

Testable implications of multiple equilibria in discrete games with correlated types

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Testable Implications of Multiple Equilibria in Discrete Games with Correlated Types¹

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Abstract

We study testable implications of multiple equilibria in discrete games with incomplete information. Unlike de Paula and Tang (2012), we allow the players' private signals to be correlated. In static games, we leverage independence of private types across games whose equilibrium selection is correlated. In dynamic games with serially correlated discrete unobserved heterogeneity, our testable implication builds on the fact that the distribution of a sequence of choices and states are mixtures over equilibria and unobserved heterogeneity. The number of mixture components is a known function of the length of the sequence as well as the cardinality of equilibria and unobserved heterogeneity support. In both static and dynamic cases, these testable implications are implementable using existing statistical tools.

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1 Introduction

In many social-economic contexts, individuals or institutions interact strategically in response to private incentives. Empirical studies typically model such interaction via static or dynamic games with incomplete information, and exploit the equilibrium implications to infer the incentives from observed states and actions. Examples include location choices in the video retail industry in Seim (2006), timing of commercials by radio stations in Sweeting (2009), choices of effort by students and teachers in classrooms in Todd and Wolpin (2018), market entry and exit of grocery stores in Grieco (2014), and the dynamic demand and supply in shipbuilding industry in Kalouptsidi (2018). Popular methods for estimating these games require a reduced-form first step that estimates the conditional choice probabilities (CCPs) in equilibrium. Subsequent steps use model-implied links between these CCPs and structural parameters to infer the latter. See Aradillas-Lopez (2010), Aradillas-López (2019), and Bajari et al. (2010) for static games; and see Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008) and Arcidiacono and Miller (2011) for dynamic games.

Given a set of model parameters, games with private types generally admit multiple equilibria. While multiplicity does not necessarily preclude model identification² and estimation protocols exist for set identified models, multiple equilibria in a data-generating process (DGP) pose challenges to estimation in both static and dynamic settings. A researcher using observational data typically has no knowledge a priori as to whether a sample is generated from a single equilibrium. With multiple equilibria in the data, the reduced-form estimator in the first step in CCP-based methods, for example, converges to a mixture of CCPs from each equilibrium represented in the sample. This mixture does not conform to the structural links used in sequential estimation.³ Therefore, detecting multiple equilibria in the sample can be an essential step for valid inference of player incentives in these settings.

In the first part of this paper, we propose a way to test multiple Bayesian Nash equilibria (BNE) in static Bayesian games where players' private types are correlated, for example, through game-level unobserved heterogeneity that is known to all players but not reported in the data. de Paula and Tang (2012) proposed a simple test for multiple equilibria in static Bayesian games where the private types are independent conditional

²As Sweeting (2009) notes, multiple equilibria in a sample can sometimes be used to aid the identification and estimation of players' payoffs in parametric likelihood models.

³Formally, a mixture of several CCP vectors, each indexed by a different equilibrium, does not generally satisfy structural equations that characterise a single equilibrium.

on observed states. That test exploited a simple equilibrium implication. With a single BNE in data, players’ strategic choices are independent conditional on covariates when we pool observations in the sample. On the other hand, with multiple equilibria in data, those choices are correlated due to their co-movement across different equilibria in the sample. The sign of such correlation conforms with the sign of interaction effects. This approach in de Paula and Tang (2012) does not apply when private types are correlated conditioning on observed covariates. In this case, strategic choices are not independent even when there is a single equilibrium.

We introduce a new method to detect multiple equilibria in static Bayesian games. This requires a working assumption that an empiricist can group the games in a sample into pairs or clusters within which equilibrium selection, if there are multiple equilibria, is correlated. For example, consider married couples making joint retirement decisions in a context of simultaneous Bayesian games. Our method can be applied if households with similar demographics and social-economic status, or located in the same geographic area, tend to be correlated in how they adopt strategies from multiple equilibria. The private types are drawn independently across games in the same pair or cluster, but are correlated within each game. Thus, strategic choices of players from two different games in the same cluster are independent under the null of single equilibrium, but are generically correlated under the alternative due to correlated equilibrium selection. Intuitively, permuting players across games within the same “cluster” allows one to emulate the ideas in de Paula and Tang (2012). This implication lends itself to a simple test for multiple equilibria, which is analogous to inference of covariate relevance in nonparametric regressions.

Aguirregabiria and Mira (2019) provided identification results in static Bayesian games with multiple equilibria and discrete unobserved heterogeneity. Their strategy requires the private types be independent from discrete unobserved heterogeneity and observed covariates, and rank conditions on equilibrium CCPs.⁴ While identifying the full model, Aguirregabiria and Mira (2019) also introduced a novel idea for detecting multiple equilibria. The idea is to compare the ex post payoffs of players recovered from each profile of CCPs as indexed by unobserved heterogeneity and equilibria. To implement this, one needs to use exclusion restrictions in the ex post payoffs, and know the actual distribution of private types.

Our approach differs from that idea in Aguirregabiria and Mira (2019) in several ways. First, we separate the task of detecting multiple equilibria from that of identifying the full model. Thus, we do not require conditions for identifying the full model, such as exclusion restrictions in ex post payoffs, rank conditions on CCPs, and assumed knowledge of

⁴See Assumption 1(B) and condition (d) in Proposition 1 in Aguirregabiria and Mira (2019).

private type distribution. Second, we allow the correlation between player choices to arise from continuous game-level unobserved heterogeneity that can be correlated with other idiosyncratic errors as well. Lastly, inference of single equilibrium using our approach is analogous to tests of covariate relevance in nonparametric regressions, and can be implemented using existing procedures such as Racine et al. (2006). It does not require sequential steps of estimating and then comparing ex post payoffs from different profiles of CCPs. We elaborate on these substantial differences in Section 2.4.

In the second part of the paper, we propose a new test for multiple Markov Perfect Equilibria (MPE) in dynamic games where private types are correlated due to Markovian, game-level discrete unobserved heterogeneity (DUH) in each period.⁵ In this case, a player’s single-period payoff depends on a time-varying DUH, through which players’ private information is correlated. If the DUH is drawn independently in each period with no serial correlation, then the idea we introduce for static games above is applicable, with each dynamic game in the sample serving as a “cluster” itself. That is, we can test the hypothesis of multiple MPE by checking whether the choices made by two players in two different time periods are correlated. However, this does not work with a serially correlated DUH, which causes correlation between those choices even under the null of single equilibrium.

To meet this challenge, we propose a new way to identify the cardinality of equilibria in dynamic games with private types and serially correlated DUH. Our method exploits the feedbacks of earlier states and choices on future outcomes over different time horizons in an equilibrium Markov process. The idea is to extract information about equilibrium selection and DUH from how these feedbacks vary with the length of history examined. We partition and pair the history of outcomes so that their joint distribution takes the form of a finite mixture. The number of components in the finite mixture is identifiable by existing methods such as Kasahara and Shimotsu (2014). More importantly, we show how the number of components in this mixture is determined by the length of history T as well as the cardinality of equilibria and the support of DUH.

To the best of our knowledge, our method uses a new source of variation that has not been exploited in the literature for detecting multiple equilibria in dynamic games. Using these results, one can conduct inference for the cardinality of equilibria in a sample, using standard rank tests such as Kleibergen and Paap (2006). This dispenses with any need for identifying a full MPE model and comparing estimates of structural elements, such as

⁵A Markov Perfect Equilibrium is a profile of time-homogeneous pure strategies that map a player’s information in each single time period to a choice. Following convention in the literature, we maintain that players do not switch between equilibria within the process of a dynamic game.

an alternative, sequential approach mentioned in Luo et al. (2019).

The rest of the paper is organized as follows. Section 2 presents the testable implications in static Bayesian games, and discusses how a test can be constructed as a test of covariate relevance in nonparametric regressions. We also illustrate the testable implications and the necessary assumptions using numerical examples. Section 3 does the same for dynamic games with private types, where the private information is correlated through serially correlated DUH. We further discuss the related literature at the end of each section.

Notation. We use uppercase letters to denote random variables and vectors, and use lowercase letters to denote their realized values. We use calligraphic letters such as \mathcal{X} for the support of random variables or vectors, and let $\#\mathcal{X}$ denote its cardinality. For a generic random vector $R \equiv (R_k : k = 1, \dots, K)$, let $R_{-k} \equiv (R_{k'} : k' = 1, 2, \dots, k-1, k+1, \dots, K)$. For a pair of sub-vectors R_1, R_2 in $R = (R_1, R_2)$, let f_{R_1}, F_{R_1} denote the marginal density and distribution of a subvector, and let $f_{R_2|R_1}, F_{R_2|R_1}$ denote the conditional density and distribution. We write $f_{R_2|R_1=r_1}$ and $F_{R_2|R_1=r_1}$, or simply $f(r_2|r_1)$ and $F(r_2|r_1)$, if there is need to be specific about the realized value conditioned on.

2 Test of Multiple Monotone Equilibria in Static Games

2.1 The model

Consider a simultaneous binary game with private information between a set of players \mathcal{I} . Each player i chooses $D_i \in \{1, 0\}$. Let X denote states that are common knowledge among players and reported in the data. For each i , let $\epsilon_i \in \mathbb{R}$ denote i 's private information, a.k.a. “type”, “shock” or “signal”, which is unknown to other players and not reported in the data. The ex post payoff for i from $D_i = 0$ is normalized to zero while that from $D_i = 1$ is:

$$v_i(D_{-i}, X) + \epsilon_i, \tag{1}$$

where v_i is real-valued single index. Additive separability is used later to establish an order among the equilibria for the game (see Lemma 1), but might be relaxed insofar as this ordering is preserved.

Assumption 1 (i) *Conditional on X , $\epsilon \equiv (\epsilon_i : i \in \mathcal{I})$ is continuously distributed with bounded, atomless density with respect to Lebesgue measure.* (ii) *The index v_i is bounded for all i .*

Assumption 1 guarantees the existence of a Bayes-Nash Equilibrium by Theorem 1 in Athey (2001) and allows for correlation between private signals conditional on X . The test in de Paula and Tang (2012) cannot be applied in this case, because players' choices are correlated even under the null of a single equilibrium in the data.

Conditional on any x , a *pure strategy* for a player i is a function $\varepsilon \in \mathbb{R} \mapsto s_i(x, \varepsilon) \in \{0, 1\}$. Let $U_i(x, \varepsilon_i; s_{-i})$ denote the difference in player i 's expected payoffs from choosing 1 and 0 when other players adopt pure strategies $s_{-i}(x, \cdot) \equiv (s_j(x, \cdot) : j \neq i)$. That is,

$$U_i(x, \varepsilon_i; s_{-i}) \equiv \int v_i(s_{-i}(x, \varepsilon_{-i}), x) dF(\varepsilon_{-i} | \varepsilon_i, x) + \varepsilon_i,$$

where $s_{-i}(x, \varepsilon_{-i})$ is shorthand for $(s_j(x, \varepsilon_j) : j \neq i)$. A *pure-strategy Bayesian Nash equilibrium* (p.s.BNE) in a game with states x is a profile $(s_i^*(x, \cdot) : i \in \mathcal{I})$ such that

$$s_i^*(x, \varepsilon_i) = \mathbf{1} \{U_i(x, \varepsilon_i; s_{-i}^*) \geq 0\} \text{ for all } i, \varepsilon_i. \quad (2)$$

A *p.s.BNE* conditional on states x is *monotone* if $s_i^*(x, \cdot)$ is non-decreasing over the support of ε_i for each i . A monotone p.s.BNE conditional on x is summarized by a vector of thresholds $t(x) \equiv (t_i(x) : i \in \mathcal{I}) \in \mathbb{R}^{\#\mathcal{I}}$ such that $s_i^*(x, \varepsilon_i) = \mathbf{1}\{\varepsilon_i \geq t_i(x)\}$. For brevity, we use the term “equilibrium” to refer to monotone p.s.BNE when there is no ambiguity from the context below.

