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cemmap working paper CWP03/21

Economic and Social Research Council

A note on global identification in structural vector autoregressions*

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This draft: 8 February 2021

Abstract

In a landmark contribution to the structural vector autoregression (SVARs) literature, Rubio-Ramírez, Waggoner, and Zha (2010, 'Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference,' *Review of Economic Studies*) shows a necessary and sufficient condition for equality restrictions to globally identify the structural parameters of a SVAR. The simplest form of the necessary and sufficient condition shown in Theorem 7 of Rubio-Ramírez et al (2010) checks the number of zero restrictions and the ranks of particular matrices without requiring knowledge of the true value of the structural or reduced-form parameters. However, this note shows by counterexample that this condition is not sufficient for global identification. Analytical investigation of the counterexample clarifies why their sufficiency claim breaks down. The problem with the rank condition is that it allows for the possibility that restrictions are redundant, in the sense that one or more restrictions may be implied by other restrictions, in which case the implied restriction contains no identifying information. We derive a modified necessary and sufficient condition for SVAR global identification and clarify how it can be assessed in practice.

Keywords: Simultaneous equation model, exclusion restrictions, redundant restrictions.
JEL codes: C01,C13,C30,C51.

*We thank Thomas Carr, Luca Fanelli, Michele Piffer, and Matthew Read for beneficial discussions and comments. Financial support from the ESRC through the ESRC Centre for Microdata Methods and Practice (CeMMAP) (grant number RES-589-28-0001) and the European Research Council (Starting grant No. 715940) is gratefully acknowledged.

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I Introduction

Rubio-Ramírez et al. (2010) (henceforth RWZ) provide necessary and sufficient conditions for the global identification of structural parameters in Structural Vector Autoregressions (SVARs) under a general class of zero restrictions imposed on the structural parameters and their (non-)linear transformations, including impulse responses. Exploiting the insights of their global identification analysis, RWZ also develop efficient and practical algorithms to perform estimation and inference for structural parameters and impulse responses. Their analytical and computational innovations have been instrumental to recent developments in the literature, including set-identified SVARs (Arias et al. (2018, 2021)), Giacomini and Kitagawa (2020), Giacomini et al. (2021b), Volpicella (2020), Amir-Ahmadi and Drautzburg (2021)), locally-identified SVARs (Bacchiocchi and Kitagawa (2020)), and SVARs with narrative restrictions (Antolín-Díaz and Rubio-Ramírez (2018), Giacomini et al. (2021a)), to list a few. RWZ provide several different versions of the necessary and sufficient conditions for global identification. Those given in Theorem 7 are particularly attractive in terms of ease of implementation. Theorem 7 of RWZ characterizes a necessary and sufficient condition for (exact) global identification through a set of rank conditions that depends only on the choice of identifying restrictions and does not require knowledge of the true value of the structural or reduced-form parameters.

This note presents a counterexample refuting the sufficiency of the rank conditions of Theorem 7 of RWZ, i.e., the rank conditions of Theorem 7 in RWZ are met but global identification fails. An analytical investigation of this counterexample reveals why these rank conditions do not guarantee global identification. We find that the rank conditions of Theorem 7 of RWZ cannot detect what we refer to as *redundancy* of imposed identifying restrictions. In this phenomenon, a set of equality restrictions on the structural parameters or impulse responses implicitly forces other structural parameters or impulse responses to zero. If it is present, some (redundant) zero restrictions are already implied by other imposed equality restrictions, so they do not contribute any further identifying information to the system. The rank conditions of Theorem 7 of RWZ, however, incorrectly count the redundant identifying restrictions as if they reduce the dimension of the admissible structural parameters, resulting in an erroneous conclusion that the model is globally identified. We argue that the redundancy of the identifying restrictions is relevant for empirical applications, rather than being of pure theoretical interest.

To modify the sufficiency claim of the rank conditions of Theorem 7 of RWZ, we provide a new necessary and sufficient condition for (exact) global identification that correctly discounts redundant identifying restrictions. RWZ propose a useful algorithm that sequentially constructs an orthonormal matrix for structural parameter identification that satisfies the identifying restrictions. Building on and modifying their algorithm, our proposed necessary and sufficient condition for global identification checks for the existence of redundant restrictions by verifying whether the orthonormal matrix generated by this sequential algorithm is unique. Verifying uniqueness boils down to checking the rank of a sequence of matrices constraining each column of the orthonormal matrix. Although this algorithm requires values of the reduced-form parameters as an input, we show that it can detect a lack of global identification even with redundant restrictions at almost any values of the reduced-form parameters. This almost-sure property is a key to facilitating the implementation of the algorithm in practice, as it justifies running the algorithm at one or a few points in the reduced-form parameter space drawn from a prior or posterior distribution or obtained as a maximum likelihood estimate.

As an alternative to their Theorem 7, Theorem 1 in RWZ presents a different form of necessary and sufficient conditions for global identification. As we illustrate in this note, its proper implementation requires a complete understanding of how the imposed identifying restrictions analytically constrain the impulse responses and the set of structural parameters. For instance, if redundant identifying restrictions are present but one is not aware which zero restrictions can be implied by others, naive implementation of the rank conditions in Theorem 1 of RWZ may also overlook a lack of global identification. To prevent this, it is important to analytically ascertain how a set of equality restrictions translate to zero restrictions for other structural objects. This is feasible for small scale SVARs, but can be less straightforward for medium or large scale SVARs. In contrast, checking our necessary and sufficient condition remains tractable and attractive even for moderate to large scale SVARs.

