universal constant $C' > 0$, we have $M^*(\psi) = 0$ with probability at least $1 - \alpha/4$ and

$$\frac{\psi (\log d)^{3/2}}{\sqrt{n} \sigma_{*,W}} \lesssim \frac{B_n (\log d)^{3/2}}{n^{1/2-1/q} \alpha^{1/q} \sigma_{*,W}}.$$

Combining all presented inequalities and applying Theorem 3.2 gives the asserted claim under conditions (M) and (E.3). \blacksquare

9. PROOFS FOR SECTION 4

Proof of Theorem 4.1. Observe that for any $w \in \mathbb{R}^d$ and $A = \prod_{j=1}^d (a_j, b_j] \in \mathcal{R}$, $\mathbb{P}(\tilde{W} \in A) = \mathbb{P}(\tilde{W} \leq b) - \mathbb{P}(\tilde{W} \leq a)$, where $a = (a_1, \dots, a_d)^T \in \mathbb{R}^d$ and $b = (b_1, \dots, b_d)^T \in \mathbb{R}^d$. Therefore, it suffices to prove the asserted claim with $\tilde{\varrho}_\Sigma$ replaced by

$$\tilde{\varrho}' = \tilde{\varrho}'_\Sigma := \sup_{r \in \mathbb{R}^d} |\mathbb{P}(\tilde{W} \leq r) - \mathbb{P}(\tilde{Z} \leq r)|, \quad \tilde{Z} \sim N(0, \tilde{\Sigma}),$$

which is what we do below.

For all $i = 1, \dots, n$, denote $\xi_i := X_i / \sqrt{n}$, so that $W = \sum_{i=1}^n \xi_i$. Also, let $Z \sim N(0, \Sigma)$ be independent of everything else. Then

$$\tilde{W} = W + G \quad \text{and} \quad \tilde{Z} = Z + G.$$

In addition, for any $r \in \mathbb{R}^d$, let $h^r: \mathbb{R}^d \rightarrow \mathbb{R}$ be the function defined by

$$h^r(x) := \mathbb{E} 1(x + G \leq r).$$

For brevity of notations, we suppress the dependence on r in what follows.

By Lemma 6.2 with $\phi = \infty$ and $\epsilon = 1$, the function h is infinitely differentiable and each derivative is bounded by a constant that only depends on the order of the derivative; in particular,

$$\sup_{x \in \mathbb{R}^d} \sum_{j,k=1}^d |\partial_{jk} h(x)| \lesssim \frac{\log d}{\sigma_{*,0}^2}, \quad \sup_{x \in \mathbb{R}^d} \sum_{j,k,l=1}^d \sup_{y \in R(0, \sigma_*, \eta)} |\partial_{jkl} h(x+y)| \lesssim \frac{(\log d)^{3/2}}{\sigma_{*,0}^3}, \quad (9.1)$$

where $\eta := 2c/\sqrt{\log d}$. The second property, local stability of the derivative, is important to obtain good dependence on δ . Since h is infinite differentiable and has bounded derivatives, we can freely interchange differentiation and integration below, without further announcement.

Now, write

$$\mathbb{P}(\tilde{W} \leq r) - \mathbb{P}(\tilde{Z} \leq r) = \mathbb{E}h(W) - \mathbb{E}h(Z). \quad (9.2)$$

Also, define the Slepian interpolant

$$F(s) := \sqrt{1-s}F + \sqrt{s}Z, \quad s \in [0, 1],$$

for any random vector F in \mathbb{R}^d . Using fundamental theorem of calculus and integration by parts, write (9.2) further as:

$$\begin{aligned} \mathbb{E}h(W) - \mathbb{E}h(Z) &= -\frac{1}{2} \int_0^1 \mathbb{E} \left\langle \nabla h(W(s)), \frac{Z}{\sqrt{s}} - \frac{W}{\sqrt{1-s}} \right\rangle ds \\ &= -\frac{1}{2} \int_0^1 \mathbb{E} \left[\langle \Sigma, \nabla^2 h(W(s)) \rangle - \left\langle \frac{W}{\sqrt{1-s}}, \nabla h(W(s)) \right\rangle \right] ds. \end{aligned} \quad (9.3)$$

To bound the integral here, we employ Stein's leave-one-out trick.

First, we have

$$\mathbb{E} \left[\left\langle \frac{W}{\sqrt{1-s}}, \nabla h(W(s)) \right\rangle \right] = \frac{1}{\sqrt{1-s}} \sum_{i=1}^n \sum_{j=1}^d \mathbb{E}[\xi_{ij} \partial_j h(W(s))].$$

Let $W^{(i)} := W - \xi_i$ for $i = 1, \dots, n$. Taylor expanding $\partial_j h(W(s))$ around $W^{(i)}(s) = W(s) - \sqrt{1-s}\xi_i$ for each i , we obtain

$$\begin{aligned} &\mathbb{E} \left[\left\langle \frac{W}{\sqrt{1-s}}, \nabla h(W(s)) \right\rangle \right] \\ &= \frac{1}{\sqrt{1-s}} \sum_{i=1}^n \sum_{j=1}^d \mathbb{E}[\xi_{ij} \partial_j h(W^{(i)}(s))] + \sum_{i=1}^n \sum_{j,k=1}^d \mathbb{E}[\xi_{ij} \xi_{ik} \partial_{jk} h(W^{(i)}(s))] + R_1(s), \end{aligned}$$

where

$$R_1(s) = \sqrt{1-s} \sum_{i=1}^n \sum_{j,k,l=1}^d \mathbb{E}[(1-U)\xi_{ij}\xi_{ik}\xi_{il}\partial_{jkl}h(W^{(i)}(s) + \sqrt{1-s}U\xi_i)],$$

and U is a uniform random variable on $[0, 1]$ independent of everything else. Using the independence between $W^{(i)}$ and ξ_i as well as $\mathbb{E}[\xi_{ij}] = 0$, we deduce

$$\mathbb{E} \left[\left\langle \frac{W}{\sqrt{1-s}}, \nabla h(W(s)) \right\rangle \right] = \sum_{i=1}^n \sum_{j,k=1}^d \mathbb{E}[\xi_{ij}\xi_{ik}] \mathbb{E}[\partial_{jk}h(W^{(i)}(s))] + R_1(s). \quad (9.4)$$

Next, we decompose $\mathbb{E}[\langle \Sigma, \nabla^2 h(W(s)) \rangle]$ as

$$\begin{aligned} &\mathbb{E}[\langle \Sigma, \nabla^2 h(W(s)) \rangle] \\ &= \mathbb{E}[\langle \Sigma - \Sigma_W, \nabla^2 h(W(s)) \rangle] + \mathbb{E}[\langle \Sigma_W, \nabla^2 h(W(s)) \rangle]. \end{aligned} \quad (9.5)$$

We have by the first inequality in (9.1)

$$|\mathbb{E}[\langle \Sigma - \Sigma_W, \nabla^2 h(W(s)) \rangle]| \lesssim \tilde{\Delta}_0. \quad (9.6)$$

Meanwhile, we further decompose $\mathbb{E} [\langle \Sigma_W, \nabla^2 h(W(s)) \rangle]$ in the following. Since $\Sigma_W = \sum_{i=1}^n \mathbb{E}[\xi_i \xi_i^T]$, we have

$$\mathbb{E} [\langle \Sigma_W, \nabla^2 h(W(s)) \rangle] = \sum_{i=1}^n \sum_{j,k=1}^d \mathbb{E}[\xi_{ij} \xi_{ik}] \mathbb{E} [\partial_{jk} h(W(s))].$$