Assumption 2 *For any x and $\varepsilon'_i > \varepsilon_i$, $U_i(x, \varepsilon_i; s_{-i}) \geq (>)0$ implies $U_i(x, \varepsilon'_i; s_{-i}) \geq (>)0$ whenever s_{-i} consists of monotone strategies.*

Assumption 2 is the single-crossing condition in games of incomplete information from Athey (2001) and holds automatically if private types are independent. Under Assumption 1 and 2, Theorem 1 in Athey (2001) implies that our game admits monotone p.s.BNE at any x . Let $\mathcal{T}(x)$ denote the complete set of threshold vectors that characterize an equilibrium given x . That is, $\mathcal{T}(x)$ is the set of all vectors $t(x)$ such that $s_i^*(x, \varepsilon_i) = \mathbf{1}\{\varepsilon_i \geq t_i(x)\}$, $i \in \mathcal{I}$ satisfies (2).

2.2 Overview of our method in a two-player game

A typical sample reports the states and choices from many games. The identities of players may vary across the games; it is only maintained that players' preference and information are drawn from the same data-generating process (DGP) across these games. Player indexes such as i and j have common meaning across the games in that the preference and information for players with the same index from different markets are

drawn from the same distribution. For example, i and j could refer to Burger King and McDonald's across several markets, or husbands and wives across several households. Our goal is to infer whether outcomes across these games are generated from more than one equilibria.

To fix ideas, consider a model with two players $\mathcal{I} \equiv \{i, j\}$ across games that share the same realization of states x , which we suppress in the notation $v_i(D_{-i})$ and throughout this subsection (except for the more general definition of adjacency) for simplicity. Assume players' actions impose the same type of externalities on each other, that is, $\text{sign}[v_i(1) - v_i(0)] = \text{sign}[v_j(1) - v_j(0)]$. (This allows for the possibility of multiple equilibria in this simple game.) $\mathcal{T} \subset \mathbb{R}^{\#\mathcal{I}}$ denotes the complete set of equilibria, and let $T \in \mathcal{T}$ denote the equilibrium selected in a game in the data.

For the rest of Section 2.2, we elaborate on the idea using a two-player model with $\mathcal{I} \equiv \{i, j\}$. First, we show this game admits a total order over the set of equilibria \mathcal{T} .

Lemma 1 *Suppose Assumptions 1 and 2 hold, and $\#\mathcal{T} < \infty$. Then $\text{sign}[(t_i - t'_i)(t_j - t'_j)] = \text{sign}(v_i(1) - v_i(0))$ for all $t \neq t'$ in \mathcal{T} .*

Proof. Let $U_i(\varepsilon_i; t_j)$ denote i 's interim payoff when j adopts a monotone pure strategy with a threshold t_j with some abuse of notation. That is, $U_i(\varepsilon_i; t_j) \equiv \int v_i(1\{\varepsilon_j \geq t_j\})dF(\varepsilon_j|\varepsilon_i) + \varepsilon_i$. Under Assumptions 1 and 2, $U_i(t_i; t_j) = 0$ for all $t \equiv (t_i, t_j) \in \mathcal{T}$. (Marginal players with private signals at the equilibrium threshold receive zero interim payoff.) Suppose $v_i(0) > v_i(1)$. If $t_j > t'_j$, then $U_i(t_i; t'_j) < 0$. This implies $U_i(\tilde{t}_i; t'_j) < 0$ for all $\tilde{t}_i < t_i$. Otherwise the single-crossing condition in Assumption 2 would fail. This then implies that $t'_i > t_i$, because $U_i(t'_i; t'_j)$ needs to be zero for $t' \in \mathcal{T}$. Hence $(t_i - t'_i)(t_j - t'_j) < 0$. By a symmetric argument, $t'_i < t_i$ if $t_j < t'_j$. The case with $t_j \neq t'_j$ and $t_i = t'_i$ are ruled out under the stated conditions. To see this, suppose $t_i = t'_i$. Then the single-crossing condition in Assumption 2 and the finiteness of \mathcal{T} implies $t_j = t'_j$, which means $t = t'$. A symmetric argument proves the claim for the case with $v_i(0) < v_i(1)$. \square

Our goal is to test the null hypothesis that “ T is degenerate in the data” (i.e., all observations in the data are drawn from a unique equilibrium) against the alternative that “ T is stochastic in the data”. More specifically, let F_T is the probability distribution over the equilibria in the game. We call T degenerate if F_T concentrates probability one on a particular equilibrium, and T stochastic if F_T allows for two or more equilibria to be selected with positive probability. To do so, we exploit pairs of games whose equilibrium selection is related under the alternative of multiple equilibria. We refer to such pairs as *adjacent games* (we briefly reinstate the conditioning variables in the definition below for generality):

Definition 1 *Let m, n index two games. When there are multiple equilibria, games m and n are adjacent if T_m and T_n are not independent (conditional on X_n, X_m): $T_m \not\perp T_n | X_n, X_m$.*

Further qualifications to such dependence are made for the results delineated in the subsections that follow. Adjacency between two games may arise when equilibria are “affiliated” or equilibria are the same across two games. We provide such examples later in this section in more general settings.

Continuing on our two-player example, and once again suppressing the states X which are presumed to be the same across games in this subsection, we make the following assumption:

Assumption 3 *$\epsilon_{i,m}$ is independent from $(\epsilon_{j,n}, T_n)$ conditional on T_m for $m \neq n$.*

This assumption allows equilibrium selection to be correlated between games, and thus allows these games to be adjacent. Under Assumption 3, we show that players’ choices from adjacent games ($D_{i,m}$ and $D_{j,n}$) are uncorrelated under the null hypothesis that equilibrium selection is degenerate, i.e., outcomes are drawn from a single equilibrium, and generically correlated under the alternative that equilibrium selection is not degenerate.

To see this, decompose the covariance $C(D_{i,m}, D_{j,n})$ between choices by two players i and j in adjacent games m and n as

$$E[C(D_{i,m}, D_{j,n} | T_m, T_n)] + C[E(D_{i,m} | T_m, T_n), E(D_{j,n} | T_n, T_m)], \quad (3)$$

where $C(\cdot | \cdot)$ and $E(\cdot | \cdot)$ denote conditional covariance and expectation respectively. Under Assumption 3,

$$\begin{aligned} E(D_{i,m} | D_{j,n}, T_m, T_n) &= \Pr\{\epsilon_{i,m} \geq T_{m,i} | 1\{\epsilon_{j,n} \geq T_{n,j}\}, T_m, T_n\} \\ &= \Pr\{\epsilon_{i,m} \geq T_{m,i} | T_m\} \\ &= E(D_{i,m} | T_m), \end{aligned}$$

where $T_{m,i}$ denotes the i -th component in the ($\#I$)-vector T_m . This implies $D_{i,m}$ is mean-independent from $D_{j,n}$ conditional on T_n, T_m . Thus $C(D_{i,m}, D_{j,n} | T_m, T_n) = 0$ and the first term in (3) is zero regardless of whether the null is true. This implication exploits the fact that, while private types are correlated within each game, they are uncorrelated across adjacent games under Assumption 3.

Next, note that the second term in (3) is the covariance between $E(D_{i,m} | T_m)$ and $E(D_{j,n} | T_n)$ under Assumption 3. Under the null, T_m has a degenerate distribution across

all games, so the covariance between $E(D_{i,m}|T_m)$ and $E(D_{j,n}|T_n)$, i.e., the second term in (3), is zero. Consequently, $C(D_{i,m}, D_{j,n}) = 0$ under the null. On the other hand, under the alternative, $E(D_{i,m}|T_m)$ and $E(D_{j,n}|T_n)$ are dependent and generally correlated as long as the equilibrium selection is correlated between adjacent games. In such a case, as we will show in Proposition 1, the sign of the covariance equals that of the interaction effect.

Given Lemma 1, we define the following total order over \mathcal{T} : for all $t, t' \in \mathcal{T}$, $t \geq t'$ if and only if $t_i \geq t'_i$.⁶ We use this order to define “increasing” or “decreasing” functions over \mathcal{T} . We show that the covariance between $D_{i,m}$ and $D_{j,n}$ is non-zero under the alternative of multiple equilibria, and its sign is equal to the sign of the interaction effect.

Assumption 4 (i) For any pair of adjacent games n and n with $m \neq n$ and any increasing function h defined over \mathcal{T} , $E[h(T_n)|T_m = t]$ is increasing in t over \mathcal{T} . (ii) $(\epsilon_{i,m}, \epsilon_{j,m})$ is independent from T_m in each game m .

Part (i) in Assumption 4 states that there is some positive association between equilibrium selection in adjacent games. It holds, for example, if T_m and T_n are affiliated as defined in Milgrom and Weber (1982). It is also satisfied trivially when adjacent games share the same equilibrium, i.e., $\Pr\{T_m = T_n\} = 1$. Part (ii) states that private information does not interfere with equilibrium selection (once conditioning on the states x suppressed in notation).

Proposition 1 Suppose Assumptions 1, 2, 3, 4 hold and $\#\mathcal{T} < \infty$. Then $C(D_{i,m}, D_{j,n}) \neq 0$ for any adjacent games m and n with $m \neq n$ under the alternative, and its sign is the same as the sign of $v_i(1) - v_i(0)$.

Proof. Recall from (3) that the covariance between $D_{i,m}$ and $D_{j,n}$ equals the covariance of $E(D_{i,m}|T_m)$ and $E(D_{j,n}|T_n)$ under Assumption 3. Applying the law of iterated expectations by conditioning on T_m , we can write the covariance between $E(D_{i,m}|T_m)$ and $E(D_{j,n}|T_n)$ as

$$\int [E(D_{i,m}|T_m = t) - E(D_{i,m})] \{E[E(D_{j,n}|T_n)|T_m = t] - E(D_{j,n})\} dF_T(t), \quad (4)$$

where F_T denote the marginal distribution of T_m , which is non-degenerate under the alternative. Suppose $v_i(1) > v_i(0)$. Then Lemma 1 implies that $t_j > t'_j$ whenever $t > t'$ (recall this total order is defined as $t_i > t'_i$). Thus $h(t) \equiv E(D_{j,n}|T_n = t) = E(1\{\epsilon_{j,n} \geq$

⁶Note from the proof for Lemma 1 that $t_i = t'_i$ implies that $t_j = t'_j$.

$t_j\}|T_n = t)$ is decreasing in t over \mathcal{T} under (ii) in Assumption 4. Part (i) in Assumption 4 then implies that $E[h(T_n)|T_m = t] \equiv E(D_{j,n}|T_m = t)$ is decreasing in t over \mathcal{T} . Next, note that $E(D_{i,m}|T_m = t) \equiv \Pr\{\epsilon_{i,m} \geq t_i|T_m = t\}$ is also decreasing in t under (ii) in Assumption 4. Therefore the covariance between $E(D_{i,m}|T_m)$ and $E(D_{j,n}|T_n)$ as expressed in (4) is the covariance between two decreasing functions of the same latent variable T_m . Hence the covariance is positive. (See Theorem 2 in Schmidt (2003).) A symmetric argument shows that $C(D_{i,m}, D_{j,n}) < 0$ when $v_i(0) > v_i(1)$. \square

Given this proposition, it is possible to detect multiplicity in the data using the correlation in actions across adjacent games. To do so, it is essential that researchers can match pairs of adjacent games in the sample into clusters, within which the equilibrium selection is known to be positively correlated. In practice, institutional details could be informative about how to construct such clusters. For example, while studying the joint retirement decisions by married couples as simultaneous Bayesian games, we may consider pairing households located in geographic regions with similar demographics in a cluster. The underlying rationale will be that regional institutional and cultural norms may induce association in the equilibrium selection in the joint retirement game for such couples. It is worth mentioning that our method can be applied even when there is some form of weak dependence in equilibrium selection across clusters. In this case, our method applies as long as some version of the Law of Large Numbers allows us to pool across weakly dependent clusters and consistently estimate the covariance between D_{im} and D_{jn} .