The rest of the paper is organized as follows. We first introduce the model and notation in Section II. In Section III, we present an example that contradicts Theorem 7 of RWZ. In Section IV we define the notion of redundant identifying restrictions and provide a modified necessary and sufficient condition for (exact) global identification. Section V concludes.

II Model

We maintain the notation used in RWZ. Let y_t be a $n \times 1$ vector of variables observed over the sample $t = 1, \dots, T$. The specification of the SVAR model is

$$y_t' A_0 = \sum_{l=1}^{p} y_{t-l}' A_l + c + \varepsilon_t', \tag{1}$$

where ε_t is a $n \times 1$ multivariate normal white noise process with null expected value and covariance matrix equal to the identity matrix I_n. The $n \times n$ matrices A_0, $A_1, \dots,$ A_p are the structural parameters and c is a $1 \times n$ vector of constant terms. The structural parameters are (A_0, A_+), where $A'_+ \equiv (A'_1, \ldots, A'_l, c')$ is a $n \times m$ matrix with $m \equiv np + 1$. We also assume that the initial conditions $y_1, \ldots, y_p$ are given and that A_0 is invertible. The set of structural parameters is denoted by $\mathbb{P}^S$, an open dense set of $\mathbb{R}^{(n+m)n}$. The structural form can be written compactly as

$$y'_t A_0 = x'_t A_+ + \varepsilon'_t \tag{2}$$

where $x'_t = \left(y'_{t-1}, \ldots, y'_{t-p}, 1\right)$.

The reduced-form representation of (2) is the standard VAR model,

$$y'_t = x'_t B + u'_t, \tag{3}$$

where $B_j = A_+ A_0^{-1}$, $u'_t = \varepsilon'_t A_0^{-1}$, and $E(u_t\, u'_t) = \Sigma = (A_0^{-1} A'_0)^{-1}$. The reduced-form parameters are (B, Σ), where Σ is a symmetric and positive definite matrix. We denote the set of reduced-form parameters by $\mathbb{P}^R \subset \mathbb{R}^{nm+n(n+1)/2}$. The relationship between the structural and reduced-form parameters is defined by the function $g : \mathbb{P}^S \to \mathbb{P}^R$, where $g\,(A_0, A_+) = (A_+ A_0^{-1}, (A_0 A'_0)^{-1})$.

The definition of global identification is the standard one provided by Rothenberg (1971); the absence of observationally equivalent parameters in the parametric space. We consider identification of the structural parameters by imposing zero restrictions on a transformation $f(\cdot)$ of the structural parameter space into the set of $k \times n$ matrices, $k \geq 1$, with domain $U \subset \mathbb{P}^S$. Such linear restrictions are represented by

$$Q_j f(A_0, A_+) e_j = 0, \qquad \text{for } j = 1, \ldots, n. \tag{4}$$

where Q_j is a $k \times k$ selection matrix for $j = 1, \ldots, n$, and e_j is the j-th column of the $n \times n$ identity matrix I_n. The rank of Q_j is denoted by q_j, which also represents the number of restrictions in the j-th column of the transformed space $f(A_0, A_+)$. As in RWZ, we order the columns of $f(A_0, A_+)$ according to

$$q_1 \geq q_2 \geq \ldots \geq q_n. \tag{5}$$

We denote the set of orthonormal matrices by $\mathcal{O}(n)$ with generic element P.

Following RWZ, we say that this transformation is admissible when the following condition holds.

Condition 1. The transformation $f(\cdot)$, with the domain U, is admissible if and only if for any $P \in \mathcal{O}\,(n)$ and $(A_0, A_+) \in U$, $f(A_0 P, A_+ P) = f(A_0, A_+)P$.

Moreover, RWZ impose the following two conditions when proving some of their results.

Condition 2. The transformation $f(\cdot)$, with the domain U, is regular if and only if U is open and f is continuously differentiable with $f'\,(A_0, A_+)$ of rank kn for all $(A_0, A_+) \in U$.

Condition 3. The transformation $f(\cdot)$, with the domain U, is strongly regular if and only if it is regular and $f(U)$ is dense in the set of $k \times n$ matrices.

To fix the sign of structural shocks, we need to impose sign normalization rules. Following RWZ, we define them as follows:

Definition 1 (Normalization rule)**.** A normalization rule can be characterized by a set $N \subset \mathbb{P}^S$ such that for any structural parameter point $(A_0, A_+) \subset \mathbb{P}^S$, there exists a unique $n \times n$ diagonal matrix D with plus or minus ones along the diagonal such that $(A_0 D, A_+ D) \in N$.

We are now able to define the set of restricted structural parameters as

$$R = \{(A_0, A_+) \in U \cap N \,|\, Q_j f(A_0, A_+) e_j = 0 \text{ for } j = 1, \ldots, n\}. \tag{6}$$

Following RWZ, we consider the following definition of identification when discussing whether or not the imposed restrictions can globally identify the structural parameters.