Taylor expanding $\partial_{jk} h(W(s))$ around $W^{(i)}(s)$, we obtain

$$\mathbb{E} [\langle \Sigma_W, \nabla^2 h(W(s)) \rangle] = \sum_{i=1}^n \sum_{j,k=1}^d \mathbb{E}[\xi_{ij} \xi_{ik}] \mathbb{E}[\partial_{jk} h(W^{(i)}(s))] + R_2(s), \quad (9.7)$$

where

$$R_2(s) = \sqrt{1-s} \sum_{i=1}^n \sum_{j,k,l=1}^d \mathbb{E}[\xi_{ij} \xi_{ik}] \mathbb{E}[\xi_{il} \partial_{jkl} h(W^{(i)}(s) + \sqrt{1-s} U \xi_i)].$$

From (9.2)–(9.7), we obtain

$$|\mathbb{P}(\tilde{W} \leq r) - \mathbb{P}(\tilde{Z} \leq r)| \lesssim \tilde{\Delta}_0 + \int_0^1 |R_1(s)| ds + \int_0^1 |R_2(s)| ds. \quad (9.8)$$

Therefore, we complete the proof once we show

$$\int_0^1 |R_1(s)| ds \lesssim \tilde{\Delta}_1 \quad \text{and} \quad \int_0^1 |R_2(s)| ds \lesssim \tilde{\Delta}_1. \quad (9.9)$$

Since $\|\xi_i\|_\infty \leq \delta \leq \sigma_{*,0} \eta / 2$ by assumption, we have

$$\begin{aligned} |R_1(s)| &\leq \sum_{i=1}^n \sum_{j,k,l=1}^d \mathbb{E} \left[|\xi_{ij} \xi_{ik} \xi_{il}| \sup_{y \in R(0, \sigma_{*,0} \eta / 2)} |\partial_{jkl} h(W^{(i)}(s) + y)| \right] \\ &= \sum_{i=1}^n \sum_{j,k,l=1}^d \mathbb{E} [|\xi_{ij} \xi_{ik} \xi_{il}|] \mathbb{E} \left[\sup_{y \in R(0, \sigma_{*,0} \eta / 2)} |\partial_{jkl} h(W^{(i)}(s) + y)| \right], \end{aligned}$$

where the last line follows from the independence between $W^{(i)}$ and ξ_i . Using $\|\xi_i\|_\infty \leq \sigma_{*,0} \eta / 2$ again, we conclude

$$\begin{aligned} |R_1(s)| &\leq \sum_{i=1}^n \sum_{j,k,l=1}^d \mathbb{E} [|\xi_{ij} \xi_{ik} \xi_{il}|] \mathbb{E} \left[\sup_{y \in R(0, \sigma_{*,0} \eta)} |\partial_{jkl} h(W(s) + y)| \right] \\ &\leq \left(\max_{1 \leq j \leq d} \sum_{i=1}^n \mathbb{E} [|\xi_{ij}|^3] \right) \sum_{j,k,l=1}^d \mathbb{E} \left[\sup_{y \in R(0, \sigma_{*,0} \eta)} |\partial_{jkl} h(W(s) + y)| \right]. \end{aligned}$$

Therefore, we obtain by the second inequality in (9.1)

$$|R_1(s)| \lesssim \max_{1 \leq j \leq d} \sum_{i=1}^n \mathbb{E} [|\xi_{ij}|^3] \frac{\log^{3/2} d}{\sigma_{*,0}^3} = \tilde{\Delta}_1. \quad (9.10)$$

This yields the first inequality in (9.9); since we can similarly prove the second one, the desired result follows from combining the bounds and noting that these bounds do not depend on r . \blacksquare

Proof of Theorem 4.2. The proof is a modification of the proof of Theorem 4.1, inspired by [26]. Here, we describe the changes, keeping all unmentioned notations the same as those in the proof of Theorem 4.1. In a nutshell, we only need to consider an additional Taylor expansion when bounding $R_1(s)$ and $R_2(s)$.

As in the proof of Theorem 4.1, it suffices to prove the asserted claim with $\tilde{\varrho}_\Sigma$ replaced by $\tilde{\varrho}'_\Sigma$. Also, note that (9.8) holds under our current assumptions by the same arguments as those in the proof of Theorem 4.1. Thus, we only need to bound

$$\int_0^1 |R_1(s)| ds \quad \text{and} \quad \int_0^1 |R_2(s)| ds.$$

To bound the former, we consider the Taylor expansion of $\partial_{jkl}h(W^{(i)}(s) + \sqrt{1-s}U\xi_i)$ around $W^{(i)}(s)$ and rewrite $R_1(s)$ as

$$R_1(s) = \sqrt{1-s} \sum_{i=1}^n \sum_{j,k,l=1}^d \mathbb{E}[(1-U)\xi_{ij}\xi_{ik}\xi_{il}\partial_{jkl}h(W^{(i)}(s))] + R'_1(s),$$

where

$$R'_1(s) = (1-s) \sum_{i=1}^n \sum_{j,k,l,q=1}^d \mathbb{E}[U(1-U)\xi_{ij}\xi_{ik}\xi_{il}\xi_{iq}\partial_{jklq}h(W^{(i)}(s) + \sqrt{1-s}UU'\xi_i)],$$

and U' is a uniform random variable on $[0, 1]$ independent of everything else. Since $W^{(i)}, \xi_i$ and U are independent, we obtain

$$R_1(s) = \sqrt{1-s} \sum_{i=1}^n \sum_{j,k,l=1}^d \mathbb{E}[1-U]\mathbb{E}[\xi_{ij}\xi_{ik}\xi_{il}]\mathbb{E}[\partial_{jkl}h(W^{(i)}(s))] + R'_1(s).$$

Thus, we conclude $R_1(s) = R'_1(s)$ by the assumption (4.2). Now, note that we have by Lemma 6.2 with $\phi = \infty$ and $\epsilon = 1$

$$\sup_{x \in \mathbb{R}^d} \sum_{j,k,l,q=1}^d \sup_{y \in R(0, \sigma_*, 0\eta)} |\partial_{jklq}h(x+y)| \lesssim \frac{(\log d)^2}{\sigma_{*,0}^4}.$$

Using this inequality instead of the second one in (9.1), we can prove $|R'_1(s)| \lesssim \tilde{\Delta}_2$ analogously to the proof of (9.10). A similar argument also yields $|R'_2(s)| \lesssim \tilde{\Delta}_2$, completing the proof. \blacksquare

Proof of Proposition 4.1. The proof is almost the same as that of [23, Proposition 1.1], except that we use Theorem 3 in [35, Chapter VIII] instead of Eq.(2.41) in [35, Chapter VIII].

It suffices to show that there is a sequence $(x_n)_{n=1}^\infty$ of real numbers such that

$$\rho := \liminf_{n \rightarrow \infty} \frac{n}{\log^2 d} \left| \mathbb{P} \left(\max_{1 \leq j \leq d} W_{n,j} \leq x_n \right) - \mathbb{P} \left(\max_{1 \leq j \leq d} Z_j \leq x_n \right) \right| > 0,$$

where

$$W_{n,j} := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{n,ij}.$$

We define the sequence $(x_n)_{n=1}^\infty$ in the same way as in the proof of Proposition 2.1. Then we can prove (7.38) and (7.39) by the same arguments as in the proof of Proposition 2.1. Moreover, since $x_n = O(\sqrt{\log d}) = o(n^{1/4})$ by assumption, Theorem 3 in [35, Chapter VIII] implies

$$\frac{P(W_1 > x_n)}{1 - \Phi_1(x_n)} = e^{\frac{\gamma}{24n} x_n^4} \left\{ 1 + O\left(\frac{x_n + 1}{\sqrt{n}}\right) \right\}.$$

Combining this with (7.38), we obtain

$$\begin{aligned} \lambda_n &= \frac{P(W_1 > x_n)}{1 - \Phi_1(x_n)} \cdot d(1 - \Phi_1(x_n)) \\ &= e^{\frac{\gamma}{24n} x_n^4} \left\{ 1 + O\left(\frac{x_n + 1}{\sqrt{n}}\right) \right\} + O\left(\frac{1}{d}\right). \end{aligned}$$