2.3 Testable implications of multiple equilibria

In this section we generalize the idea in Section 2.2 to a full model with heterogeneous states and three or more players. Recall that $T_m \equiv (T_{i,m} : i \in \mathcal{I}) \in \mathcal{T}(X_m)$ denotes the equilibrium selected in game m in the data. Let $F_{T_m, T_n|x, x'}$ denote the *joint* equilibrium selection conditional on $(X_m, X_n) = (x, x')$ in the DGP. That is, for any x and x' , $F_{T_m, T_n|x, x'}$ is a joint distribution with support $\mathcal{T}(x) \times \mathcal{T}(x')$. Our goal is to test the null that “ $F_{T_m, T_n|x, x'}$ is degenerate for all (x, x') ” against the alternative that “ $F_{T_m, T_n|x, x'}$ is nondegenerate at least for some (x, x') ”.

Assumption 5 *For any pair of adjacent games m and n with $m \neq n$, $\epsilon_{i,m}$ is independent of $(\epsilon_{-i,n}, T_n, X_n)$ conditional on (T_m, X_m) for all $i \in \mathcal{I}$.*

Assumption 5 extends the conditional independence condition (Assumption 3) in Section 2.2 by accounting for observed states. It accommodates dependence between (T_m, T_n, X_m, X_n) , and also allows for correlation between private types $\epsilon_{i,m}$, $\epsilon_{-i,m}$ and observed states X_m .

Assumption 6 *There exists $i, j \in \mathcal{I}$ such that the covariance between $E(D_{i,m}|T_m, X_m)$ and $E(D_{j,n}|T_n, X_n)$ is non-zero conditional on some $(X_m, X_n) = (x, x')$.*

Assumption 6 states there is nonzero correlation between equilibrium selection and thus implies adjacency of the two games. This extends an implication of Assumption 4 in Section 2.2 to the case with heterogeneous states. It does not hold if (T_m, X_m) is independent from (T_n, X_n) or if X_n and X_m are correlated but T_m and T_n are independent conditional on X_m and X_n . To further illustrate its content, we provide two examples of sufficient conditions for Assumption 6 later in this subsection.

Proposition 2 *Suppose Assumptions 1, 2, 5 hold. Under the null, $C(D_{i,m}, D_{j,n}|X_m, X_n) = 0$ for all i, j . Under the alternative, $C(D_{i,m}, D_{j,n}|X_m, X_n) \neq 0$ for some i, j and $(X_m, X_n) = (x, x')$ if Assumption 6 holds.*

Proof of Proposition 2. Applying the Law of Total Covariance conditioning on (T_m, T_n) , we have

$$\begin{aligned} C(D_{i,m}, D_{j,n}|X_m, X_n) &= \underbrace{C[E(D_{i,m}|T_m, T_n, X_m, X_n), E(D_{j,n}|T_m, T_n, X_m, X_n)|X_m, X_n]}_{\equiv A} \\ &+ \underbrace{E[C(D_{i,m}, D_{j,n}|T_m, T_n, X_m, X_n)|X_m, X_n]}_{\equiv B}. \end{aligned} \quad (5)$$

Monotone pure-strategy Bayesian Nash equilibria exist under Assumptions 1 and 2. Assumption 5 implies that for i, j ,

$$\begin{aligned} &E(D_{i,m}|D_{j,n}, T_m, T_n, X_m, X_n) \\ &= \Pr\{\epsilon_{i,m} \geq T_{m,i} | 1\{\epsilon_{j,n} \geq T_{n,j}\}, T_m, T_n, X_m, X_n\} \\ &= \Pr\{\epsilon_{i,m} \geq T_{m,i} | T_m, X_m\} = E(D_{i,m}|T_m, X_m). \end{aligned} \quad (6)$$

Thus $D_{i,m}$ is mean-independent from $(D_{j,n}, T_n, X_n)$ conditional on (T_m, X_m) . Therefore, the second term ‘‘B’’ on the right-hand side of (5) is zero regardless of whether the null is true. Under Assumption 5, the first term ‘‘A’’ equals the covariance between $E(D_{i,m}|T_m, X_m)$ and $E(D_{j,n}|T_n, X_n)$ conditional on (X_m, X_n) . Under the null, (T_m, T_n) are degenerate once conditioning on (X_m, X_n) . Thus the term ‘‘A’’ is zero under the null. Under the alternative, the covariance between $E(D_{i,m}|T_m, X_m = x)$ and $E(D_{j,n}|T_n, X_n = x')$ is non-zero for some (x, x') by Assumption 6. \square

We conclude this section with examples of primitive conditions that imply Assumption 6. The first generalizes affiliated equilibrium selection in Section 2.2 to the case with heterogeneous observable covariates.

Example 1. (Affiliated Equilibrium Selection in Adjacent Games) Consider a game with $\mathcal{I} \equiv \{i, j\}$ where both players have the same sign of externality over each other under all states. That is, $\text{sign}(\Delta v_i(x)) = \text{sign}(\Delta v_j(x))$ for all x , where $\Delta v_i(x) \equiv v_i(1, x) - v_i(0, x)$. By conditioning the proof in Lemma 1 on any x with $\#\mathcal{T}(x) < \infty$, we can show that under Assumptions 1, 2 and 5,

$$\text{sign}[(t_i - t'_i)(t_j - t'_j)] = \text{sign}[\Delta v_i(x)] \text{ for all } t \neq t' \text{ in } \mathcal{T}(x). \quad (7)$$

As noted in Section 2.2, this allows us to define total orders over $\#\mathcal{T}(x)$ for each x . Suppose for some (x, x') with $\text{sign}(\Delta v_i(x)) = \text{sign}(\Delta v_i(x'))$ and $\#\mathcal{T}(x), \#\mathcal{T}(x') < \infty$:

[Monotone Equilibrium Selection (MES)] For some $x, x', E[h(T_n, X_n)|T_m = t, X_m = x, X_n = x']$ is increasing in t over $\mathcal{T}(x)$ for any h that is increasing in its first argument.

This condition generalises Assumption 4 by allowing for heterogeneous states. It holds, for example, if T_m and T_n are affiliated conditional on $(X_m, X_n) = (x, x')$ as defined in Milgrom and Weber (1982). As in Section 2.2, we maintain that private types does not affect equilibrium selection directly, i.e., ϵ_i is independent from T given X .

We show that the MES condition implies Assumption 6. Therefore, it also implies $C(D_{i,m}, D_{j,n}|X_m = x, X_n = x') \neq 0$ under the alternative. Let $E(D_i|T = t, X = x) \equiv \phi_i(t, x)$ and $E(D_j|T = t, X = x') \equiv \phi_j(t, x')$. The Law of Iterated Expectation implies that the covariance between $E(D_{i,m}|T_m, X_m)$ and $E(D_{j,n}|T_n, X_n)$ conditional on $(X_m, X_n) = (x, x')$ is:⁷

$$\int \left(\begin{array}{c} \{\phi_i(t, x) - E[\phi_i(T_m, x)|x, x']\} \times \\ \{E[\phi_j(T_n, x')|T_m = t, x, x'] - E[\phi_j(T_n, x')|x, x']\} \end{array} \right) dF_{T_m|X_m, X_n}(t|x, x'). \quad (8)$$

Here the expectation $E[\phi_i(T_m, x)|x, x']$ is with respect to T_m given $(X_m, X_n) = (x, x')$. First, consider the case where $\Delta v_i(x) > 0$ and $\Delta v_i(x') > 0$. By (7), we can define total orders over $\mathcal{T}(x)$ and $\mathcal{T}(x')$ as “ $t > t'$ if and only if $t_i > t'_i$ ”. Because ϵ_i is independent from T given X , $\phi_i(t, x) = \Pr\{\epsilon_i \geq t_i|X = x\}$ is decreasing in t over $\mathcal{T}(x)$. Besides, $\phi_j(t, x')$ is also decreasing in t over $\mathcal{T}(x')$, because (7) implies that $t_j > t'_j$ whenever $t_i > t'_i$. The MES condition then implies $E[\phi_j(T_n, x')|T_m = t, x, x']$ is decreasing in t . Hence (8) is the

⁷To derive this expression, consider a generic random vector (W, Y, Z) . Let $\mu_f(z) \equiv E[f(W, Z)|Z = z]$ and $\mu_g(z) \equiv E[g(Y, Z)|Z = z]$. Then $C(f(W, Z), g(Y, Z)|Z = z)$ equals

$$\begin{aligned} & E\{[f(W, z) - \mu_f(z)][g(Y, z) - \mu_g(z)]|Z = z\} = E[E\{[f(W, z) - \mu_f(z)][g(Y, z) - \mu_g(z)]|Y, Z = z\}|Z = z] \\ & = E\{[E\{f(W, z)|Y, Z = z\} - \mu_f(z)] \times [g(Y, z) - \mu_g(z)] | Z = z\}. \end{aligned}$$

The claim in the text holds with $W \equiv T_n, Y \equiv T_m, f(W, Z) \equiv E(D_{j,n}|T_n, X_n), g(Y, Z) \equiv E(D_{i,m}|T_m, X_m)$ and $Z \equiv (X_m, X_n)$.

covariance between two decreasing functions of T_m conditional on $(X_m, X_n) = (x, x')$, and is positive. (See Theorem 2 in Schmidt (2003).) A symmetric argument shows that (8) is negative when $\Delta v_i(x) < 0$ and $\Delta v_i(x') < 0$. Hence Assumption 6 holds for adjacent games with $X_m = x$ and $X_n = x'$. \square

In the second example, the adjacent games share the same equilibrium selection with probability one, and Assumption 6 is satisfied.

Example 2. (Same Equilibrium in Adjacent Games) Consider a game with three players or more ($\#\mathcal{I} \geq 3$). Assumption 6 holds if adjacent games are known to select the same equilibrium:

[Same Equilibrium Selection (SES)] $\Pr\{T_m = T_n | X_m = X_n\} = 1$ for any two adjacent games m and n .

The SES condition accommodates multiple equilibria across disjoint pairs or clusters of adjacent games in the data.⁸ Under this condition, we can define $\tilde{F}_{T|x}(\cdot)$ as the marginal distribution of equilibria across pairs of adjacent games with $(X_m, X_n) = (x, x)$. The conditional covariance between $D_{i,m}$ and $D_{i,n}$ is:⁹

$$\int \left(\begin{array}{l} \{\phi_i(t, x) - E[\phi_i(T_m, x) | X_m = x, X_n = x]\} \times \\ \{\phi_i(t, x) - E[\phi_i(T_n, x) | X_m = x, X_n = x]\} \end{array} \right) d\tilde{F}_{T|x}(t),$$

which is the variance of $\phi_i(T, x)$ with T drawn from $\tilde{F}_{T|x}$. Such a variance is strictly positive under the alternative because $\tilde{F}_{T|x}$ is non-degenerate. Hence Assumption 6 holds with $i = j$ and $(X_m, X_n) = (x, x)$. We can also replace $D_{i,n}$ with $D_{j,n}$ in this example. In this case, the covariance is also nonzero generically, because $\phi_i(T, x)$ and $\phi_j(T, x)$ are functions of the same variable. \square

Building on these ideas, one can implement a statistical test for the null hypothesis of single equilibrium in the DGP, using tools that already exist in the econometrics literature. Let $\{m, n\}$ label two adjacent games. With $D_{i,m}, D_{j,n}$ being binary, zero conditional

⁸It is worth mentioning that one can also use this condition to test multiple Markovian Perfect equilibria (MPE) in dynamic games with zero discount factors, where private types are correlated through unobserved heterogeneity that is serially independent. In this case, one can treat the “stage games” from each period in the same Markov process as static games within the same “cluster”, and the SES condition simply means players do not switch between MPEs within a process.