Definition 2 (Exact identification)**.** Consider an SVAR with restrictions represented by R. The SVAR is exactly identified if and only if, for almost any reduced-form parameter point (B, Σ), there exists a unique structural parameter point $(A_0, A_+) \in R$ such that $g(A_0, A_+) = (B, \Sigma)$.

In this definition, if the set of structural parameters under the restrictions R constrains the reduced-form parameters, the domain of the reduced-form parameters for which the almost-sure property is required is restricted to $\tilde{\mathbb{P}}^R \subset \mathbb{P}^R$, where $\tilde{\mathbb{P}}^R$ is the set of reduced-form parameters generated by the structural parameters satisfying R. For instance, if $f(\cdot)$ maps the structural parameters to long-run impulse responses, its domain U restricts the reduced-form VARs to being invertible. Then, $\tilde{\mathbb{P}}^R$ corresponds to the set of reduced-form parameters constrained to invertible VARs.

III An Illustrative Counterexample

In the setting described in the previous section, RWZ shows a variety of necessary and sufficient conditions for the identifying restrictions R with admissible $f(\cdot)$ to globally identify the structural parameters. Among those, the necessary and sufficient condition for exact identification presented in Theorem 7 of RWZ is particularly attractive, as it reduces verification of exact identification to a simple exercise of computing the ranks of the matrices Q_j, $1 \leq j \leq n$. So that our exposition is self-contained, we present Theorem 7 of RWZ here:

Theorem 7 in RWZ: *Consider an SVAR with admissible and strongly regular restrictions represented by R.*[1] *The SVAR is exactly identified if and only if* $q_j = n - j$ *for* $1 \le j \le n$.

The first result in this note is that the "if" statement of this theorem is false, as shown by the following counterexample.

III.1 A counterexample

Consider a trivariate SVAR characterized by the following restrictions

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad IR_0 = \begin{pmatrix} \times & 0 & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix} \tag{7}$$

where $IR_0 = (A_0^{-1})'$ is the contemporaneous impulse response matrix, the symbol '$\times$' indicates that no restriction is imposed, and '0' represents a zero (or exclusion) restriction. The function $f(A_0, A_+)$ will be

$$f(A_0, A_+) = \begin{pmatrix} A_0 \\ IR_0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \\ \times & 0 & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{8}$$

The matrices of restrictions defined in (4) can be specified as

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

According to Theorem 7 in RWZ, the SVAR is exactly (globally) identified, as the ranks of the restriction matrices follow $q_1 = n - 1 = 2$, $q_2 = n - 2 = 1$ and $q_3 = n - 3 = 0$. However, analytical investigation shows the current set of identifying restrictions fails to achieve global identification.

[1] Admissible and strongly regular restrictions represented by R mean $f(\cdot)$ in (6) is admissible and strongly regular.

Let us express the reduced-form covariance matrix and its Cholesky decomposition as

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \Rightarrow \Sigma_{tr} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}. \tag{10}$$

Imposing triangularity on A_0, we can obtain A_0 and $IR_0 = A_0^{-1\prime}$ as

$$A_0' = \Sigma_{tr}^{-1} = \begin{pmatrix} \frac{1}{l_{11}} & 0 & 0 \\ -\frac{l_{21}}{l_{11}l_{22}} & \frac{1}{l_{22}} & 0 \\ \frac{l_{21}l_{32}-l_{22}l_{31}}{l_{11}l_{22}l_{33}} & -\frac{l_{32}}{l_{22}l_{33}} & \frac{1}{l_{33}} \end{pmatrix} \Rightarrow IR_0 = A_0^{-1\prime} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}. \tag{11}$$

Consider applying Algorithm 1 in RWZ to determine an orthogonal matrix P that maps the (A_0, A_+) parameters under triangularity to the one satisfying the imposed restrictions.

First, $f(A_0, A_+)$ is

$$f(A_0, A_+) = \begin{pmatrix} A_0 \\ IR_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{l_{11}} & -\frac{l_{21}}{l_{11}l_{22}} & \frac{l_{21}l_{32}-l_{22}l_{31}}{l_{11}l_{22}l_{33}} \\ 0 & \frac{1}{l_{22}} & -\frac{l_{32}}{l_{22}l_{33}} \\ 0 & 0 & \frac{1}{l_{33}} \\ l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}. \tag{12}$$

As in RWZ, let $\bar{Q}_1$ and $\bar{Q}_2$ be the matrices of indicators for the restricted elements of $f(A_0, A_+)$ obtained by removing the row vectors of zeros from Q_1 and Q_2. Algorithm 1 in RWZ suggests calculating

$$\tilde{Q}_1 = \bar{Q}_1 f(A_0, A_+) = \begin{pmatrix} 0 & \frac{1}{l_{22}} & -\frac{l_{32}}{l_{22}l_{33}} \\ 0 & 0 & \frac{1}{l_{33}} \end{pmatrix}, \tag{13}$$

and finding a unit-length vector that is orthogonal to the row vectors of $\tilde{Q}_1$. The QR decomposition of $\tilde{Q}_1$ and a sign normalization lead to $p_1 = (1, 0, 0)'$ as a unique unit vector satisfying $\tilde{Q}_1 p_1 = 0$, so we can pin down the first column vector of P.