Using the Maclaurin expansion of the exponential function, we obtain

$$\lambda_n = 1 + \frac{\gamma}{24n} x_n^4 + o\left(\frac{x_n^4}{n}\right) + O\left(\frac{x_n + 1}{\sqrt{n}}\right) + O\left(\frac{1}{d}\right)$$

and thus

$$e^{-\lambda_n + 1} = 1 - \frac{\gamma}{24n} x_n^4 + o\left(\frac{x_n^4}{n}\right) + O\left(\frac{x_n + 1}{\sqrt{n}}\right) + O\left(\frac{1}{d}\right).$$

Note that we particularly have $\lambda_n = O(1)$. Thus, (7.39) yields

$$\rho \geq \liminf_{n \rightarrow \infty} \frac{n}{\log^2 d} |e^{-\lambda_n} - e^{-1}| = \liminf_{n \rightarrow \infty} e^{-1} \frac{n}{\log^2 d} \frac{|\gamma| x_n^4}{24} = e^{-1} \frac{|\gamma|}{6}$$

because $x_n/\sqrt{2 \log d} \rightarrow 1$ as well as $d^{-1} = o((\log d)^{-1}) = o(n^{-1} \log^2 d)$ and $n/\log^3 d \rightarrow 0$ by assumption. This completes the proof. \blacksquare

10. PROOFS FOR SECTION 6

Throughout this section, for brevity of notations, we often drop super-indices ϕ , ϵ , r , and Σ in the functions $g^\phi(\cdot)$, $m^{r,\phi}(\cdot)$, and $\rho^{r,\phi,\epsilon,\Sigma}(\cdot)$ and simply write $g(\cdot)$, $m(\cdot)$, and $\rho(\cdot)$ instead. Also, we use \lesssim to denote inequalities that hold up to a universal constant.

Proof of Lemma 6.1. First, note that the asserted claim for general $r \in \mathbb{R}^d$ follows from the asserted claim for $r = 0$ because

$$\rho^{r,\phi,\epsilon,\Sigma}(w) = \rho^{0,\phi,\epsilon,\Sigma}(w - r).$$

Similarly, the asserted claim for general Σ follows from the asserted claim for $\Sigma = I_d$. Indeed, define $\Sigma^1 = \Sigma - \sigma_*^2 I_d$ and $\Sigma^2 = \sigma_*^2 I_d$ and let Z^1 and Z^2 be independent random vectors in \mathbb{R}^d such that $Z^1 \sim N(0, \Sigma^1)$ and $Z^2 \sim N(0, \Sigma^2)$. Then $Z \sim N(0, \Sigma)$ is equal in distribution to $Z^1 + Z^2$. Hence,

$$\rho^{r,\phi,\epsilon,\Sigma}(w) = \mathbb{E} m^{r,\phi}(w + \epsilon Z^1 + \epsilon Z^2) = \mathbb{E} \rho^{r,\phi,\epsilon\sigma_*, I_d}(w + \epsilon Z^1),$$

and so, by Jensen's inequality,

$$\begin{aligned} & \sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon\sigma_*\eta)} |\partial_{j_1, \dots, j_v} \rho^{r,\phi,\epsilon,\Sigma}(w + y)| \\ & \leq \mathbb{E} \left[\sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon\sigma_*\eta)} |\partial_{j_1, \dots, j_v} \rho^{r,\phi,\epsilon\sigma_*, I_d}(w + \epsilon Z^1 + y)| \right] \\ & = \sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon\sigma_*\eta)} |\partial_{j_1, \dots, j_v} \rho^{r,\phi,\epsilon\sigma_*, I_d}(w + y)| \end{aligned}$$

Therefore, in what follows, we set $r = 0$ and $\Sigma = I_d$.

Next, we prepare some notation. Let

$$\mathbb{R}_-^d = \{w \in \mathbb{R}^d : w \leq 0\}.$$

Also, let $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ denote the pdf of the standard normal distribution on \mathbb{R}^d . In addition, let $\varphi_1: \mathbb{R} \rightarrow \mathbb{R}$ and $\Phi_1: \mathbb{R} \rightarrow \mathbb{R}$ denote the pdf and the cdf of the standard normal distribution on \mathbb{R} . Moreover, for all $j = 1, \dots, d$, let $\pi_j: \mathbb{R}^d \rightarrow \mathbb{R}$ be the function defined by

$$\pi_j(w) = 1 \left\{ j = \arg \max_{1 \leq k \leq d} w_k \right\}, \quad w \in \mathbb{R}^d,$$

where $\arg \max_{1 \leq k \leq d} w_k$ is equal to the smallest $l = 1, \dots, d$ such that $w_l = \max_{1 \leq k \leq d} w_k$. Here, it is useful to note that the functions $\pi_j(\cdot)$ satisfy

$$\sum_{j=1}^d \pi_j(w) = 1 \tag{10.1}$$

for all $w \in \mathbb{R}^d$. Finally, let $\Psi: \mathbb{R}^d \rightarrow \mathbb{R}$ be the function defined by

$$\Psi(w) = \int_{-\infty}^0 \prod_{j=1}^d \Phi_1(t - w_j) dt, \quad w \in \mathbb{R}^d.$$

It is straightforward to check that

$$\partial_{j_1 \dots j_u} \Psi(w) = (-1)^u \int_{\mathbb{R}_-^d} \pi_{j_1}(s) \partial_{j_2, \dots, j_u} \varphi(s - w) ds \quad (10.2)$$

for all $u = 1, \dots, v$, $j_1, \dots, j_u = 1, \dots, d$, and $w \in \mathbb{R}^d$.

For the rest of the proof, we proceed in two steps. In the first step, we prove that

$$\begin{aligned} & \sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon\eta)} |\partial_{j_1 \dots j_v} \rho(w + y)| \\ & \leq \frac{2\phi}{\epsilon^{v-1}} \sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \eta)} |\partial_{j_1 \dots j_v} \Psi(w + y)|. \end{aligned} \quad (10.3)$$

In the second step, we prove that

$$\sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \eta)} |\partial_{j_1 \dots j_v} \Psi(w + y)| \lesssim (\log d)^{(v-1)/2}. \quad (10.4)$$

Combining these steps gives the asserted claim of the lemma.

Step 1. Here, we prove (10.3). Fix any $w \in \mathbb{R}^d$ and $j_1, \dots, j_v = 1, \dots, d$, and observe that

$$\partial_{j_1} \rho(w) = \int_{\mathbb{R}^d} \partial_{j_1} m(w + \epsilon z) \varphi(z) dz = \frac{1}{\epsilon^d} \int_{\mathbb{R}^d} \partial_{j_1} m(s) \varphi\left(\frac{s - w}{\epsilon}\right) ds,$$

where the second equality holds by the change of variables $z \mapsto s = w + \epsilon z$. Therefore,

$$\begin{aligned} \partial_{j_1, \dots, j_v} \rho(w) &= \frac{(-1)^{v-1}}{\epsilon^{d+v-1}} \int_{\mathbb{R}^d} m_{j_1}(s) \partial_{j_2, \dots, j_d} \varphi\left(\frac{s - w}{\epsilon}\right) ds \\ &= \frac{(-1)^{v-1}}{\epsilon^{v-1}} \int_{\mathbb{R}^d} m_{j_1}(w + \epsilon z) \partial_{j_2, \dots, j_d} \varphi(z) dz, \end{aligned}$$

where the second equality holds by the reverse change of variables $s \mapsto z = (s - w)/\epsilon$. In addition,

$$m_{j_1}(w + \epsilon z) = g' \left(\max_{1 \leq j \leq d} (w_j + \epsilon z_j) \right) \pi_{j_1}(w + \epsilon z)$$

for almost all z with respect to the Lebesgue measure on \mathbb{R}^d . Thus, given that