⁹Recall that agents labelled by the same index across games are not required to be the same exact individual. It is only maintained that these individuals with identical indexes share the same preference, and are drawn from the same component in the DGP.

covariance is equivalent to conditional mean-independence.¹⁰ The null hypothesis of a unique equilibrium in the data, formulated as a zero conditional covariance restriction above, is thus equivalent to:

$$H_0 : E(D_{i,m}|X_m, X_n, D_{j,n}) = E(D_{i,m}|X_m, X_n) \text{ a.e.} \quad (9)$$

This equality fails with positive probability under the alternative of multiple equilibria. Significance tests in such nonparametric models have been studied extensively in the literature, both for continuous and discrete covariates. See, for example, Fan and Li (1996), Racine (1997), Chen and Fan (1999), Delgado et al. (2001), Lavergne (2001), and more recently, in Racine et al. (2006).

2.4 Relation to literature

Aguirregabiria and Mira (2019) studied static games with incomplete information when players' private information are independent conditional on some unobserved heterogeneity known to all players but not measured in the sample. To identify the full model in its generality, they treated equilibrium selection as a component in the game-level unobserved heterogeneity. They proposed an eigendecomposition method to recover conditional choice probabilities (CCPs) given unobserved heterogeneity, and then used exclusion restrictions similar to Bajari et al. (2010) to identify players' ex post payoff functions from these CCPs.

In their Section 4.3, Aguirregabiria and Mira (2019) proposed a novel idea to test multiple equilibria by checking whether the payoff functions recovered from the component CCPs in the finite mixture leads to distinctive payoff functions. The main idea is that, if there are multiple equilibria, then some of the component CCPs recovered from the finite mixture would lead to the same payoff functions. To implement this idea, one needs to follow sequential steps: (1) recover component CCPs given unobserved heterogeneity and equilibrium selection using eigendecomposition, (2) estimate payoffs from component CCPs, and (3) comparing payoff estimates calculated from different CCPs. The first two steps involve nonparametric estimation, and the last step requires developing asymptotic

¹⁰Abstracting from covariates for simplicity and letting $p_{i,m} = E(D_{i,m})$ and $p_{j,n} = E(D_{j,n})$, this obtains since $C(D_{i,m}, D_{j,n}) = E[(D_{i,m} - p_{i,m})(D_{j,n} - p_{j,n})] = E[(E(D_{i,m}|D_{j,n}) - p_{i,m})(D_{j,n} - p_{j,n})]$ by the Law of Iterated Expectations. Further developing this expression we get that it is equal to $p_{j,n}[E(D_{i,m}|D_{j,n} = 1) - E(D_{i,m})](1 - p_{j,n}) + (1 - p_{j,n})[E(D_{i,m}|D_{j,n} = 0) - E(D_{i,m})](0 - p_{j,n}) = p_{j,n}(1 - p_{j,n})[E(D_{i,m}|D_{j,n} = 1) - E(D_{i,m}|D_{j,n} = 0)]$. If the covariance is zero, then $E(D_{i,m}|D_{j,n} = 1) - E(D_{i,m}|D_{j,n} = 0) = E(D_{i,m})$.

theory of a test statistic that properly accounts for the estimation error in the first two steps.

Unlike Aguirregabiria and Mira (2019), our goal in this paper is not to fully identify discrete Bayesian games with unobserved heterogeneity and multiple equilibria. Rather, our objective is to construct a robust test for multiple equilibria in settings where players' information are generally correlated. These include, but are not restricted to, the case considered in Aguirregabiria and Mira (2019) where players' private signals are independent conditional on game-level unobserved heterogeneity. Moreover, our method is easy to implement. It formulates the task of inferring multiple equilibria as a test for the relevance of discrete covariates in nonparametric expectation. Thus one can directly tap into existing methods in the literature mentioned above to define a test statistic using kernel regressions, and characterize its asymptotic properties.

2.5 Numerical examples

In our simulation exercises, we investigate how the covariance between choices varies across different designs of static Bayesian games with binary choices. We first report results from games with two players $\{i, j\}$. In all designs, the vector of observed states $X \in \mathbb{R}^2$ consists of a discrete X_1 , which is uniformly distributed over a discrete support $\{1, 2, 3, 4\}$, and a continuous X_2 , which is standard uniform over $[0, 1]$. We experiment with *three* specifications of ex post payoffs for player i from choosing action 1:

Specification 1. $\beta_1 X_1 + \beta_2 X_2 + \delta D_j + \epsilon_i$.

Specification 2. $\beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + \beta_4 X_2^2 + \delta D_j + \epsilon_i$.

Specification 3. $\beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + \beta_4 X_2^2 + \beta_5 \sqrt{X_1} + \delta D_j + \epsilon_i$.

Ex post payoff for player j in each design is specified in a similar way, with subscripts i and j swapped in D_j and ϵ_i . The slope coefficients are $\beta_1 = 1/4$, $\beta_2 = 1/5$, $\beta_3 = 1/10$, $\beta_4 = -1/5$, $\beta_5 = -1/10$ and $\delta = -2$. The pair of private information components follows a bivariate normal distribution with zero mean, a unit variance, and a correlation coefficient $\rho \neq 0$. For each specification of ex post payoffs, we experiment with two designs with $\rho = 0.5$ and 0.7 respectively. For each design, we solve for multiple pure-strategy BNE, characterized as paired thresholds conditional on X .

Given these parameter values and using the law of total covariances (conditioning on equilibrium selection), we can calculate the covariance between $D_{i,m}$ and $D_{j,n}$ from adjacent games conditional on state X . Table 1 reports the covariance between $D_{i,m}$

and $D_{j,n}$, averaged over the support of X , for various designs of the data-generating process. The signs of these covariances are consistent with the negative interaction effect $\delta < 0$. As the data-generating process moves farther way from the null hypothesis of single equilibrium ($\varphi = 0$ with φ denoting the probability for mixing between two equilibria), the magnitude of these covariances also increases.

Table 1. Covariance between $D_{i,m}$ and $D_{j,n}$ in 2-player games

φ	$\rho = 0.50$			$\rho = 0.70$		
	Spec#1	Spec#2	Spec#3	Spec#1	Spec#2	Spec#3
0.1	-0.0303	-0.0300	-0.0286	-0.0366	-0.0364	-0.0348
0.2	-0.0538	-0.0534	-0.0509	-0.0651	-0.0647	-0.0618
0.3	-0.0706	-0.0701	-0.0668	-0.0854	-0.0849	-0.0812
0.4	-0.0807	-0.0801	-0.0764	-0.0976	-0.0970	-0.0928
0.5	-0.0840	-0.0835	-0.0796	-0.1017	-0.1010	-0.0966

Next, we do a similar investigation for games with three players indexed by i, j, k . The ex post payoff for player 1 from choosing action i is specified as

$$X\beta_i + \delta_{ij}D_j + \delta_{ik}D_k + \epsilon_k;$$

and likewise for the other two players j and k with coefficients $(\beta_j, \delta_{ji}, \delta_{jk})$ and $(\beta_k, \delta_{ki}, \delta_{kj})$ respectively. The vector of private signals $(\epsilon_i, \epsilon_j, \epsilon_k)$ is tri-variate normal with zero mean and unit variance. The correlation coefficients are $\rho_{ij} = 0.75$ and $\rho_{ik} = \rho_{jk} = 0.8$. For simplicity, we let $\beta_i = \beta_j$, $\delta_{ki} = \delta_{kj} = -3.1$, $\delta_{ik} = \delta_{jk} = -3.25$ and $\delta_{ij} = \delta_{ji} = -0.95$ in all designs. Thus players i and j are ex ante identical in all designs. The observed states $X = (X_1, X_2)$ follows the same distribution as in two-player cases above. We also experiment with three specifications of the index $X\beta_i$ in ex post payoffs, with $\beta_i = \beta_j = (1.73, -0.97, 0.155, -0.16, -0.01)$ and $\beta_k = (1.91, -1.645, -0.295, 0.29, 0.75)$, where the five components correspond to $(\beta_1, \dots, \beta_5)$ in Specification 1-3 above. Similar to the case with two players, our simulation uses DGPs that mix between extreme equilibria (in terms of threshold for $D_k = 1$), with adjacent games sharing the same equilibrium selection.

Table 2. Analytic covariances in 3-player games

φ	$Cov(D_{k,m}, D_{k,n})$			$Cov(D_{k,m}, D_{j,n})$		
	Spec#1	Spec#2	Spec#3	Spec#1	Spec#2	Spec#3
0.1	0.0441	0.0529	0.0335	-0.0261	-0.0275	-0.0218
0.2	0.0785	0.0941	0.0596	-0.0464	-0.0489	-0.0388
0.3	0.1030	0.1235	0.0782	-0.0609	-0.0641	-0.0509
0.4	0.1177	0.1412	0.0894	-0.0696	-0.0733	-0.0582
0.5	0.1226	0.1470	0.0931	-0.0725	-0.0763	-0.0606

Similar to the two-player cases, the magnitude of these covariances, averaged over the state space, increases as the data-generating process moves farther from the null hypothesis (that is, as φ increases between 0 and 0.5). The sign of the covariance of the same-type players $D_{k,m}, D_{k,n}$ is positive, while that between different types of players is negative.¹¹

3 Test of Multiple Equilibria in Dynamic Games

Dynamic games with private information has been studied extensively in the literature. See Aguirregabiria and Mira (2007), Bajari et al. (2007), Pesendorfer and Schmidt-Dengler (2008) and Arcidiacono and Miller (2011) for example. As in the case with static games, presence of multiple Markov Perfect equilibria in the sample poses challenges to estimation and inference based on conditional choice probabilities.

We introduce new tests for multiple equilibria in the dynamic games with incomplete information, where players' information are correlated in any given period through a time-varying, serially correlated unobserved heterogeneity. The idea for the test is related to that in the static case in that it also amounts to investigating the correlation between decisions made in "adjacent" games. Specifically, in the dynamic setting, different periods play the role of adjacent games and we form adjacent pairs of actions by matching players across different periods but within the same dynamic game.

We also note that the ideas below can be used for static games if one assumes that the discount factor is zero and takes a 'cluster' to correspond to one such dynamic game (with zero discount factor).

¹¹Though we do not have analytical results for this pattern with three or more players, this is what one would expect to obtain in two-player games with strategic substitution (i.e., negative interaction parameters).

3.1 The model

Let \mathcal{I} denote a set of players. Each $i \in \mathcal{I}$ makes a sequence of discrete actions $D_{i,t} \in \mathcal{D}$ indexed by time periods $t = 1, 2, \dots, \infty$. In each period t , player i observes a vector of *common* states (X_t, ξ_t) and *private* shocks $\epsilon_{i,t} \equiv (\epsilon_{i,t}^d : d \in \mathcal{D})$. While X_t is reported in the sample, ξ_t is a time-varying discrete unobserved game heterogeneity (DUH) not recorded in the data. Let $D_t \equiv (D_{i,t} : i \in \mathcal{I})$ and $\epsilon_t \equiv (\epsilon_{i,t} : i \in \mathcal{I})$ for each t .

A player's payoff at time t is given by a real-valued function $\pi_i(D_t, X_t, \xi_t, \epsilon_{i,t})$. In each period t , players make choices simultaneously to maximize

$$E \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \pi_i(D_s, X_s, \xi_s, \epsilon_{i,s}) \mid x_t, \xi_t, \epsilon_{i,t} \right\},$$

where $\beta \in (0, 1)$ is a constant discount factor known to all players. We maintain that (X_t, ξ_t, ϵ_t) follows a controlled first-order Markov process with a time-homogeneous transition density h that satisfies the following conditional independence:

$$h(X_{t+1}, \xi_{t+1}, \epsilon_{t+1} \mid X_t, \xi_t, \epsilon_t, D_t) = g(\epsilon_{t+1} \mid X_{t+1}, \xi_{t+1}) f(X_{t+1}, \xi_{t+1} \mid X_t, \xi_t, D_t), \quad (10)$$

where the private signals are independent conditional on common states:

$$g(\epsilon_{t+1} \mid X_{t+1}, \xi_{t+1}) = \prod_{i \in \mathcal{I}} g_i(\epsilon_{i,t+1} \mid X_{t+1}, \xi_{t+1}),$$

with $g_i(\cdot \mid \cdot, \cdot)$ denoting a conditional marginal density of private signals. Note that this specification of the law of transition permits the contemporary private signals $\epsilon_{i,t}, \epsilon_{j,t}$ to be correlated through unobserved ξ_t when conditioning on X_t alone. In what follows, we suppress the time subscript t in $x_t, \xi_t, D_{i,t}, D_t$ and $\epsilon_{i,t}$ to simplify notation.