Next, to find the second column vector p_2 of P, we form the matrix

$$\tilde{Q}_2 = \begin{pmatrix} \bar{Q}_2 f(A_0, A_+) \\ p_1' \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ \hdashline 1 & 0 & 0 \end{pmatrix} \tag{14}$$

and search for a unit vector p_2 satisfying $\tilde{Q}_2 p_2 = 0$. Since the rank of $\tilde{Q}_2$ is one for any value of

l_{11}, we cannot pin down a unique p_2 (up to the sign normalization). From a geometric point of view, any vector belonging to the unit circle in $\mathbb{R}^3$ orthogonal to the unit vector $p_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}'$ is admissible as p_2. This implies that given any reduced-form parameter value of Σ, the imposed restrictions fail to pin down a unique orthogonal matrix P, implying that, contrary to the claim in Theorem 7 of RWZ, global identification does not hold in this example.

Some packaged algorithms for the QR decomposition, including the Matlab function $qr(\cdot)$, yield an orthogonal vector p_2 irrespective of whether it is unique or not. That is, if $\tilde{Q}_2$ is not full-rank, these algorithms implicitly select one unit vector p_2 from infinitely many admissible ones. As a result, an application of the "if" statement of Theorem 7 and naive implementation of Algorithm 1 in RWZ may fail to detect the failure of global identification and mislead subsequent impulse response analysis.

III.2 Analytical investigation

To understand why the "if" statement of Theorem 7 of RWZ breaks down and how it can be modified, it is useful to determine analytically the special feature of the identifying restrictions specified in (7).

We begin with the inversion of the A_0 matrix; the determinant of A_0 is

$$|A_0| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \tag{15}$$

and the adjunct matrix is

$$\text{Adj}(A_0) = \begin{pmatrix} a_{22}a_{33} - a_{32}a_{23} & -(a_{12}a_{33} - a_{32}a_{13}) & a_{12}a_{23} - a_{22}a_{13} \\ -(a_{21}a_{33} - a_{31}a_{23}) & a_{11}a_{33} - a_{31}a_{13} & -(a_{11}a_{23} - a_{21}a_{13}) \\ a_{21}a_{32} - a_{31}a_{22} & -(a_{11}a_{32} - a_{31}a_{12}) & a_{11}a_{22} - a_{21}a_{12} \end{pmatrix}. \tag{16}$$

The inverse is $A_0^{-1} = |A_0|^{-1}\,\text{Adj}(A_0)$. Substituting the two zero restrictions on A_0, $a_{21} = 0$ and $a_{31} = 0$, into $A_0^{-1\prime}$ leads to

$$A_0^{-1\prime} = \frac{1}{a_{11}(a_{22}a_{33} - a_{23}a_{32})} \begin{pmatrix} a_{22}a_{33} - a_{32}a_{23} & 0 & 0 \\ -(a_{12}a_{33} - a_{32}a_{13}) & a_{11}a_{33} & -(a_{11}a_{32} - a_{31}a_{12}) \\ a_{12}a_{23} - a_{22}a_{13} & -a_{11}a_{23} & a_{11}a_{22} \end{pmatrix} = IR_0. \tag{17}$$

It is evident that the two restrictions on A_0 imply two zero restrictions on IR_0, $(A_0^{-1\prime})_{[1,2]} = (A_0^{-1\prime})_{[1,3]} = 0$. One of these, $(A_0^{-1\prime})_{[1,2]} = 0$, is exactly the zero restriction specified for IR_0 in (7). In other words, we intended to impose the three restrictions, but the two imposed on A_0 imply the third imposed on IR_0, so this third restriction was redundant. Due to this redundancy, the third restriction does not further constrain the admissible orthonormal matrix P, which translates into rank deficiency of $\tilde{Q}_2$.

Although this redundancy phenomenon can occur in some realistic applications,[2] whether or not any of the imposed set of restrictions are redundant cannot be directly assessed by the simple rank conditions in Theorem 7 of RWZ. As a way to uncover such redundancy, one may want to examine how a set of zero restrictions imposed on one structural object translates to zero restrictions on other objects. In Section IV below, we modify the necessary and sufficient condition of Theorem 7 of RWZ by offering a systematic way to detect redundancy of the imposed identifying restrictions.

III.3 Detecting the failure of global identification

In their Theorem 6, RWZ provides an alternative necessary and sufficient condition for exact identification of SVARs. If we properly take into account that the imposed zero restrictions imply zero restrictions on other objects, this alternative approach can correctly detect a lack of global identification. We illustrate how in our example.

For $1 \leq j \leq n$ and any $k \times n$ matrix X, let $M_j(X)$ be a $(k+j) \times n$ matrix defined by

$$M_j(X) = \begin{pmatrix} \multicolumn{2}{c}{Q_j X} \\ I_{j \times j} & O_{j \times (n-j)} \end{pmatrix},$$

where Q_j is a $k \times k$ matrix defined in (4). Theorem 6 of RWZ provides a necessary and sufficient condition for exact identification through the rank conditions for $M_j(f(A_0, A_+))$.