$$g'(t) = \begin{cases} \phi & \text{if } t \in (0, 1/\phi), \\ 0 & \text{if } t \notin (0, 1/\phi), \end{cases}$$

denoting

$$A_1 = \left\{ z \in \mathbb{R}^d : w + \epsilon z \leq 0 \right\}, \quad A_2 = \left\{ z \in \mathbb{R}^d : w + \epsilon z \leq 1/\phi \right\},$$

we have

$$\partial_{j_1, \dots, j_v} \rho(w) = \frac{(-1)^{v-1} \phi}{\epsilon^{v-1}} \int_{A_2 \setminus A_1} \pi_{j_1}(w + \epsilon z) \partial_{j_2, \dots, j_v} \varphi(z) dz,$$

and so

$$\begin{aligned} |\partial_{j_1, \dots, j_v} \rho(w)| &\leq \frac{\phi}{\epsilon^{v-1}} \left| \int_{A_2} \pi_{j_1}(w + \epsilon z - 1/\phi) \partial_{j_2, \dots, j_v} \varphi(z) dz \right| \\ &\quad + \frac{\phi}{\epsilon^{v-1}} \left| \int_{A_1} \pi_{j_1}(w + \epsilon z) \partial_{j_2, \dots, j_v} \varphi(z) dz \right|, \end{aligned}$$

where we used $\pi_{j_1}(w + \epsilon z) = \pi_{j_1}(w + \epsilon z - 1/\phi)$. Therefore,

$$\begin{aligned} &\sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \eta)} |\partial_{j_1, \dots, j_v} \rho(w + y)| \\ &\leq \frac{2\phi}{\epsilon^{v-1}} \sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \eta)} \left| \int_{A_1} \pi_{j_1}(w + y + \epsilon z) \partial_{j_2, \dots, j_v} \varphi(z) dz \right|. \end{aligned} \quad (10.5)$$

Moreover,

$$\begin{aligned} &\int_{A_1} \pi_{j_1}(w + y + \epsilon z) \partial_{j_2, \dots, j_v} \varphi(z) dz \\ &= \int_{\{s \in \mathbb{R}^d : \epsilon s \leq 0\}} \pi_{j_1}(\epsilon s) \partial_{j_2, \dots, j_v} \varphi(s - (w + y)/\epsilon) ds \\ &= \int_{\mathbb{R}_-^d} \pi_{j_1}(s) \partial_{j_2, \dots, j_v} \varphi(s - (w + y)/\epsilon) ds \\ &= (-1)^v \partial_{j_1, \dots, j_v} \Psi((w + y)/\epsilon), \end{aligned} \quad (10.6)$$

where the first equality holds by the change of variables $z \mapsto s = z + (w + \epsilon)/\epsilon$ and the third by (10.2). Combining (10.5) and (10.7) gives the asserted claim of this step.

Step 2. Here, we prove (10.4). To do so, we proceed by induction on v . For $v = 1$, we have for all $w \in \mathbb{R}^d$, $y \in R(0, \eta)$, and $j = 1, \dots, d$ that

$$|\partial_j \Psi(w + y)| = \int_{-\infty}^0 \left(\prod_{l: l \neq j} \Phi_1(t - w_l - y_l) \right) \varphi_1(t - w_j - y_j) dt.$$

To bound the integral on the right-hand side here, consider the partition

$$(-\infty, 0] = \mathbb{T}_j \cup \mathbb{T}_j^c,$$

where

$$\mathbb{T}_j = \left\{ t \in (-\infty, 0] : |t - w_j| \leq (2 \log d)^{1/2} + \eta \right\}, \quad \mathbb{T}_j^c = (-\infty, 0] \setminus \mathbb{T}_j.$$

Then

$$\begin{aligned}
& \int_{\mathbb{T}_j} \left(\prod_{l: l \neq j} \Phi_1(t - w_l - y_l) \right) \varphi_1(t - w_j - y_j) dt \\
& \leq \int_{\mathbb{T}_j} \left(\prod_{l: l \neq j} \Phi_1(t - w_l + \eta) \right) \varphi_1(t - w_j + \eta) \frac{\varphi_1(t - w_j - y_j)}{\varphi_1(t - w_j + \eta)} dt \\
& \lesssim \int_{\mathbb{T}_j} \left(\prod_{l: l \neq j} \Phi_1(t - w_l + \eta) \right) \varphi_1(t - w_j + \eta) dt \\
& \leq \int_{-\infty}^0 \left(\prod_{l: l \neq j} \Phi_1(t - w_l + \eta) \right) \varphi_1(t - w_j + \eta) dt = -\partial_j \Psi(w - \eta),
\end{aligned}$$

where the second inequality holds for all $y \in R(0, \eta)$ because $\eta \leq K/\sqrt{\log d}$. Also,

$$\begin{aligned}
& \int_{\mathbb{T}_j^c} \left(\prod_{l: l \neq j} \Phi_1(t - w_l - y_l) \right) \varphi_1(t - w_j - y_j) dt \\
& \leq \int_{\mathbb{T}_j^c} \varphi_1(t - w_j - y_j) dt \lesssim \int_{(2 \log d)^{1/2}}^{+\infty} \varphi_1(t) dt \lesssim \varphi_1\left((2 \log d)^{1/2}\right) \lesssim d^{-1}.
\end{aligned}$$

Combining these bounds, we obtain

$$\begin{aligned}
\sum_{j=1}^d \sup_{y \in R(0, \eta)} |\partial_j \Psi(w + y)| & \lesssim 1 - \sum_{j=1}^d \partial_j \Psi(w - \eta) \\
& = 1 + \sum_{j=1}^d \int_{\mathbb{R}_-^d} \pi_j(s) \varphi(s - w + \eta) ds \\
& = 1 + \int_{\mathbb{R}_-^d} \varphi(s - w + \eta) ds \leq 2,
\end{aligned}$$

where the last line follows from (10.1). This gives (10.4) for $v = 1$.

Now, fix $v \geq 2$. By induction, we can assume that

$$\max_{1 \leq u \leq v-1} \sup_{w \in \mathbb{R}^d} \sum_{j_1, \dots, j_u=1}^d \sup_{y \in R(0, \eta)} |\partial_{j_1, \dots, j_u} \Psi(w + y)| \lesssim (\log d)^{(u-1)/2}. \quad (10.8)$$

Also, define

$$\mathcal{J} = \{(j_1, \dots, j_v) \in \{1, \dots, d\}^v : \text{all } j_1, \dots, j_v \text{ are different}\}$$

and $\mathcal{J}^c = \{1, \dots, d\}^v \setminus \mathcal{J}$. Like in the $v = 1$ case, we can check that for all $w \in \mathbb{R}^d$, $y \in R(0, \eta)$, and $(j_1, \dots, j_v) \in \mathcal{J}$, we have

$$|\partial_{j_1 \dots j_v} \Psi(w + y)| \lesssim d^{-v} + (-1)^v \partial_{j_1 \dots j_v} \Psi(w - \eta).$$

Therefore,

$$\begin{aligned} & \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \eta)} |\partial_{j_1 \dots j_v} \Psi(w + y)| \\ & \lesssim 1 + \sum_{(j_1, \dots, j_v) \in \mathcal{J}^c} \sup_{y \in R(0, \eta)} |\partial_{j_1 \dots j_v} \Psi(w + y)| + (-1)^v \sum_{j_1, \dots, j_v=1}^d \partial_{j_1 \dots j_v} \Psi(w - \eta). \end{aligned}$$

Here, for all $w \in \mathbb{R}^d$,

$$\begin{aligned} \left| \sum_{j_1, \dots, j_v=1}^d \partial_{j_1 \dots j_v} \Psi(w) \right| &= \sum_{j_1, \dots, j_v=1}^d \int_{\mathbb{R}_-^d} \pi_{j_1}(s) \partial_{j_2, \dots, j_v} \varphi(s - w) ds \\ &= \sum_{j_2, \dots, j_v=1}^d \int_{\mathbb{R}_-^d} \partial_{j_2, \dots, j_v} \varphi(s - w) ds \lesssim (\log d)^{(v-1)/2} \end{aligned}$$

by (10.1) and Lemma 2.2 in [23].