Let $\sigma_i(x, \xi, \epsilon_i) \rightarrow \mathcal{D}$ denote a pure Markovian strategy adopted by a player i ; and let $\sigma \equiv (\sigma_i : i \in \mathcal{I})$. Let $p_i^{(\sigma)}(d_{-i} \mid x, \xi)$ denote the probability for $D_{-i} \equiv (D_j : j \in \mathcal{I} \setminus \{i\}) = d_{-i}$ conditional on i 's information (x, ξ) when the strategy profile is σ ;¹² and let

$$f_i^{(\sigma)}(x', \xi' \mid x, \xi, d_i) \equiv \sum_{d_{-i}} p_i^{(\sigma)}(d_{-i} \mid x, \xi) f(x', \xi' \mid x, \xi, d_i, d_{-i})$$

denote the transition matrix given i 's information. Let $\tilde{V}_i^{(\sigma)}(x, \xi, \epsilon_i)$ denote payoff for player i if it behaves optimally now and onwards given other firms' strategies in σ . By the Bellman's principle of optimality,

$$\begin{aligned} & \tilde{V}_i^{(\sigma)}(x, \xi, \epsilon_i) \\ = & \max_{d_i \in \mathcal{D}} \left\{ \Pi_i^{(\sigma)}(d_i, x, \xi, \epsilon_i) + \beta \int \int \left[\int \tilde{V}_i^{(\sigma)}(x', \xi', \epsilon'_i) g_i(\epsilon'_i \mid x', \xi') d\epsilon'_i \right] f_i^{(\sigma)}(x', \xi' \mid x, \xi, d_i) d\xi' dx' \right\} \end{aligned}$$

¹²Note ϵ_i is not in i 's information set due to independence between ϵ_i and ϵ_{-i} conditional on (x, ξ) .

where $g_i(\varepsilon_i|x, \xi)$ denotes the conditional distribution of i 's private shocks and

$$\Pi_i^{(\sigma)}(d_i, x, \xi, \varepsilon_i) \equiv \sum_{d_{-i}} p_i^{(\sigma)}(d_{-i}|x, \xi) \pi_i(d_{-i}, d_i, x, \xi, \varepsilon_i)$$

is i 's expected payoff in time t given other players' strategies in σ . Define an integrated value function for i as:

$$V_i^{(\sigma)}(x, \xi) \equiv \int \tilde{V}_i^{(\sigma)}(x, \xi, \varepsilon_i) g_i(\varepsilon_i|x, \xi) d\varepsilon_i.$$

For a given strategy profile σ , the integrated value function is characterized as the unique solution to the following fixed point equation:¹³

$$\begin{aligned} & V_i^{(\sigma)}(x, \xi) \\ = & \int \max_{d_i \in \mathcal{D}} \left\{ \Pi_i^{(\sigma)}(d_i, x, \xi, \varepsilon_i) + \beta \int \int V_i^{(\sigma)}(x', \xi') f_i^{(\sigma)}(x', \xi'|x, \xi, d_i) d\xi' dx' \right\} g_i(\varepsilon_i|x, \xi) d\varepsilon_i, \quad \forall x \in \mathcal{X}. \end{aligned}$$

A *Markov Perfect Equilibrium* (MPE) is a profile of strategies σ^* such that for any i and (x, ε_i) ,

$$\sigma_i^*(x, \xi, \varepsilon_i) = \arg \max_{d_i \in \mathcal{D}} \left(\Pi_i^{(\sigma^*)}(d_i, x, \xi, \varepsilon_i) + \beta \int \int V_i^{(\sigma^*)}(x', \xi') f_i^{(\sigma^*)}(x', \xi'|x, \xi, d_i) d\xi' dx' \right)$$

for all $i \in \mathcal{I}$ and (x, ξ, ε_i) . In general, the model admits multiple MPE. Our goal is to test the null hypothesis that the data-generating process is rationalized by a single MPE.

3.2 Detecting multiple equilibria

Consider a sample that consists of a large number of independent dynamic games. For each game, the sample reports states and choices over a finite number of periods $t = 1, 2, \dots, T$. We maintain the following assumption about the data-generating process.

Assumption 7 *Within each game, the players follow a fixed profile of MPE strategies σ^* in all periods $t = 1, 2, \dots, T$.*

To fix ideas, consider a simplified model with no states X_t . An MPE in this case is a profile of functions $(\sigma_i : i \in \mathcal{I})$, with each σ_i mapping from $(\xi_t, \varepsilon_{i,t})$ to $D_{i,t}$. Let \mathcal{M} denote the discrete and finite set of MPE admitted by the model, and let $M \equiv \#\mathcal{M}$ denote its cardinality. The equilibrium selection mechanism $\varphi(\cdot)$ is a probability mass function with

¹³It can be shown by verifying the Blackwell sufficient conditions that, for any given i and σ , the right-hand side is a contraction mapping in the functional space for integrated value functions.

support \mathcal{M} . Let K denote the cardinality of the support of ξ_t , and let $\lambda(\cdot)$ denote the probability mass of ξ_t .¹⁴

Let's partition the set of players \mathcal{I} into $\mathcal{I}_A \cup \mathcal{I}_B$, and define

$$D_t \equiv (D_{A,t}, D_{B,t}) \text{ with } D_{A,t} \equiv (D_{i,t})_{i \in \mathcal{I}_A} \text{ and } D_{B,t} \equiv (D_{j,t})_{j \in \mathcal{I}_B}.$$

Let $\delta_A \equiv (\#\mathcal{D})^{\#\mathcal{I}_A}$ and $\delta_B \equiv (\#\mathcal{D})^{\#\mathcal{I}_B}$ denote the cardinality of the support of $D_{A,t}$ and $D_{B,t}$

In what follows, let \sum_m and \sum_ξ denote the summation of m and ξ over their respective support. Using the law of total probability, decompose the joint probability mass of D_t as

$$\begin{aligned} P(D_t) &= \sum_m \varphi(m) \left[\sum_\xi \lambda(\xi) P_m(D_{A,t}, D_{B,t} | \xi) \right] \\ &= \sum_{m,\xi} \varphi(m) \lambda(\xi) P_m(D_{A,t} | \xi) P_m(D_{B,t} | \xi), \end{aligned} \quad (11)$$

where $P_m(\cdot | \xi)$ denotes the conditional probability mass implied by a single MPE $m \in \mathcal{M}$ conditional on ξ .

For convenience, let $\{\omega_j\}_{j=1,\dots,J}$ denote elements of the joint support of the discrete vector (m, ξ_t) . By construction, $J \equiv MK$. Let $\theta_j \equiv \Pr\{(m, \xi_t) = \omega_j\}$. In matrix notation, the joint probability mass of $(D_{A,t}, D_{B,t})$ is summarized by

$$\mathbf{P} \equiv \underbrace{\begin{pmatrix} q_{A,1} & \cdots & q_{A,J} \end{pmatrix}}_{Q_A} \underbrace{\begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \theta_J \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} q'_{B,1} \\ \vdots \\ q'_{B,J} \end{pmatrix}}_{Q'_B} = \sum_j \theta_j q_{A,j} q'_{B,j}, \quad (12)$$

where $q_{A,j}$ is a δ_A -by-1 column vector $\{P_m(D_{A,t} = a | \xi_t) : a \in \mathcal{D}_A\}$ with $(m, \xi_t) = \omega_j$, and $q_{B,j}$ is a δ_B -by-1 column vector $\{P_m(D_{B,t} = b | \xi_t) : b \in \mathcal{D}_B\}$ with $(m, \xi_t) = \omega_j$.

Assumption 8 $\min\{\delta_A, \delta_B\} > MK$; and both Q_A and Q_B have full rank.

The rank condition requires that the vectors of conditional choice probabilities in \mathbf{P} are not linearly dependent. Such conditions are common in structural models with discrete unobserved heterogeneity. For example, Assumption 2.1 and 2.2 in Hu (2008) uses such rank conditions to identify general nonlinear models with misclassification errors. Assumption 2 in Hu and Shum (2012) use such conditions to identify dynamic models with unobserved state variables. In both cases, the rank conditions are introduced to

¹⁴In general $\lambda(\cdot)$ need not be time-homogeneous. We suppress such dependence to simplify notation.

ensure linear independence between component distributions in a mixture model. In our case, this rank conditions essentially rule out pathologies where the choices probabilities conditional on unobserved states are linear combinations of each other. In Section 3.6, we provide numerical examples that satisfy these rank conditions in MPE. Our calculation also verifies that Assumption 8 hold generically in the sense that continuous variation in the model parameters almost surely implied equilibrium choice probabilities that satisfy the rank conditions.

This condition is easier to hold when the number of players or the number of choices are large relative to the cardinality of the support of unobserved heterogeneity and equilibria. For example, with $M = 2$, $K = 4$, and players faces 3 choices. Then we'll need two players in sets A and B each. In the next subsections, we examine situations when the set of players and/or alternatives is not sufficiently large.

Under Assumption 8, \mathbf{P} is a finite mixture with each component being a rank-one matrix $q_{A,j}q'_{B,j}$ and mixing weights given by w_j . Assumption 8 also implies that it is not possible to decompose the δ_A -by- δ_B matrix \mathbf{P} into other observationally equivalent forms of finite mixtures with fewer components.¹⁵ It then follows that MK can be generically identified as the rank of \mathbf{P} under maintained conditions.

Next, construct a $\#(\mathcal{I})$ -vector of actions across two periods

$$\tilde{D}_t \equiv (D_{A,t-1}, D_{B,t+1}).$$

By construction, the support of \tilde{D}_t and D_t are both $\mathcal{D}^{\#\mathcal{I}}$ with cardinality $\delta_A\delta_B = (\#\mathcal{D})^{\#\mathcal{I}}$. With slight abuse of notation, let $\lambda_m(\xi, \xi') \equiv \Pr\{\xi_t = \xi, \xi_{t+1} = \xi' | m\}$ denote the joint probability mass for time-varying and serially correlated DUH ξ_t and ξ_{t+1} implied in a single MPE indexed by $m \in \mathcal{M}$; and let $P_m(\cdot | \xi, \xi')$ denote the CCPs given $\xi_t = \xi$ and $\xi_{t+1} = \xi'$ implied in that single MPE. Similarly, let $\{\tilde{\omega}_j\}_{j=1, \dots, J'}$ denote elements of the joint support of the discrete vector (m, ξ_t, ξ_{t+1}) . By construction $J' \equiv MK^2$. The joint probability mass of (\tilde{D}_t, D_t) is

$$P(\tilde{D}_t, D_t) = \sum_{m, \xi, \xi'} \varphi(m) \lambda_m(\xi, \xi') P_m(\tilde{D}_t, D_t | \xi, \xi'),$$

where

$$\begin{aligned} P_m(\tilde{D}_t, D_t | \xi, \xi') &= P_m(D_{B,t+1}, D_{A,t-1}, D_t | \xi, \xi') \\ &= P_m(D_{B,t+1} | D_t, D_{A,t-1}, \xi, \xi') P_m(D_t | D_{A,t-1}, \xi, \xi') P_m(D_{A,t-1} | \xi, \xi') \\ &= P_m(D_{B,t+1} | \xi') P_m(D_t | \xi', \xi) P_m(D_{A,t-1} | \xi). \end{aligned} \tag{13}$$

¹⁵To see this, suppose one can write an alternative decomposition $\mathbf{P} = \hat{Q}_1 \hat{\Lambda} \hat{Q}'_2$, where $\hat{\Lambda}$ is a diagonal matrix with dimension $\tilde{J} < MK$. This would imply the rank of \mathbf{P} is strictly less than MK . On the other hand, Assumption 8 and (12) imply that \mathbf{P} must have full rank $J = MK$. Contradiction.