Theorem 6 in RWZ: *Consider an SVAR with admissible and strongly regular restrictions represented by R. The SVAR is exactly identified if and only if the total number of restrictions is equal to $n(n-1)/2$ and for some $(A_0, A_+) \in R$, $M_j(f(A_0, A_+))$ is of rank n for $1 \leq j \leq n$.*

In the current example, the total number of restrictions imposed is 3 and it meets the condition for the total number of restrictions with $n = 3$. We hence focus on checking the rank condition for $M_j(f(A_0, A_+))$, $j = 1, 2, 3$. In this check, we substitute the following matrices into $f(A_0, A_+)$:

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad IR_0 = \begin{pmatrix} \times & 0 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}, \tag{18}$$

[2] Many influential empirical papers combine restrictions on both contemporaneous relationships among the endogenous variables and the contemporaneous impulse responses. Examples include Blanchard (1989), Blanchard and Perotti (2002), Bernanke (1986).

where the symbol '×' denotes the parameters in Eq. (17). We obtain, if $a_{22}a_{33} - a_{32}a_{23} \neq 0$,

$$M_1\left(f(A_0, A_+)\right) = \begin{pmatrix} 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \\ \hdashline 1 & 0 & 0 \end{pmatrix} \qquad \text{rank}\left(M_1\right) = 3$$

$$M_2\left(f(A_0, A_+)\right) = \begin{pmatrix} a_{22}a_{33} - a_{32}a_{23} & 0 & 0 \\ \hdashline 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{rank}\left(M_2\right) = 2 < 3. \tag{19}$$

Hence, the rank condition of Theorem 6 in RWZ fails. This is consistent with the conclusion in our analysis above; the imposed restrictions uniquely pin down the first column vector of P, but not the second column vector of P. Thus, plugging in the expression of $f(A_0, A_+)$ obtained analytically under the imposed restrictions, the rank condition of Theorem 6 of RWZ correctly detects the failure of global identification due to the redundancy among the imposed identifying restrictions.

It is important to note that understanding analytically the whole set of constraints implied by the imposed restrictions is crucial to correctly performing the check of the rank condition in Theorem 6 of RWZ. For instance, in the current example, if we were not aware of the redundancy issue of the identifying restrictions and incorrectly let the $(1, 3)$-element of $M_2(f(A_0, A_+))$ be an unknown potentially nonzero free parameter, we would have erroneously claimed that $M_2(f(A_0, A_+))$ were of rank 3 and concluded that the exact identification holds. If the dimension of the SVAR is large, exhaustively investigating and figuring out the entire set of constraints implied by the zero restrictions on $f(A_0, A_+)$ is challenging. In such a case, immediate implementation of the rank conditions of Theorem 6 of RWZ is limited.

IV Modified necessary and sufficient condition for exact identification

In this section we provide a modified necessary and sufficient condition for exact identification that eliminates the redundancy issue that invalidates Theorem 7 of RWZ. Our proposal relies on the sequential feature of Algorithm 1 in RWZ and checks the rank condition for uniqueness of the j-th column vector p_j for each $j = 1, \ldots, n$.

Given the reduced-form parameter (B, Σ), set (A_0, A_+) to be an unrestricted set of structural parameters satisfying $\Sigma = (A_0')^{-1}(A_0)^{-1}$ and $B = A_+ A_0^{-1}$, such as $A_0' = \Sigma_{tr}^{-1}$ and $A_+ = B(\Sigma_{tr}^{-1})'$.

Let

$$\tilde{Q}_1 = Q_1 f(A_0, A_+), \text{ and } \tilde{Q}_j = \begin{pmatrix} Q_j f(A_0, A_+) \\ p_1' \\ \vdots \\ p_{j-1}' \end{pmatrix} \text{ for } j = 2, \ldots, n. \tag{20}$$

By Theorem 5 and Algorithm 1 of RWZ, the exact identification of SVARs follows if and only if, for almost every reduced-form parameters (B, Σ), the orthogonality conditions $\tilde{Q}_j p_j = 0$ combined with the sign normalization restrictions pin down a unique orthogonal matrix P.

For P to be uniquely determined, it is necessary to have $q_j = n - j$ for all $1 \leq j \leq n$. This is, however, not a sufficient condition, because if any of the orthogonal vectors $(p_1, \ldots, p_{j-1})$ is linearly dependent on the row vectors of $Q_j f(A_0, A_+)$, a rank-deficient $\tilde{Q}_j$ fails to pin down a unique p_j. This is exactly the mechanism that caused the systematic failure of global identification in our illustrative counterexample. To rule out such rank-deficiency in the characterization of the global identification condition, we introduce the following concept:

Definition 3 (Non-redundant restrictions)**.** Given reduced-form parameter (B, Σ), let $A_0' = \Sigma_{tr}^{-1}$ and $A_+ = B(\Sigma_{tr}^{-1})'$. Identifying restrictions for a SVAR that are represented by zero restrictions $Q_j f(A_0, A_+) e_j = 0$, $j = 1, \ldots, n$, are *non-redundant* at given reduced-form parameter point, (B, Σ) if for every $j = 2, \ldots, n$, orthogonal vectors $(p_1, \ldots, p_{j-1})$ are linearly independent of the row vectors of $Q_j f(A_0, A_+)$, i.e., $\tilde{Q}_j$ defined in (20) is full row-rank for all $j = 2, \ldots, n$.