Hence, it remains to prove that

$$\sum_{(j_1, \dots, j_v) \in \mathcal{J}^c} \sup_{y \in R(0, \eta)} |\partial_{j_1 \dots j_v} \Psi(w + y)| \lesssim (\log d)^{(v-1)/2}. \quad (10.9)$$

To do so, for all $(j_1, \dots, j_v) \in \mathcal{J}^c$, let $N(j_1, \dots, j_v)$ denote the number of different indices among v indices j_1, \dots, j_v . Then

$$\mathcal{J}^c = \mathcal{J}_1 \cup \dots \cup \mathcal{J}_{v-1},$$

where

$$\mathcal{J}_u = \{(j_1, \dots, j_v) \in \mathcal{J}^c : N(j_1, \dots, j_v) = u\}, \quad u = 1, \dots, v-1.$$

Thus,

$$\sum_{(j_1, \dots, j_v) \in \mathcal{J}^c} \sup_{y \in R(0, \eta)} |\partial_{j_1 \dots j_v} \Psi(w + y)| = \sum_{u=1}^{v-1} \sum_{(j_1, \dots, j_v) \in \mathcal{J}_u} \sup_{y \in R(0, \eta)} |\partial_{j_1 \dots j_v} \Psi(w + y)|.$$

Next, fix any $u = 1, \dots, v-1$ and consider the corresponding sum on the right-hand side of the equality above. Fix any $(j_1, \dots, j_v) \in \mathcal{J}_u$. By the definition of \mathcal{J}_u , there are exactly u different indices among v indices j_1, \dots, j_v . Denote them by o_1, \dots, o_u and assume that they appear k_1, \dots, k_u times, respectively, where $k_1 + \dots + k_u = v$. Then, denoting

$$o = (o_1, \dots, o_u), \quad \mathcal{J}^o = \{1, \dots, d\} \setminus \{o_1, \dots, o_u\},$$

we have for all $w \in \mathbb{R}^d$ that

$$\begin{aligned} |\partial_{j_1 \dots j_v} \Psi(w)| &= \left| \int_{-\infty}^0 \prod_{j \in \mathcal{J}^o} \Phi_1(t - w_j) \prod_{i=1}^u \partial^{k_i} \Phi_1(t - w_{o_i}) dt \right| \\ &\lesssim \int_{-\infty}^0 \prod_{j \in \mathcal{J}^o} \Phi_1(t - w_j) \prod_{i=1}^u \left((|t - w_{o_i}|^{k_i-1} + 1) \varphi_1(t - w_{o_i}) \right) dt. \end{aligned}$$

To bound the integral on the right-hand side here, consider the partition

$$(-\infty, 0] = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \dots \cup \mathcal{T}_u,$$

where

$$\mathcal{T}_0 = \left\{ t \in (-\infty, 0] : \forall_{i=1}^u |t - w_{o_i}| \leq (4v^2 \log d)^{1/2} \right\}$$

and

$$\mathcal{T}_i = \left\{ t \in (-\infty, 0] \setminus \mathcal{T}_0 : i = \arg \max_{1 \leq k \leq u} |t - w_{o_k}| \right\}, \quad i = 1, \dots, u.$$

Then

$$\int_{\mathcal{T}_0} \prod_{j \in \mathcal{J}^o} \Phi_1(t - w_j) \prod_{i=1}^u \left((|t - w_{o_i}|^{k_i-1} + 1) \varphi_1(t - w_{o_i}) \right) dt \lesssim (\log d)^{(v-u)/2} |\partial_{o_1 \dots o_u} \Psi(w)|$$

and, for all $i = 1, \dots, u$,

$$\begin{aligned} &\int_{\mathcal{T}_i} \prod_{j \in \mathcal{J}^o} \Phi_1(t - w_j) \prod_{i=1}^u \left((|t - w_{o_i}|^{k_i-1} + 1) \varphi_1(t - w_{o_i}) \right) dt \\ &\lesssim \int_{\mathcal{T}_i} |t - w_{o_i}|^{v-u} \varphi_1(t - w_{o_i}) dt \lesssim \int_{(4v^2 \log d)^{1/2}}^{+\infty} t^v \varphi_1(t) dt \\ &\lesssim \int_{(4v^2 \log d)^{1/2}}^{+\infty} \exp(v \log t - t^2/2) dt \lesssim \int_{(4v^2 \log d)^{1/2}}^{+\infty} \exp(-t^2/4) dt \\ &= \sqrt{2} \int_{(2v^2 \log d)^{1/2}}^{+\infty} \exp(-t^2/2) dt \lesssim \varphi_1 \left((2v^2 \log d)^{1/2} \right) \lesssim d^{-v}. \end{aligned}$$

Combining these bounds, we obtain

$$\begin{aligned} &\sum_{(j_1, \dots, j_v) \in \mathcal{J}^u} \sup_{y \in R(0, \eta)} |\partial_{j_1 \dots j_v} \Psi(w + y)| \\ &\lesssim 1 + (\log d)^{(v-u)/2} \sum_{o_1, \dots, o_u=1}^d \sup_{y \in R(0, \eta)} |\partial_{o_1 \dots o_u} \Psi(w + y)| \\ &\lesssim 1 + (\log d)^{(v-u)/2} (\log d)^{(u-1)/2} \lesssim (\log d)^{(v-1)/2}, \end{aligned}$$

where the third line follows from (10.8). Therefore, given that $u = 1, \dots, v - 1$ here is arbitrary, (10.9) follows, which gives the asserted claim of this step and completes the proof of the lemma. \blacksquare

Proof of Lemma 6.2. As in the proof of Lemma 6.1, it suffices to consider the case with $r = 0$ and $\Sigma = I_d$, which is what we do below. Our argument relies on the following elementary identity.

Lemma 10.1. *For any random vector W in \mathbb{R}^d , we have*

$$\mathbb{E}m(W) = \mathbb{E}m^{0,\phi}(W) = \phi \int_0^{\phi^{-1}} \mathbb{P}(W \leq s) ds.$$

Proof. For any random variable τ , we have, with \mathbb{P}^τ being the law of τ ,

$$\begin{aligned} \mathbb{E}[g(\tau)] &= \int_{\mathbb{R}} g(t) \mathbb{P}^\tau(dt) = - \int_{\mathbb{R}} \left(\int_t^\infty g'(s) ds \right) \mathbb{P}^\tau(dt) \\ &= - \int_{-\infty}^\infty \left(\int_{\mathbb{R}} 1_{(-\infty, s]}(t) \mathbb{P}^\tau(dt) \right) g'(s) ds \\ &= - \int_{-\infty}^\infty \mathbb{P}(\tau \leq s) g'(s) ds = \phi \int_0^{\phi^{-1}} \mathbb{P}(\tau \leq s) ds. \end{aligned}$$

Applying this identity with $\tau = \max_{1 \leq j \leq d} W_j$, we obtain

$$\mathbb{E}[m(W)] = \phi \int_0^{\phi^{-1}} \mathbb{P} \left(\max_{1 \leq j \leq d} W_j \leq s \right) ds = \phi \int_0^{\phi^{-1}} \mathbb{P}(W \leq s) ds.$$