The last equality above uses several implications of the assumptions maintained above: (i) $D_{i,t}$ is determined by $(\xi_t, \epsilon_{i,t})$ in each single MPE; (ii) $\epsilon_{i,t}$ is independent of past histories of $(\epsilon_s, \xi_s)_{s \leq t-1}$ once conditional on ξ_t , and (iii) ξ_{t+1} is independent of past $(\xi_s, \epsilon_s)_{s \leq t-1}$ once conditional on ξ_t .

Let $\tilde{\mathbf{P}}$ denote a $\delta_A \delta_B$ -by- $\delta_A \delta_B$ matrix that summarizes the joint probability mass of (\tilde{D}_t, D_t) , with rows indexing the realization of \tilde{D}_t and columns indexing the realization of D_t . Let $\tilde{\theta}_j \equiv \Pr\{(m, \xi_t, \xi_{t+1}) = \tilde{\omega}_j\}$. In matrix notation,

$$\tilde{\mathbf{P}} \equiv \underbrace{\begin{pmatrix} \tilde{q}_1 & \cdots & \tilde{q}_{J'} \end{pmatrix}}_{\tilde{Q}} \underbrace{\begin{pmatrix} \tilde{\theta}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{\theta}_{J'} \end{pmatrix}}_{\tilde{A}} \underbrace{\begin{pmatrix} q'_1 \\ \vdots \\ q'_{J'} \end{pmatrix}}_{Q'} = \sum_j \tilde{\theta}_j \tilde{q}_j q'_j, \quad (14)$$

where \tilde{q}_j is a $\delta_A \delta_B$ -by-1 column vector

$$\{P_m(D_{A,t-1} = d_A | \xi_t = \xi) P_m(D_{B,t+1} = d_B | \xi_{t+1} = \xi') : (d_A, d_B) \in \mathcal{D}^{\#I}\}$$

with $(m, \xi, \xi') = \tilde{\omega}_j$; and q_j is a $\delta_A \delta_B$ -by-1 column vector

$$\{P_m(D_t = d | \xi_t = \xi, \xi_{t+1} = \xi') : d \in \mathcal{D}^{\#I}\}$$

with $(m, \xi, \xi') = \tilde{\omega}_j$.

Assumption 9 *Both \tilde{Q} and Q have full rank MK^2 .*

Similar to Assumption 8, this is a restriction on the choice probabilities conditional on DUH ξ and equilibrium index m . Also note that by construction, $\min\{\delta_A, \delta_B\} > MK$ implies that $\delta_A \delta_B > MK^2$. It follows from Kasahara and Shimotsu (2016) that MK^2 is generically identified as the rank of the $\delta_A \delta_B$ -by- $\delta_A \delta_B$ square matrix $\tilde{\mathbf{P}}$ under maintained conditions. Consequently, both M and K are identified from knowledge of $MK = \text{rank}(\mathbf{P})$ and $MK^2 = \text{rank}(\tilde{\mathbf{P}})$. The cardinality of equilibria in the DGP is revealed as $[\text{rank}(\mathbf{P})]^2 / \text{rank}(\tilde{\mathbf{P}})$.

Proposition 3 *Suppose Assumptions 7, 8 and 9 hold in the dynamic games with private information above. Then the number of MPE in the data-generating process is $[\text{rank}(\mathbf{P})]^2 / \text{rank}(\tilde{\mathbf{P}})$.*

For this method to work, it is essential that we choose to partition the history and focus on the joint probability mass of \tilde{D}_t and D_t in $\tilde{\mathbf{P}}$. If we were to partition the history

differently by defining $\tilde{\mathbf{P}}$ as the joint probability mass of D_t and D_{t+1} , then its finite mixture representation would only consist of MK components.

The basic idea in this subsection can be extended to accommodate the transition of observable states in addition to DUH. To do so, condition the identification method on realization of X_{t+1} and X_t . The mixing weights in \mathbf{P} and $\tilde{\mathbf{P}}$ become $\Pr\{\xi_t = \xi | X_t = x, m\}$ and $\Pr\{\xi_{t+1} = \xi', \xi_t = \xi | X_{t+1} = x', X_t = x, m\}$ respectively. The derivation in (11) and (13) carries through after conditioning on “ $X_t = x$ ” and “ $X_t = x$ and $X_{t+1} = x'$ ” respectively.

We have thus formulated the test for multiple MPE in the data-generating process as a question of inferring the rank of a consistently estimable matrix. Kleibergen and Paap (2006) proposed a rank test which uses the singular value decomposition and has a pivotal limiting distribution under the null. Kasahara and Shimotsu (2014) used this test statistic to construct a consistent estimator for the number of components in a finite mixture model, following a sequential testing algorithm in Robin and Smith (2000). The estimator in Kasahara and Shimotsu (2014) can be used for our purpose of testing multiple MPE. The asymptotic properties of the rank test and the estimator are presented in Kleibergen and Paap (2006) and Kasahara and Shimotsu (2014). Both papers documented evidence of the test and estimator’s finite sample performance via various simulation exercises.

3.3 Extension: binary decisions with many players

The method in Section 3.2 requires the support of choice profiles be larger than the joint support of DUH and equilibria (that is, $(\#\mathcal{D})^{\#\mathcal{I}_A}, (\#\mathcal{D})^{\#\mathcal{I}_B} > MK$). If choices are binary $\mathcal{D} \equiv \{0, 1\}$, one can extend the logic to test multiple equilibria provided there are sufficiently many players in a game.

To fix ideas, let’s first consider a model with no X_t as before. Suppose the set of players in each dynamic game is partitioned into two types labelled by 1 and 2. Players with the same type are ex ante identical in that they share the same ex post preference and have idiosyncratic shocks drawn independently from the same distribution. Let n_1 and n_2 denote the number of type-1 and type-2 players so that $\#\mathcal{I} = n = n_1 + n_2$. A type-symmetric MPE is characterized by $\sigma_\tau^*(\xi, \varepsilon_i) \rightarrow \mathcal{D}$ for types $\tau = 1, 2$.

We can modify the method in Section 3.2 for this setting. Let $m_{\tau t}$ denote the number of type- τ players choosing action 1 in period t . Then we can construct a $(n_1 + 1)$ -by- $(n_2 + 1)$ matrix of the joint probability mass of (m_{1t}, m_{2t}) , denoted \mathbf{S} , with the rows and columns indexed by the possible values of m_{1t} and m_{2t} , and (i, j) -th element being probability for $m_{1t} = i + 1$ and $m_{2t} = j + 1$. Likewise we can construct a \tilde{n} -by- \tilde{n} matrix for the joint

probability mass of (m_{1t}, m_{2t}) and $(m_{1,t-1}, m_{2,t+1})$, denoted $\tilde{\mathbf{S}}$, where $\tilde{n} \equiv (n_1 + 1)(n_2 + 1)$. Applying the law of total probability and exploiting the conditional independence assumptions, we can decompose \mathbf{S} and $\tilde{\mathbf{S}}$ both into products of three matrices in forms similar to (12) and (14). If $n_1, n_2 > MK - 1$ and proper rank conditions hold, then the ranks of \mathbf{S} and $\tilde{\mathbf{S}}$ are equal to the dimensions of diagonal matrices in the products, or MK and MK^2 respectively. Thus the cardinality of equilibria in the DGP is identified.

3.4 Extension: binary decisions with two players

The remaining challenge is to test for multiple MPE when there are only two players and binary decisions $\#\mathcal{D} = 2$ and $\#\mathcal{I} = 2$. In this case, we need to use five consecutive time periods. Let W_t denote a discretization of (D_t, X_t) defined by partitioning the support of X_t into intervals.

Consider the following joint probability mass function for a fixed realization of the second-period choices and states \bar{w}_2 :

$$\begin{aligned} P(w_3, \bar{w}_2, w_1) &= \sum_{m, \xi_2} \varphi(m) \lambda(\xi_2) P_m(w_3, \bar{w}_2, w_1 | \xi_2) \\ &= \sum_{m, \xi_2} \varphi(m) \lambda(\xi_2) \frac{P_m(w_3 | \bar{w}_2, \xi_2, w_1) P_m(\bar{w}_2, \xi_2, w_1)}{\lambda(\xi_2)} \\ &= \sum_{m, \xi_2} \varphi(m) P_m(w_3 | \bar{w}_2, \xi_2) P_m(\bar{w}_2, \xi_2, w_1), \end{aligned}$$

where $\varphi(\cdot)$ is the equilibrium selection; $\lambda(\cdot)$ is the marginal probability mass function of ξ_2 under MPE $m \in \mathcal{M}$. The last equality above uses conditional independence conditions maintained on the state and DUH transitions. Let $\delta_{(1)}$ and $\delta_{(3)}$ denote the cardinality of the marginal support of W_1 and W_3 respectively.

Let $\mathbf{P}_{w_3, \bar{w}_2, w_1}$ denote a $\delta_{(3)}$ -by- $\delta_{(1)}$ matrix that summarizes the joint probability mass function of (W_3, W_2, W_1) when the realization of W_2 is fixed at \bar{w}_2 . Let the rows in $\mathbf{P}_{w_3, \bar{w}_2, w_1}$ be indexed by elements on the support of W_3 and the columns by elements on the support of W_1 . In matrix notation,

$$\mathbf{P}_{w_3, \bar{w}_2, w_1} \equiv (\Phi_{\bar{w}_2}) \Pi (\Psi_{\bar{w}_2})'$$

where Π is a J -by- J diagonal matrix with non-zero diagonal entries being $\{\varphi(m) : m \in \mathcal{M}\}$ (each $\varphi(m)$ is repeated K times on the diagonal); $\Phi_{\bar{w}_2}$ is a $\delta_{(3)}$ -by- J matrix with its (i, j) -th component being $P_m(w_3 | \bar{w}_2, \xi_2)$ where $(m, \xi_2) = \omega_j$ and w_3 is the i -th element in support of W_3 ; and likewise $\Psi_{\bar{w}_2}$ is a $\delta_{(1)}$ -by- J matrix with its (i, j) -th component being

$P_m(\bar{w}_2, \xi_2, w_1)$ where $(m, \xi_2) = \omega_j$ and w_1 is the i -th element in support of W_1 . With $\delta_{(3)}, \delta_{(1)} > J = MK$ and $\Phi_{\bar{w}_2}, \Psi_{\bar{w}_2}$ both having full rank, we identify $J = MK$ as the rank of $\mathbf{P}_{w_3, \bar{w}_2, w_1}$.