If the imposed zero restrictions are non-redundant and the rank condition of Theorem 7 in RWZ holds, we can guarantee

$$\text{rank}\,(\tilde{Q}_j) = \text{rank} \begin{pmatrix} Q_j f(A_0, A_+) \\ p_1' \\ \vdots \\ p_{j-1}' \end{pmatrix} = n - 1 \tag{21}$$

for all $j = 1, \ldots, n$. We can therefore solve for an orthonormal matrix P uniquely by sequentially solving $\tilde{Q}_j p_j = 0$, for $j = 1, \ldots, n$. If non-redundancy of the imposed restrictions holds for almost any reduced-form parameter point (B, Σ), we can achieve exact identification. We hence obtain the following theorem that modifies Theorem 7 of RWZ. We provide a proof in the Appendix.

Theorem 1 (A necessary and sufficient condition for exact identification)**.** *Consider an SVAR with admissible and strongly regular restrictions represented by R. The SVAR is exactly identified if and only if $q_j = n - j$ for $j = 1, \ldots, n$ and the restrictions are non-redundant at almost any reduced-form parameter (B, Σ).*

In comparison to Theorem 7 of RWZ, our Theorem 1 adds the almost-sure non-redundancy condition of the imposed restrictions as a part of necessary and sufficient condition. Accordingly, the modified necessary and sufficient condition of our Theorem 1 may not appear as simple as checking the ranks of Q_j matrices. However, the next theorem, which extends Theorem 3 of RWZ to the current setting, leads to an easy-to-implement procedure for assessing the almost-sure non-redundancy condition:

Theorem 2. *Consider an SVAR with admissible and regular restrictions represented by R that satisfies $q_j = n - j$ for $1 \leq j \leq n$. Let $\tilde{\mathbb{P}}^R \subset \mathbb{P}^R$ be the set of reduced-form parameters (B, Σ) generated by the structural parameters satisfying R. Let K be the set of reduced-form parameters $(B, \Sigma) \in \tilde{\mathbb{P}}^R$ that satisfy the non-redundancy condition, i.e., the rank conditions of Definition 3 holds. Either K is empty or the complement of K is of measure zero in $\tilde{\mathbb{P}}^R$.*

A practical implication of this theorem is that we can assess exact identification of SVARs by checking the rank conditions of non-redundancy at some finite number of points of $(B, \Sigma) \in \tilde{\mathbb{P}}^R$ drawn from a probability distribution supporting $\tilde{\mathbb{P}}^R$. Such probability distribution can be a prior or posterior distribution for the reduced-form parameters in a Bayesian VAR. Building on and modifying Algorithm 1 of RWZ, the next algorithm correctly judges if exact identification holds or not, almost surely in terms of the sampling probability therein.

Algorithm 1. *Consider an SVAR with admissible and strongly regular restrictions represented by R that satisfies $q_j = n - j$, for $j = 1, \ldots, n$. Let (B_m, Σ_m), $m = 1, \ldots, M$ be M number of draws of the reduced-form parameters from a probability distribution supporting $\tilde{\mathbb{P}}^R$. M does not have to be large and a small integer $M \geq 2$ should suffice.*

For each $m = 1, \ldots, M$, perform the following steps:

1. *Let $A_0' = \Sigma_{tr,m}^{-1}$ and $A_+ = B_m(\Sigma_{tr,m}^{-1})'$, where $\Sigma_{tr,m}$ is the lower-triangular Cholesky factor of Σ_m.*

2. *For each $j = 1, \ldots, n$, sequentially, check the rank conditions for non-redundancy, i.e., check if $\mathrm{rank}(\tilde{Q}_j) = n - 1$ holds, where $\tilde{Q}_1 = Q_1 f(A_0, A_+)$ and*

$$\tilde{Q}_j = \begin{pmatrix} Q_j f(A_0, A_+) \\ p_1' \\ \vdots \\ p_{j-1}' \end{pmatrix} \tag{22}$$

   for $j = 2, \ldots, n$, and p_j is an $n \times 1$ vector satisfying $\tilde{Q}_j p_j = 0$ which is unique (up to sign normalization) if $\mathrm{rank}(\tilde{Q}_{j'}) = n - 1$ holds for all preceding $j' = 1, \ldots, j - 1$.

If at least one drawn reduced-form parameter point passes Step 2 of the current algorithm, we conclude that the imposed identifying restrictions R achieve exact identification. If none of the drawn reduced-form parameter points passes Step 2, we conclude that the imposed identifying restrictions do not achieve exact identification.

The constructions of the orthonormal vectors $p_1, \ldots, p_n$ by solving $\tilde{Q}_j p_j = 0$ sequentially for $j = 1, \ldots, n$, as incorporated in Step 2 of Algorithm 1, is proposed in Algorithm 1 of RWZ. For the purpose of checking exact identification, the important feature of our algorithm is the step of checking $\text{rank}(\tilde{Q}_j) = n - 1$ for all $j = 1, \ldots, n$. This extra step, which is absent in Algorithm 1 of RWZ, is necessary to detect failure of exact identification due to redundancy of the identifying restrictions.