This completes the proof. \blacksquare

Using Lemma 10.1, we can rewrite $\rho(w)$ as

$$\begin{aligned} \rho(w) &= \phi \int_0^{\phi^{-1}} \mathbb{P}(w + \epsilon Z \leq s) ds \\ &= \phi \int_0^{\phi^{-1}} \left\{ \int_{\mathbb{R}^d} 1_{\mathbb{R}_-^d}(w - s + \epsilon z) \varphi(z) dz \right\} ds \\ &= \phi \int_0^{\phi^{-1}} \left\{ \int_{\mathbb{R}^d} 1_{\mathbb{R}_-^d}(\epsilon z) \varphi(z - \epsilon^{-1}(w - s)) dz \right\} ds \\ &= \phi \int_0^{\phi^{-1}} \left\{ \int_{\mathbb{R}_-^d} \varphi(z - \epsilon^{-1}(w - s)) dz \right\} ds. \end{aligned}$$

Thus, we have

$$\partial_{j_1, \dots, j_v} \rho(w) = \frac{(-1)^v \phi}{\epsilon^v} \int_0^{\phi^{-1}} \left\{ \int_{\mathbb{R}_-^d} \partial_{j_1, \dots, j_v} \varphi(z - \epsilon^{-1}(w - s)) dz \right\} ds. \quad (10.10)$$

Combining this identity with Lemma 2.2 in [23] gives the asserted claim. \blacksquare

Proof of Lemma 6.3. As in the beginning of the proof of Lemma 6.1, define $\Sigma^1 = \Sigma - \sigma_*^2 I_d$ and $\Sigma^2 = \sigma_*^2 I_d$ and let Z^1 and Z^2 be independent random vectors in \mathbb{R}^d such that $Z^1 \sim N(0, \Sigma^1)$ and $Z^2 \sim N(0, \Sigma^2)$. Then

$$\begin{aligned} & \sup_{w \in (A_2 \setminus A_1)^c} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \sigma_* \eta)} |\partial_{j_1, \dots, j_v} \rho^{r, \phi, \epsilon, \Sigma}(w + y)| \\ & \leq \mathbb{E} \left[\sup_{w \in (A_2 \setminus A_1)^c} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \sigma_* \eta)} |\partial_{j_1, \dots, j_v} \rho^{r, \phi, \epsilon \sigma_*, I_d}(w + \epsilon Z^1 + y)| \right]. \end{aligned}$$

Also, by the union and Chernoff's bounds,

$$\mathbb{P}(\|Z^1\|_\infty > \kappa) \leq 2de^{-\kappa^2/2} \leq 2de^{-(\kappa-\eta)^2/4}.$$

Thus, by Lemma 6.2,

$$\begin{aligned} & \mathbb{E} \left[\sup_{w \in (A_2 \setminus A_1)^c} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \sigma_* \eta)} |\partial_{j_1, \dots, j_v} \rho^{r, \phi, \epsilon \sigma_*, I_d}(w + \epsilon Z^1 + y)| \right] \\ & \lesssim \sup_{w \in (A_2' \setminus A_1')^c} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \sigma_* \eta)} |\partial_{j_1, \dots, j_v} \rho^{r, \phi, \epsilon \sigma_*, I_d}(w + y)| + \frac{(\log d)^{v/2}}{(\epsilon \sigma_*)^v} \times de^{-(\kappa-\eta)^2/4}, \end{aligned}$$

where

$$A_1' = \{w \in \mathbb{R}^d : w \leq r - \epsilon \kappa\}, \quad A_2' = \{w \in \mathbb{R}^d : w \leq r + \epsilon \kappa + \phi^{-1}\}.$$

Combining these inequalities shows that the asserted claim for general Σ follows from the asserted claim for $\Sigma = I_d$ with replacing A_1 and A_2 by A_1' and A_2' , respectively. In what follows, we therefore set $\Sigma = I_d$. Similarly, again as in the proof of Lemma 6.1, we set $r = 0$.

Further, note that identity (10.10) derived in the proof of Lemma 6.2 did not rely on any specific assumptions of Lemma 6.2, and so remains valid under current assumptions. We will use this identity below.

Next, for any $w \in \mathbb{R}^d$ and $s \in [0, \phi^{-1}]$, if $\max_j w_j < -\epsilon \kappa$, we have $\max_j (w_j - s) < -\epsilon \kappa - s < -\epsilon \kappa$; if $\max_j w_j > \epsilon \kappa + \phi^{-1}$, we have $\max_j (w_j - s) > \epsilon \kappa + \phi^{-1} - s > \epsilon \kappa$. So, we deduce from (10.10) that

$$\begin{aligned} & \sup_{w \in (A_2' \setminus A_1')^c} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \eta)} |\partial_{j_1, \dots, j_v} \rho(w + y)| \\ & \leq \sup_{w \in \mathbb{R}^d : \max_j w_j \notin [-\epsilon \kappa, \epsilon \kappa]} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \epsilon \eta)} \frac{\phi}{\epsilon^v} \int_0^{\phi^{-1}} \left| \int_{\mathbb{R}^d} \partial_{j_1, \dots, j_v} \varphi(z - \epsilon^{-1}(w + y)) dz \right| ds \end{aligned}$$

$$\leq \sup_{x \in \mathbb{R}^d: \max_j x_j \notin [-\kappa, \kappa]} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \eta)} \frac{1}{\epsilon^v} \left| \int_{\mathbb{R}_-^d} \partial_{j_1, \dots, j_v} \varphi(z - x - y) dz \right|.$$

Thus, we complete the proof once we show

$$\sup_{x \in \mathbb{R}^d: \max_j x_j \notin [-\kappa, \kappa]} \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \eta)} \left| \int_{\mathbb{R}_-^d} \partial_{j_1, \dots, j_v} \varphi(z - x - y) dz \right| \lesssim d^v e^{-(\kappa-\eta)^2/4}. \quad (10.11)$$

To prove (10.11), we introduce some additional notations. For an integer $\nu \geq 0$, the ν -th Hermite polynomial is denoted by H_ν : $H_\nu(t) = (-1)^\nu \varphi_1(t)^{-1} \varphi_1^{(\nu)}(t)$. We denote by t_ν the maximum root of H_ν when $\nu \geq 1$. For example, $t_1 = 0, t_2 = 1, t_3 = \sqrt{3}$. It is evident that H_ν is positive and strictly increasing on (t_ν, ∞) . We also have

$$t_1 < t_2 < \dots. \quad (10.12)$$

See e.g. [38, Theorem 3.3.2]. When $\nu \geq 1$, we define the function h_ν on \mathbb{R} by $h_\nu(t) = H_{\nu-1}(t) \varphi_1(t)$, $t \in \mathbb{R}$. A simple computation shows $h'_\nu(t) = -h_{\nu+1}(t)$; hence h_ν is strictly decreasing on $[t_\nu, \infty)$. Moreover, since h_ν is either even or odd, we have

$$|h_\nu(-t)| = |h_\nu(t)| \quad \text{for any } t \in \mathbb{R}. \quad (10.13)$$

Finally, for every $u \in \{1, \dots, v\}$, we set

$$\begin{aligned} \mathcal{N}_u(v) &:= \{(\nu_1, \dots, \nu_u) \in \mathbb{Z}^u : \nu_1, \dots, \nu_u \geq 1, \nu_1 + \dots + \nu_u = v\}, \\ \mathcal{J}_u(d) &:= \{(j_1, \dots, j_u) \in \{1, \dots, d\}^u : j_1, \dots, j_u \text{ are mutually different}\}. \end{aligned}$$

We turn to the main body of the proof of (10.11). We have for any $x \in \mathbb{R}^d$

$$\begin{aligned} & \sum_{j_1, \dots, j_v=1}^d \sup_{y \in R(0, \eta)} \left| \int_{\mathbb{R}_-^d} \partial_{j_1, \dots, j_v} \varphi(z - x - y) dz \right| \\ & \lesssim \sum_{u=1}^v \sum_{(\nu_1, \dots, \nu_u) \in \mathcal{N}_u(v)} \sum_{(j_1, \dots, j_u) \in \mathcal{J}_u(d)} \sup_{y \in R(0, \eta)} \left| \int_{\mathbb{R}_-^d} \partial_{j_1}^{\nu_1} \dots \partial_{j_u}^{\nu_u} \varphi(z - x - y) dz \right|. \end{aligned}$$