Next, consider the following joint probability mass function for a fixed pair of realization in the second and fourth period (\bar{w}_4, \bar{w}_2) :

$$P(w_5, \bar{w}_4, w_3, \bar{w}_2, w_1) = \sum_{m, \xi_4, \xi_2} \varphi(m) \lambda_m(\xi_4, \xi_2) P_m(w_5, \bar{w}_4, w_3, \bar{w}_2, w_1 | \xi_4, \xi_2),$$

where $\lambda_m(\cdot, \cdot)$ is the joint distribution of (ξ_2, ξ_4) under MPE $m \in \mathcal{M}$; and for each MPE m and conditioning on (ξ_4, ξ_2) , the joint probability mass of the history is

$$\begin{aligned} & P_m(w_5, \bar{w}_4, w_3, \bar{w}_2, w_1 | \xi_4, \xi_2) \\ &= P_m(w_5 | \bar{w}_4, \xi_4, w_3, \bar{w}_2, \xi_2, w_1) P_m(\bar{w}_4, w_3, \bar{w}_2 | w_1, \xi_4, \xi_2) P_m(w_1 | \xi_4, \xi_2). \end{aligned} \quad (15)$$

By our maintained assumptions,

$$P_m(w_5 | \bar{w}_4, \xi_4, w_3, \bar{w}_2, \xi_2, w_1) = P_m(w_5 | \bar{w}_4, \xi_4),$$

$$\begin{aligned} P_m(\bar{w}_4, w_3, \bar{w}_2 | w_1, \xi_4, \xi_2) &= \frac{P_m(\bar{w}_4, \xi_4 | w_3, \bar{w}_2, \xi_2, w_1) P_m(w_3 | \bar{w}_2, \xi_2, w_1) P_m(\bar{w}_2, \xi_2 | w_1)}{P_m(\xi_4, \xi_2 | w_1)} \\ &= \frac{P_m(\bar{w}_4, \xi_4 | w_3, \bar{w}_2, \xi_2) P_m(w_3 | \bar{w}_2, \xi_2) P_m(\bar{w}_2, \xi_2 | w_1)}{P_m(\xi_4, \xi_2 | w_1)}, \end{aligned}$$

and

$$P_m(w_1 | \xi_4, \xi_2) = \frac{P_m(\xi_4, \xi_2 | w_1) P(w_1)}{\lambda_m(\xi_2, \xi_4)}.$$

Substituting this into (15), we get

$$\begin{aligned} & P(w_5, \bar{w}_4, w_3, \bar{w}_2, w_1) \\ &= \sum_{m, \xi_4, \xi_2} \varphi(m) P_m(w_5 | \bar{w}_4, \xi_4) P_m(\bar{w}_4, \xi_4, w_3 | \bar{w}_2, \xi_2) P_m(\bar{w}_2, \xi_2, w_1). \end{aligned}$$

Let $\delta_{(5,1)}$ denote the cardinality of the support of (W_5, W_1) .

Let $\tilde{\mathbf{P}}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$ denote a $\delta_{(5,1)}$ -by- $\delta_{(3)}$ matrix that summarizes the joint probability mass function of $(W_5, W_4, W_3, W_2, W_1)$ when (W_4, W_2) are fixed at (\bar{w}_4, \bar{w}_2) . Let the rows in $\tilde{\mathbf{P}}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$ be indexed by elements on the joint support of (W_5, W_1) and the columns by elements on the marginal support of W_3 . In matrix notation,

$$\tilde{\mathbf{P}}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1} \equiv \left(\tilde{\Phi}_{\bar{w}_4, \bar{w}_2} \right) \tilde{\Pi} \left(\tilde{\Psi}_{\bar{w}_4, \bar{w}_2} \right)'$$

where $\tilde{\Pi}$ is a J' -by- J' diagonal matrix with non-zero diagonal entries being $\{\varphi(m) : m \in \mathcal{M}\}$ (each $\varphi(m)$ is repeated K^2 times on the diagonal); $\tilde{\Phi}_{\bar{w}_4, \bar{w}_2}$ is a $\delta_{(5,1)}$ -by- J' matrix with its (i, j) -th component being $P_m(w_5 | \bar{w}_4, \xi_4) P_m(\bar{w}_2, \xi_2, w_1)$ where $(m, \xi_2, \xi_4) = \tilde{w}_j$ and (w_5, w_1) being the i -th element in support of (W_5, W_1) , and likewise $\tilde{\Psi}_{\bar{w}_4, \bar{w}_2}$ is a $\delta_{(3)}$ -by- J' matrix with its (i, j) -th component being $P_m(\bar{w}_4, \xi_4, w_3 | \bar{w}_2, \xi_2)$ where $(m, \xi_2, \xi_4) = \tilde{w}_j$ and w_3 is the i -th element in support of W_3 . With $\delta_{(5,1)}, \delta_{(3)} > J'$ and $\left(\tilde{\Phi}_{\bar{w}_4, \bar{w}_2}\right)$ and $\left(\tilde{\Psi}_{\bar{w}_4, \bar{w}_2}\right)$ being full rank, we identify $J' = MK^2$ as the rank of $\tilde{\mathbf{P}}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$.

For this method to work, it is crucial that we partition the history into (W_1, W_5) and W_3 while indexing the rows and columns $\tilde{\mathbf{P}}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$. To see how, suppose we had defined $\tilde{\mathbf{P}}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$ differently as a $\delta_{(5,3)}$ -by- $\delta_{(1)}$ matrix with rows indexing (w_5, w_3) and columns index w_1 . Then it would have rank $J = MK$ as opposed to $J' = MK^2$.

3.5 Related literature

Otsu et al. (2016) tested the null hypothesis that choices observed in a finite number of games/markets $s = 1, 2, \dots, S < \infty$ are generated by the same MPE. With the number of time periods T in each game approaching infinity, they proposed an asymptotic test based on the estimation and comparison of conditional choice probabilities across the finite number of games. In comparison, we focus on a different scenario where a sample is large in the number of independent dynamic games/markets (that is, asymptotics is defined as $S \rightarrow \infty$) while the number of time periods T observed in each game is small and finite (say $T \leq 3$).

Hu and Shum (2012) used an eigendecomposition method to fully identify general dynamic models with unobserved heterogeneity. Luo et al. (2019) extended this method to deal with multiple equilibria in dynamic games by treating equilibrium selection as a component in discrete unobserved heterogeneity on the game level. The key in their paper was to define a set of conditioning events so that the conditional distribution of observed outcomes admitted a finite-mixture (or eigenvalue decomposition) representation with MK components, with K being cardinality of unobserved heterogeneity and M being cardinality of equilibria in the DGP. Luo et al. (2019) then proposed to test multiple equilibria by checking whether player payoffs identified from the component CCPs in the finite mixture (which themselves need to be recovered from outcome distributions via eigenvalue decomposition) are distinctively different. This idea was built on the same insight that Aguirregabiria and Mira (2019) introduced for static games: if there are multiple equilibria in the DGP, then player payoffs backed out from some of the component

CCPs in the mixture would be identical.¹⁶

In contrast, the insight in our new approach is different. We separate the task of detecting multiple equilibria from identifying the full model. By partitioning and pairing outcome histories with different lengths, we can represent their joint distribution as finite mixtures whose cardinality of components *vary with the length of history in addition to M and K* . This is a new source of variation that has not been exploited in the literature to the best of our knowledge. More importantly, it allows us to conduct simple inference for equilibrium cardinality through standard rank test such as Kleibergen and Paap (2006), thus dispensing with the sequential steps discussed in Luo et al. (2019).¹⁷

3.6 Numerical examples

Example 1. (Multiple players with binary choices.) Consider a dynamic game of imperfect information between n players. Each player i belongs to one of two types (labeled as 1 or 2) and makes a binary decision $d_{i,t}$ in each period t . Same-type players have identical ex post payoffs and have idiosyncratic shocks drawn independently from the same distribution. Let n_1 and n_2 denote the number of type-1 and type-2 players respectively.

A time-varying state variable $\xi_t \in \{0, 1\}$ is commonly observed by all players in each period t . In addition, each player observes a vector of private signals in each period: $\varepsilon_{i,t} \equiv (\varepsilon_{i,t,s})_{s=0,1} \in \mathbb{R}^2$. Let $\varepsilon_t \equiv (\varepsilon_{i,t})_{i \leq n}$. The state-and-signal transition is such that $\Pr\{\xi_{t+1}, \varepsilon_{t+1} | \xi_t, \varepsilon_t, d_t\} = \Pr\{\varepsilon_{t+1}\} \Pr\{\xi_{t+1} | \xi_t, d_t\}$, where $d_t \equiv (d_{i,t})_{i \leq n}$ and $\varepsilon_{i,t}$ are i.i.d. across $i \leq n$ and independent from ξ_t . For each i and t , $\varepsilon_{i,t}$ is bivariate normal with zero mean and an identity covariance matrix. The law of transition for ξ_t and the ex post payoffs depend on d_t only through the number of type-1 and type-2 players who chooses 1 in period t , denoted $m_t \equiv (m_{1t}, m_{2t})$. The transition $\Pr\{\xi_{t+1} = 1 | \xi_t = \xi, m_t\} = p_{\xi,1}$ if $m_{1t}/n_1 \geq m_{2t}/n_2$; and $p_{\xi,0}$ otherwise. The ex post payoff of a type- τ player i choosing $d_{i,t} = d_i \in \{0, 1\}$ is given by

$$\pi_\tau(d_i, d_{-i,t}, \xi_t) + \varepsilon_{i,t,d_i},$$

¹⁶Luo et al. (2019) described the test in sequential steps: (1) recover component CCPs given DUH and equilibrium selection using eigendecomposition, (2) estimate payoffs from component CCPs, and (3) comparing payoff estimates calculated from different CCPs. The costs for nonparametrically implementation of these steps are high. See our discussion about analogous costs for static games in the second paragraph of Section 2.4. Luo et al. (2019) did not provide any test statistic or asymptotic theory for implementing this test in their paper.

¹⁷In our method, we only need to recover the ranks of the joint distribution of outcome histories. As a result, we do not need to actually perform an eigendecomposition. This means we do not need the finite mixture to take a form of QAQ^{-1} . Expressing the finite mixture in a much less restrictive form $QA\tilde{Q}$ is totally fine for our purpose.

where $\pi_\tau(d_i, d_{-i,t}, \xi) = c_\tau(d_i, \xi) + \delta_{\tau 1}(d_i) (m_{1t}/n_1) + \delta_{\tau 2}(d_i) (m_{2t}/n_2)$ for type $\tau = 1, 2$.

Consider a game with ten players with $n_1 = n_2 = 5$ and a discount factor $\beta = 0.75$. The other game parameters are specified as follows:

$c_{\tau=1}(d_i, \xi)$		$c_{\tau=2}(d_i, \xi)$		$\delta_{\tau=1, \tilde{\tau}}(d_i)$		$\delta_{\tau=2, \tilde{\tau}}(d_i)$			
$d_i = 0$	$d_i = 1$	$d_i = 0$	$d_i = 1$	$d_i = 0$	$d_i = 1$	$d_i = 0$	$d_i = 1$		
$\xi = 0$	2.6	-0.8	1.4	-1.8	$\tilde{\tau} = 1$	2.5	4	2.5	6.5
$\xi = 1$	1.4	-1.8	2.6	-0.8	$\tilde{\tau} = 2$	2.5	6.5	2.5	4

and

$$p_{0,0} = 0.375; p_{0,1} = 0.725; p_{1,0} = 0.675; p_{1,1} = 0.125.$$

These parameter values reflect certain economic interpretation: First, for different types of players, the marginal impacts of unobserved states ξ on $c_\tau(d_i, \xi)$ move in different directions. Second, when $d_i = 1$, $\delta_{\tau, \tilde{\tau}}$ reflects higher complementarity from other types of players choosing the same action. Third, the transition of states depends on the choice profiles summarized by $(m_{1t}, m_{2,t})$, which has a substantial impact on the likelihood of transition to higher states. We also allow the sign of such impact depend on the current state as well.

We solve for conditional choice probabilities in type-symmetric MPE. The game admits two MPE's that lead to different vectors of CCPs:

Conditional Choice Probabilities (CCPs) in MPE

	Eq#1		Eq#2	
	$\tau = 1$	$\tau = 2$	$\tau = 1$	$\tau = 2$
$\xi = 0$	0.473	0.533	0.041	0.022
$\xi = 1$	0.375	0.309	0.304	0.334

To illustrate the method in Section 3.2, suppose both equilibria are selected with positive probability in the data generating process. With A being the set of type-1 players and B being the set of type-2 players, the joint probability mass \mathbf{P} in (12) is a 2^5 -by- 2^5 with rank $MK = 4$. Furthermore, the other joint probability mass $\tilde{\mathbf{P}}$ in (14) is 2^{10} -by- 2^{10} matrix with rank $MK^2 = 8$. It is verified that this holds because the rank conditions in Assumptions 8 and 9 are satisfied.