V Conclusion

Based on a counterexample, this note demonstrates that the sufficiency claim in Theorem 7 of RWZ, commonly used by applied macro-economists because of its simplicity, is not correct. Analytical investigation of this counterexample reveals the issue of redundancy among the identifying restrictions, which the rank conditions of Theorem 7 of RWZ overlook. To rectify this, we present a new set of necessary and sufficient conditions for exact identification and a computational algorithm that can correctly detect redundant identifying restrictions and is easy to implement in practice. We recommend this procedure to any researchers who wish to check global identification of SVARs under their choice of equality identifying restrictions.

References

Amir-Ahmadi, P. and T. Drautzburg (2021): "Identification and inference with ranking restrictions," *Quantitative Economics*, Forthcoming.

Antolín-Díaz, J. and J. Rubio-Ramírez (2018): "Narrative Sign Restrictions for SVARs," *American Economic Review*, 108, 2802–29.

Arias, J., J. Rubio-Ramírez, and D. Waggoner (2018): "Inference Based on SVARs Identified with Sign and Zero Restrictions: Theory and Applications," *Econometrica*, 86, 685–720.

Arias, J. E., J. F. Rubio-Ramírez, and D. F. Waggoner (2021): "Inference in Bayesian Proxy-SVARs," *Journal of Econometrics*, Forthcoming.

Bacchiocchi, E. and T. Kitagawa (2020): "Locally- but not globally-identified SVARs," *Cemmap working paper.*

Bernanke, B. (1986): "Alternative explanations of the money-income correlation," *Carnegie-Rochester Conference Series on Public Policy*, 25, 49–99.

Blanchard, O. (1989): "A traditional interpretation of macroeconomic Fluctuations," *American Economic Review*, 79, 1146–1164.

Blanchard, O. and R. Perotti (2002): "An empirical characterization of the dynamic effects of changes in government spending and taxes on output," *The Quarterly Journal of Economics*, 117, 1329–1368.

Giacomini, R. and T. Kitagawa (2020): "Robust Bayesian Inference for Set-identified Models," *Econometrica*, Forthcoming.

Giacomini, R., T. Kitagawa, and M. Read (2021a): "Identication and inference under narrative restrictions," *unpublished manuscript.*

——— (2021b): "Robust Bayesian Inference in Proxy SVARs," *Journal of Econometrics*, Forthcoming.

Rothenberg, T. (1971): "Identification in parametric models," *Econometrica*, 39, 577–591.

Rubio-Ramírez, J., D. Waggoner, and T. Zha (2010): "Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference," *Review of Economic Studies*, 77, 665–696.

Volpicella, A. (2020): "SVARs identification through bounds on the forecast error variance," *Journal of Business Economics and Statistics*, Forthcoming.

A Appendix: Proofs of Theorems

The proof proceeds via a sequence of lemmas modifying those shown in RWZ.

Lemma 1. *If $q_j = n - j$ for $j = 1, \ldots, n$ and all the restrictions are non-redundant, then for every $(A_0, A_+) \in U$, there exists a $P \in \mathcal{O}(n)$ such that $(A_0 P, AP) \in R$.*

Proof. This lemma modifies Lemma 5 in RWZ by adding the non-redundancy condition. Under the non-redundancy condition, for every $j = 1, \ldots, n$, the rank of $\tilde{Q}_j$ in Eq. (22) is equal to $n - 1$. Algorithm 1 of RWZ is therefore guaranteed to yield a unique unit vector p_j. □

The next lemma modifies Lemma 6 in RWZ by explicitly assuming non-redundancy.

Lemma 2. *If $q_j = n - j$ for $j = 1, \ldots, n$ and all the restrictions are non-redundant, then there exists $(A_0, A_+) \in R$, such that $M_j(f(A_0, A_+))$ is of rank n for $j = 1, \ldots, n$.*

Proof. The proof proceeds similarly to the proof of Lemma 6 in RWZ, except for the necessary modification due to the additional assumption we impose for non-redundancy. If the imposed restrictions are non-redundant as assumed in the current lemma, there are exactly $q_j = n - j$ number of independent restrictions operating for the structural parameters for $j = 1, \ldots, n$. Let $\mathcal{V}_j$ be the column space of $(Q_j f(A_0, A_+))'$ for $(A_0, A_+) \in (U \cap N)$. Moreover, let $\mathcal{V}_j^{\perp}$ be the linear subspace of $\mathbb{R}^n$ that is orthogonal to $\mathcal{V}_j$. In case of no restrictions for certain j, let $\mathcal{V}_j^{\perp}$ be the whole $\mathbb{R}^n$. Because U is an open dense set, and given the assumption that $q_j = \text{rank}\,(Q_j) = n - j$, it is possible to find some values of $(A_0, A_+) \in (U \cap N)$ such that $\text{rank}\,(Q_j f(A_0, A_+)) = q_j = n - j$, which implies $\dim(\mathcal{V}_j) = n - j$ and $\dim(\mathcal{V}_j^{\perp}) = j$. For any $f(A_0, A_+) \in (U \cap N)$, let $P_{1:i} = (p_1, p_2, \ldots, p_i)$ be an $n \times i$ matrix of orthogonal vectors in $\mathbb{R}^n$. Moreover, let $\mathcal{P}_i$ be the linear subspace of $\mathbb{R}^n$ generated by the columns of $P_{1:i}$ and let $\mathcal{P}_i^{\perp}$ be the linear subspace of $\mathbb{R}^n$ orthogonal to the column vectors of $\mathcal{P}_i$. The dimension of $\mathcal{P}_i$ is clearly equal to i, while that of $\mathcal{P}_i^{\perp}$ is $n - i$. Now, given $(A_0, A_+) \in (U \cap N)$, according to Algorithm 1, it is possible to define the elements in $P \in \mathcal{O}(n)$ in the following recursive way:

$$
\begin{aligned}
p_1 \in \mathcal{H}_1 &\equiv \mathcal{V}_1^{\perp} \\
p_2 \in \mathcal{H}_2 &\equiv \mathcal{V}_2^{\perp} \cap \mathcal{P}_1^{\perp} \\
p_3 \in \mathcal{H}_3 &\equiv \mathcal{V}_3^{\perp} \cap \mathcal{P}_2^{\perp} \\
&\vdots \\
p_j \in \mathcal{H}_j &\equiv \mathcal{V}_j^{\perp} \cap \mathcal{P}_{j-1}^{\perp} \\
&\vdots \\
p_n \in \mathcal{H}_n &\equiv \mathcal{V}_n^{\perp} \cap \mathcal{P}_{n-1}^{\perp}
\end{aligned}
$$

where $\mathcal{H}_j \in \mathbb{R}^n$ is the set of feasible p_j given the restrictions on the j-th column of $f(A_0, A_+)$ and the set of previous orthogonal vectors collected in $P_{1:(j-1)}$. Given the assumption of non redundant restrictions, for $j = 1, \ldots, n$, $\text{rank}\,(\tilde{Q}_j) = n - 1$, and according to Lemma 1, we obtain that $\dim(\mathcal{H}_j) = 1$. Moreover, being $(p_1, \ldots, p_{j-1})$ mutually orthogonal by construction, $\dim(\mathcal{P}_{j-1}) = j - 1$. Thus, because $\dim(\mathcal{P}_{j-1}) = j - 1$ and $\dim(\mathcal{V}_j) = n - j$, in order for $\text{rank}\,(\tilde{Q}_j)$ to be equal to $n - 1$, the vector spaces $\mathcal{V}_j$ and $\mathcal{P}_{j-1}$ must be disjoint. For $j = 1, \ldots, n$, thus, the number of restrictions effectively operating in the columns of $f(A_0, A_+)$ is equal to $n - j = q_j$.

Having proved this, the remaining part of the proof follows exactly as the proof of Lemma 6 in RWZ. □

Lemma 3. *If $q_j = n - j$ for $j = 1, \ldots, n$ and all the restrictions are non-redundant, then the SVAR is exactly identified.*

Proof. This lemma modifies Lemma 7 in RWZ by adding the non-redundancy condition. We have seen that without the further assumption of non-redundant restrictions, the set of all $f(A_0, A_+) \in U$ such that there exists an orthogonal matrix $P \neq I_n$ with $(A_0P, A_+P) \in R$, denoted by G in RWZ, could be of strictly positive measure. According to Lemma 1 and Algorithm 1, the situation of G having a positive measure is precluded. Accordingly, under the assumption of $q_j = n - j$, for $j = 1, \dots, n$, and non-redundancy of the restrictions, the claim of exact identification follows as in the proof of Lemma 7 in RWZ. □

The previous Lemmas 1 to 3 allow to prove the *sufficient* part of the condition in Theorem 1. We now move to the other direction and show the following lemma.

Lemma 4. *If the SVAR is exactly identified, then $q_j = n-j$ for $j = 1, \dots, n$ and all the restrictions are non-redundant.*

Proof. This lemma modifies Lemma 9 in RWZ by explicitly claiming non-redundancy in its conclusion. The first part of the proof, which consists of proving that an exactly identified SVAR presents a pattern of restrictions of the form $q_j = n - j$, for $j = 1, \dots, n$, is essentially the same. What we need to prove is that if the model is exactly identified, then all the restrictions are non redundant. Using the result in Theorem 5 in RWZ, we can say that if an SVAR is exactly identified, than for almost every structural parameter point $(A_0, A_+) \in U$ there exists a unique $P \in \mathcal{O}(n)$ such that $(A_0P, A_+P) \in R$. Moreover, we have seen that such a P matrix can be obtained through our Algorithm 1. However, sequentially for each $j = 1, \dots, n$, for the algorithm to obtain a unique p_j, the rank of $\tilde{Q}_j$ must be equal to $n - 1$, proving thus the result. □

Proof of Theorem 1. The claim follows from Lemma 3 and Lemma 4. □

Proof of Theorem 2. This theorem is based on Lemma 2 given above and Theorem 3 in RWZ. In fact, according to Lemma 2, if $q_j = n - j$ for $j = 1, \dots, n$ and all the restrictions are non-redundant, then the rank condition in Theorem 1 in RWZ is also met. Getting at this point, the proof of Theorem 3 in RWZ can be applied as it is and leads to the conclusion of the current theorem. □