Therefore, we obtain the desired result once we show

$$\sup_{x \in \mathbb{R}^d: \max_j x_j > \kappa} \sum_{(j_1, \dots, j_u) \in \mathcal{J}_u(d)} \sup_{y \in R(0, \eta)} \left| \int_{\mathbb{R}_-^d} \partial_{j_1}^{\nu_1} \dots \partial_{j_u}^{\nu_u} \varphi(z - x - y) dz \right| \lesssim d^v e^{-(\kappa-\eta)^2/4} \quad (10.14)$$

and

$$\sup_{x \in \mathbb{R}^d: \max_j x_j < -\kappa} \sum_{(j_1, \dots, j_u) \in \mathcal{J}_u(d)} \sup_{y \in R(0, \eta)} \left| \int_{\mathbb{R}_-^d} \partial_{j_1}^{\nu_1} \dots \partial_{j_u}^{\nu_u} \varphi(z - x - y) dz \right| \lesssim d^v e^{-(\kappa-\eta)^2/4} \quad (10.15)$$

for any $(\nu_1, \dots, \nu_u) \in \mathcal{N}_u(v)$ with $u \in \{1, \dots, v\}$.

For every $x \in \mathbb{R}^d$, set

$$I(x) := \sum_{(j_1, \dots, j_u) \in \mathcal{J}_u(d)} \sup_{y \in R(0, \eta)} \left| \int_{\mathbb{R}_-^d} \partial_{j_1}^{\nu_1} \cdots \partial_{j_u}^{\nu_u} \varphi(z - x - y) dz \right|.$$

Then we have

$$I(x) = \sum_{(j_1, \dots, j_u) \in \mathcal{J}_u(d)} \prod_{q=1}^u \sup_{y \in R(0, \eta)} |h_{\nu_q}(-x_{j_q} - y_{j_q})| \prod_{k: k \neq j_1, \dots, j_u} \Phi_1(-x_k - y_k).$$

In particular, we obtain $I(x) \lesssim d^v$. Therefore, it is enough to prove (10.14)–(10.15) when $\kappa - \eta > t_v \vee 1$ because we can take a sufficiently large constant C'_v depending only on v so that $I(x) \leq C'_v d^v e^{-(\kappa - \eta)^2/4}$ for all $x \in \mathbb{R}^d$ and $\kappa - \eta \leq t_v \vee 1$.

Let us consider the case $\max_j x_j > \kappa$. Then there is an $l \in \{1, \dots, d\}$ such that $x_l > \kappa$. For any positive integer $\nu \leq v$, h_ν is decreasing on $[t_\nu, \infty)$ (recall (10.12)), so we have $|h_\nu(-x_l - y_l)| \lesssim (\kappa - \eta)^v e^{-(\kappa - \eta)^2/2}$ for any $y \in R(0, \eta)$ (recall (10.13) and $\kappa - \eta > t_\nu \vee 1$). Hence, noting that $|h_\nu|$ is bounded by a constant depending only on ν , we obtain

$$\begin{aligned} I(x) &\lesssim \sum_{(j_1, \dots, j_u) \in \mathcal{J}_u(d)} (\kappa - \eta)^v e^{-(\kappa - \eta)^2/2} \vee \Phi_1(-\kappa + \eta) \\ &\leq d^v \kappa^v e^{-\kappa^2/2} \vee \Phi_1(-\kappa + \eta). \end{aligned}$$

Since $\Phi_1(-\kappa + \eta) \leq e^{-(\kappa - \eta)^2/2}$ by Chernoff's bound and $\sup_{t > 0} t^v e^{-t^2/4} < \infty$, we obtain (10.14).

Next consider the case $\max_j x_j < -\kappa$. Then we have $-x_j > \kappa$ for all $j = 1, \dots, d$. Since $\kappa - \eta > t_\nu$ and h_ν is decreasing on $[t_\nu, \infty)$ for any positive integer $\nu \leq v$, we obtain

$$\begin{aligned} I(x) &\leq \sum_{(j_1, \dots, j_u) \in \mathcal{J}_u(d)} \prod_{q=1}^u h_{\nu_q}(\kappa - \eta) \lesssim d^u \prod_{q=1}^u (\kappa - \eta)^{\nu_q - 1} e^{-(\kappa - \eta)^2/2} \\ &\lesssim d^v (\kappa - \eta)^{v-1} e^{-(\kappa - \eta)^2/2} \lesssim d^v e^{-(\kappa - \eta)^2/4}, \end{aligned}$$

where the third inequality follows from $\sum_q \nu_q = v$ and $1 \leq u \leq v$, and the last inequality follows from $\sup_{t > 0} t^{v-1} e^{-t^2/4} < \infty$. Hence we obtain (10.15) and complete the proof. \blacksquare

11. AUXILIARY LEMMAS

Lemma 11.1. *Let $Z = (Z_1, \dots, Z_d)^T$ be a centered Gaussian random vector in \mathbb{R}^d with a non-singular covariance matrix $\Sigma = (\Sigma_{jk})_{j,k=1}^d$. Then for any $j = 1, \dots, d$, $\epsilon > 0$, $w \in \mathbb{R}^d$,*

and bounded and measurable $h: \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\mathbb{E}h(w + \epsilon Z)Z_j = \epsilon \sum_{k=1}^d \partial_k h_\epsilon(w) \Sigma_{jk},$$

where $h_\epsilon: \mathbb{R}^d \rightarrow \mathbb{R}$ is given by $h_\epsilon(w) = \mathbb{E}h(w + \epsilon Z)$ for all $w \in \mathbb{R}^d$.

Remark 11.1. This lemma is a version of Stein's identity suitable for non-differentiable functions h . Although the lemma seems to be rather well known, we provide its proof below for reader's convenience. ■

Proof. Observe that the asserted claim for general $\epsilon > 0$ follows from the asserted claim for $\epsilon = 1$ by rescaling of the vector Z . Therefore, we only consider the case $\epsilon = 1$.

Now, fix any $j = 1, \dots, d$, $w \in \mathbb{R}^d$, and bounded and measurable $h: \mathbb{R}^d \rightarrow \mathbb{R}$. Then, denoting $A = \Sigma^{-1/2}$, so that $V = AZ \sim N(0, I_d)$, we have

$$\begin{aligned} h_\epsilon(w) &= \mathbb{E}h(w + Z) = \mathbb{E}h(w + A^{-1}V) \\ &= \int h(w + A^{-1}v) \varphi(v) dv = |A| \int h(s) \varphi(A(s - w)) ds, \end{aligned}$$

where φ is the pdf of the standard normal distribution on \mathbb{R}^d and $|A|$ is the determinant of A . Thus, differentiating under the integral, which is allowed by Corollary A.10 in [22], for all $k = 1, \dots, d$,

$$\begin{aligned} \partial_k h_\epsilon(w) &= -|A| \int h(s) \sum_{l=1}^d A_{kl} \partial_l \varphi(A(s - w)) ds \\ &= - \int h(w + A^{-1}v) \sum_{l=1}^d A_{kl} \partial_l \varphi(v) dv = \int h(w + A^{-1}v) \sum_{l=1}^d A_{kl} v_l \varphi(v) dv. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k=1}^d \partial_k h_\epsilon(w) \Sigma_{jk} &= \int h(w + A^{-1}v) \sum_{l=1}^d \sum_{k=1}^d \Sigma_{jk} A_{kl} v_l \varphi(v) dv \\ &= \int h(w + A^{-1}v) \sum_{l=1}^d (\Sigma^{1/2})_{jl} v_l \varphi(v) dv = \mathbb{E}h(w + Z)Z_j, \end{aligned}$$

where the last equality follows from $Z = \Sigma^{1/2}V$. The asserted claim follows. ■

Lemma 11.2. Let X , Y , and Z be independent random vectors in \mathbb{R}^d . Denote

$$\zeta := \sup_{r \in \mathbb{R}^d} \left| \mathbb{P}(X \leq r) - \mathbb{P}(Y \leq r) \right| \quad \text{and} \quad \gamma := \sup_{r \in \mathbb{R}^d} \left| \mathbb{P}(X + Z \leq r) - \mathbb{P}(Y + Z \leq r) \right|$$

and let $\epsilon > 0$ be such that $\alpha := \mathbb{P}(Z \in R(0, \epsilon)) > 1/2$. Then

$$\zeta \leq \frac{\gamma + \alpha\tau}{2\alpha - 1},$$

where

$$\tau := \sup_{r \in \mathbb{R}^d} \left| \mathbb{P}(Y \leq r + 2\epsilon) - \mathbb{P}(Y \leq r) \right|.$$

Remark 11.2. This result is an adaptation of Lemma 2.4 from [23] with hopefully easier to follow notations and is a version of Lemma 11.4 from [9]. We provide a proof here for reader's convenience. \blacksquare

Proof. Note that

$$\zeta = \max \left(\sup_{r \in \mathbb{R}^d} \left(\mathbb{P}(X \leq r) - \mathbb{P}(Y \leq r) \right), \sup_{r \in \mathbb{R}^d} \left(\mathbb{P}(Y \leq r) - \mathbb{P}(X \leq r) \right) \right)$$

and consider the case

$$\zeta = \sup_{r \in \mathbb{R}^d} \left(\mathbb{P}(X \leq r) - \mathbb{P}(Y \leq r) \right). \quad (11.1)$$

In this case, for any $r \in \mathbb{R}^d$, we have

$$\mathbb{P}(X + Z \leq r + \epsilon) - \mathbb{P}(Y + Z \leq r + \epsilon) = \mathcal{I}_{1,r} + \mathcal{I}_{2,r}, \quad (11.2)$$

where

$$\begin{aligned} \mathcal{I}_{1,r} &= \mathbb{E} \left[\left(1\{X + Z \leq r + \epsilon\} - 1\{Y + Z \leq r + \epsilon\} \right) 1\{Z \in R(0, \epsilon)\} \right], \\ \mathcal{I}_{2,r} &= \mathbb{E} \left[\left(1\{X + Z \leq r + \epsilon\} - 1\{Y + Z \leq r + \epsilon\} \right) 1\{Z \notin R(0, \epsilon)\} \right]. \end{aligned}$$

Here, denoting $\zeta_r := \mathbb{P}(X \leq r) - \mathbb{P}(Y \leq r)$, we have

$$\begin{aligned} \mathcal{I}_{1,r} &\geq \mathbb{E} \left[\left(1\{X \leq r\} - 1\{Y \leq r + 2\epsilon\} \right) 1\{Z \in R(0, \epsilon)\} \right] \\ &= \mathbb{E} \left[1\{X \leq r\} - 1\{Y \leq r + 2\epsilon\} \right] \mathbb{E} \left[1\{Z \in R(0, \epsilon)\} \right] \geq (\zeta_r - \tau)\alpha \end{aligned}$$

and

$$\mathcal{I}_{2,r} = \mathbb{E} \left[\mathbb{E} \left[1\{X + Z \leq r + \epsilon\} - 1\{Y + Z \leq r + \epsilon\} \mid Z \right] 1\{Z \notin R(0, \epsilon)\} \right] \geq -\zeta(1 - \alpha).$$

Therefore, taking the supremum over $r \in \mathbb{R}^d$ in (11.2) and recalling (11.1), we have

$$\gamma \geq \zeta(2\alpha - 1) - \tau\alpha.$$

Rearranging the terms in this inequality gives the asserted claim under (11.1), and since the case

$$\zeta = \sup_{r \in \mathbb{R}^d} \left(\mathbb{P}(Y \leq r) - \mathbb{P}(X \leq r) \right)$$

is similar, the proof is complete. \blacksquare

Lemma 11.3 (Nazarov's inequality). *Let $Z = (Z_1, \dots, Z_d)^T$ be a centered Gaussian random vector in \mathbb{R}^d such that $\mathbb{E}Z_j^2 \geq 1$ for all $j = 1, \dots, d$ with $d \geq 3$. Then for any $z \in \mathbb{R}^d$ and any $\varepsilon > 0$,*

$$\mathbb{P}(Z \leq z + \varepsilon) - \mathbb{P}(Z \leq z) \leq C\varepsilon\sqrt{\log d},$$

where $C > 0$ is a universal constant.

Proof. See Lemma A.1 in [15]. ■

Lemma 11.4. *Let X_1, \dots, X_n be independent centered random vectors in \mathbb{R}^d with $d \geq 2$. Define the following quantities: $Z := \max_{1 \leq j \leq d} |\sum_{i=1}^n X_{ij}|$, $M := \max_{1 \leq i \leq n} \max_{1 \leq j \leq d} |X_{ij}|$, and $\sigma^2 := \max_{1 \leq j \leq d} \sum_{i=1}^n \mathbb{E}[X_{ij}^2]$. Then*

$$\mathbb{E}[Z] \leq C \left(\sigma\sqrt{\log d} + \sqrt{\mathbb{E}[M^2] \log d} \right),$$

where $C > 0$ is a universal constant.

Proof. See Lemma 8 in [14] ■

Lemma 11.5. *Assume the setting of Lemma 11.4. (i) For every $\eta > 0, \beta \in (0, 1]$ and $t > 0$,*

$$\mathbb{P}\{Z \geq (1 + \eta)\mathbb{E}[Z] + t\} \leq \exp\{-t^2/(3\sigma^2)\} + 3 \exp\{-(t/(C\|M\|_{\psi_\beta}))^\beta\},$$

where $C > 0$ is a constant depending only on η, β . (ii) For every $\eta > 0, s \geq 1$ and $t > 0$,

$$\mathbb{P}\{Z \geq (1 + \eta)\mathbb{E}[Z] + t\} \leq \exp\{-t^2/(3\sigma^2)\} + C'\mathbb{E}[M^s]/t^s,$$

where $C' > 0$ is a constant depending only on η and s .

Proof. See Theorem 4 in [1] for case (i) and Theorem 2 in [2] for case (ii). ■

Lemma 11.6. *Let X_1, \dots, X_n be independent random vectors in \mathbb{R}^d with $d \geq 2$ such that $X_{ij} \geq 0$ for all $i = 1, \dots, n$ and $j = 1, \dots, d$. Define $Z := \max_{1 \leq j \leq d} \sum_{i=1}^n X_{ij}$ and $M := \max_{1 \leq i \leq n} \max_{1 \leq j \leq d} X_{ij}$. Then*

$$\mathbb{E}[Z] \leq C \left(\max_{1 \leq j \leq d} \mathbb{E} \left[\sum_{i=1}^n X_{ij} \right] + \mathbb{E}[M] \log d \right),$$

where $C > 0$ is a universal constant.

Proof. See Lemma 9 in [14]. ■

Lemma 11.7. *Assume the setting of Lemma 11.6. (i) For every $\eta > 0, \beta \in (0, 1]$ and $t > 0$,*

$$\mathbb{P}\{Z \geq (1 + \eta)\mathbb{E}[Z] + t\} \leq 3 \exp\{-(t/(C\|M\|_{\psi_\beta}))^\beta\},$$

where $C > 0$ is a constant depending only on η, β . (ii) For every $\eta > 0, s \geq 1$ and $t > 0$,

$$\mathbb{P}\{Z \geq (1 + \eta)\mathbb{E}[Z] + t\} \leq C'\mathbb{E}[M^s]/t^s,$$

where $C' > 0$ is a constant depending only on η, s .

Proof. See Lemma E.4 in [15]. ■

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