It is worth mentioning that this idea for testing multiple MPE remains valid when applied to lower-dimension matrices that aggregate over the rows and columns in \mathbf{P} and $\tilde{\mathbf{P}}$. For instance, one may well replace A and B by arbitrarily picked subsets of type-1 or type-2 players, say, three from each type. Then one can construct two joint prob mass

matrices similar to \mathbf{P} and $\tilde{\mathbf{P}}$ for these players only. These matrices will be 2^3 -by- 2^3 and 2^6 -by- 2^6 in dimensions, and have ranks 4 and 8 respectively.

We also illustrate how to implement the alternative method in Section 3.3. To do so, construct a 6-by-6 matrix of joint probability masses \mathbf{S} for (m_{1t}, m_{2t}) , i.e., the number of type-1 and type-2 players choosing 1. Likewise construct a 6^2 -by- 6^2 matrix of joint probability masses $\tilde{\mathbf{S}}$ for $m_t \equiv (m_{1t}, m_{2t})$ and $\tilde{m}_t \equiv (m_{1,t-1}, m_{2,t+1})$. Both matrices admit a diagonalized forms similar to (12) and (14), with the diagonal matrices in the middle of the decompositions being the same as Λ and $\tilde{\Lambda}$ in (12) and (14). It is verified that the outer matrices in both decompositions, a.k.a. component mass functions conditional on equilibrium selection and unobserved states, satisfy the appropriate rank conditions. Therefore the probability mass matrices \mathbf{S} and $\tilde{\mathbf{S}}$ have ranks $MK = 4$ and $MK^2 = 8$ respectively. As noted, one can also perform the test by replacing $\tilde{\mathbf{S}}$ with lower-dimension transformation, such as \tilde{n} -by- \tilde{n} summary matrix (with $8 < \tilde{n} < 36$) formed by some linear combinations of rows and columns in $\tilde{\mathbf{S}}$.

Given the equilibrium choice probabilities reported in Example 1A above, the numeric values of the probability mass matrices are

$$\mathbf{S} = \begin{pmatrix} 0.1588 & 0.0285 & 0.0152 & 0.0113 & 0.0053 & 0.0011 \\ 0.0449 & 0.0390 & 0.0510 & 0.0443 & 0.0221 & 0.0047 \\ 0.0176 & 0.0485 & 0.0770 & 0.0728 & 0.0381 & 0.0084 \\ 0.0098 & 0.0351 & 0.0616 & 0.0618 & 0.0333 & 0.0074 \\ 0.0034 & 0.0135 & 0.0256 & 0.0268 & 0.0147 & 0.0033 \\ 0.0005 & 0.0022 & 0.0044 & 0.0047 & 0.0026 & 0.0006 \end{pmatrix};$$

and

$$\tilde{\mathbf{S}}^* = \begin{pmatrix} 0.0615 & 0.0249 & 0.0154 & 0.0339 & 0.0132 & 0.0105 & 0.0071 & 0.0011 & 0.0008 \\ 0.0272 & 0.0153 & 0.0177 & 0.0300 & 0.0108 & 0.0127 & 0.0072 & 0.0013 & 0.0010 \\ 0.0249 & 0.0134 & 0.0194 & 0.0332 & 0.0128 & 0.0179 & 0.0110 & 0.0020 & 0.0017 \\ 0.0407 & 0.0289 & 0.0443 & 0.0694 & 0.0250 & 0.0335 & 0.0188 & 0.0034 & 0.0027 \\ 0.0120 & 0.0116 & 0.0188 & 0.0273 & 0.0092 & 0.0114 & 0.0058 & 0.0010 & 0.0007 \\ 0.0147 & 0.0126 & 0.0203 & 0.0340 & 0.0132 & 0.0179 & 0.0111 & 0.0020 & 0.0016 \\ 0.0079 & 0.0067 & 0.0111 & 0.0162 & 0.0055 & 0.0074 & 0.0038 & 0.0007 & 0.0005 \\ 0.0011 & 0.0012 & 0.0019 & 0.0034 & 0.0014 & 0.0020 & 0.0013 & 0.0002 & 0.0002 \\ 0.0008 & 0.0007 & 0.0011 & 0.0020 & 0.0008 & 0.0012 & 0.0007 & 0.0001 & 0.0001 \end{pmatrix}$$

The rows and columns of \mathbf{S} correspond to the number of type-1 and type-2 players choosing 1 respectively (ordered from $m = 0, 1, \dots, 5$). The 9-by-9 matrix $\tilde{\mathbf{S}}^*$ is a coarsening of the

original 36-by-36 matrix $\tilde{\mathbf{S}}$. It is constructed by adding up every four adjacent rows and four adjacent columns in $\tilde{\mathbf{S}}$. The rank of \mathbf{S} equals $4 = MK$; the rank(s) of $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{S}}^*$ are both equal to $8 = MK^2$.

Example 2. (Two players with binary decisions.) Consider a dynamic game between two forward-looking players i and j who make dynamic, optimal binary decisions $d_{it}, d_{jt} \in \{0, 1\}$ in each period t . Players observe both states that evolve over time: $z_t \in \{0, 1\}$ and $\xi_t \in \{-1, 1\}$, with z_t reported in the data while ξ_t is not. At time t , player i observes a vector of private signals $\varepsilon_{it} \equiv (\varepsilon_{it1}, \varepsilon_{it0}) \in \mathbb{R}^2$. Let $\varepsilon_t \equiv (\varepsilon_{it}, \varepsilon_{jt})$, and $\varepsilon_{it}, \varepsilon_{jt}, z_t$ be independent conditional on ξ_t . Let $s_t \equiv (z_t, \xi_t)$ denote the vector of states, and its transition satisfies

$$\Pr(s_{t+1}, \varepsilon_{t+1} | s_t, \varepsilon_t, d_{i,t}, d_{j,t}) = \Pr(\varepsilon_{t+1} | s_{t+1}) \Pr(s_{t+1} | s_t, d_{i,t}, d_{j,t}),$$

Conditional on ξ_t , ε_{it} and ε_{jt} are both bivariate normal with mean $(\xi_t, 0)$ and identity covariance. The state transition is

$$\begin{aligned} \Pr\{z_{t+1} = 1, \xi_{t+1} | z_t, \xi_t, d_{i,t}, d_{j,t}\} &= E(z_{t+1} | z_t, \xi_t, d_{i,t}, d_{j,t}) \Pr(\xi_{t+1} | \xi_t); \\ E(z_{t+1} | z_t, \xi_t, d_{i,t}, d_{j,t}) &= \delta(z_t, \xi_t, d_{it} + d_{jt}). \end{aligned}$$

The evolution of ξ_t is serially correlated, but does not depend on observed state or choices d_{it}, d_{jt} . In this model, dynamics exist because d_{it}, d_{jt} affect the transition of observed states z_t . Ex post payoff of i from $d_i = \tau$ in time t is $\pi(\tau, d_j, z) + \varepsilon_{it\tau}$, where $\pi(d_{it}, d_{jt}, z_t) = c_z/2$ if $d_{it} = d_{jt}$; $\pi(d_i, d_j, z) = \alpha c_z$ if $d_{it} > d_{jt}$; and $\pi(d_{it}, d_{jt}, z_t) = (1 - \alpha)c_z$ otherwise.

The specification is chosen to mimic a stylized game of dynamic labor force participation decisions by a married couple i and j in a household. In this case, $z_t \in \{0, 1\}$ reflects the level of household savings at time t (with $z_t = 1$ denoting high savings); and ξ_t is a series of time-varying household shocks. Dynamics exist in because the transition of savings depends on both husband and wife's decision to participate in the labor force. On the other hand, the transition of shocks do not depend on the couple's decisions. In each period, c_z is the per-period income or consumption budget, where $c_1 > c_0$ reflect positive association with savings. The share $\alpha \in (0, 1)$ is the share claimed by household members, which depends on their labor participation decisions.

We solve this game with a discount factor $\beta = 0.7$, and parameter values specified as follows: $\alpha = 0.6$, $c_0 = 1.5$, $c_1 = 3$, $\Pr\{\xi_{t+1} = -1 | \xi_t = -1\} = 0.5$, $\Pr\{\xi_{t+1} = -1 | \xi_t = 1\} =$

0.3, and $E(z_{t+1}|z_t, \xi_t, d_{it}, d_{jt})$ specified as

	$d_{it} + d_{jt} = 0$		$d_{it} + d_{jt} = 1$		$d_{it} + d_{jt} = 2$	
	$\xi = -1$	$\xi = 1$	$\xi = -1$	$\xi = 1$	$\xi = -1$	$\xi = 1$
$z_t = 0$	0.0625	0.1875	0.4375	0.5625	0.6875	0.8125
$z_t = 1$	0.1250	0.25	0.5	0.625	0.75	0.8750

which, in the joint labor participation example above, suggests transition to higher states (savings) is more likely when both players participate.

This specification leads to two symmetric MPE in which the conditional choice probabilities for participation conditional on common states are:

	<i>MPE#1</i>		<i>MPE#2</i>	
	$\xi = -1$	$\xi = 1$	$\xi = -1$	$\xi = 1$
$z_t = 0$	0.272	0.786	0.310	0.826
$z_t = 1$	0.216	0.822	0.347	0.852

Plugging in these numbers in the definition of $\mathbf{P}_{w_3, \bar{w}_2, w_1}$ and $\tilde{\mathbf{P}}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$ in Section 3.4 verifies the rank conditions for all realization of \bar{w}_2, \bar{w}_4 .

Consider P_{w_3, \bar{w}_2, w_1} . The value of \bar{w}_2 is $\bar{z}_2 = 1$, $\bar{d}_{i2} = 1$ $\bar{d}_{j2} = 0$. The dimension of the matrix is 8-by-8, where the rows and columns correspond to a specific action and observed state in period 3 and period 1 respectively. See below the matrix, P_{w_3, \bar{w}_2, w_1} when the equilibrium selection probability is 0.4. Its rank is equal to $4 = MK$.

$[z_3, d_{i3}, d_{j3}]$	$[z_1, d_{i1}, d_{j1}]$	[0, 0, 0]	[0, 1, 0]	[0, 0, 1]	[0, 1, 1]	[1, 0, 0]	[1, 1, 0]	[1, 0, 1]	[1, 1, 1]
[0, 0, 0]		0.0027	0.0103	0.0106	0.0266	0.0038	0.0086	0.0106	0.0261
[0, 1, 0]		0.0019	0.0077	0.0079	0.0205	0.0027	0.0065	0.0077	0.0200
[0, 0, 1]		0.0020	0.0079	0.0082	0.0212	0.0028	0.0066	0.0081	0.0208
[0, 1, 1]		0.0047	0.0190	0.0195	0.0526	0.0066	0.0160	0.0192	0.0515
[1, 0, 0]		0.0027	0.0107	0.0110	0.0290	0.0038	0.0089	0.0110	0.0284
[1, 1, 0]		0.0020	0.0082	0.0084	0.0231	0.0028	0.0071	0.0081	0.0225
[1, 0, 1]		0.0025	0.0099	0.0102	0.0277	0.0035	0.0081	0.0101	0.0271
[1, 1, 1]		0.0065	0.0266	0.0274	0.0776	0.0090	0.0222	0.0268	0.0758

Next, consider $\tilde{P}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$. The value of \bar{w}_2 and \bar{w}_4 are $\bar{z}_2 = 1$, $\bar{d}_{i2} = 1$ $\bar{d}_{j2} = 0$ and $\bar{z}_4 = 1$, $\bar{d}_{i4} = 1$ $\bar{d}_{j4} = 0$ respectively. The original dimension of the matrix is 64-by-8, with the rows correspond to a specific action and observed state in period 1 and period 5 and the columns correspond to a specific action and observed state in period 3. As noted above, to reduce dimension, we collapsed $\tilde{P}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$ into a 8-by-8 matrix, by adding the rows that share the same d_{i5}, z_5 and d_{j1} . See below the collapsed matrix

$\tilde{P}_{w_5, \bar{w}_4, w_3, \bar{w}_2, w_1}$, when the equilibrium selection probability is 0.4, and hence. Its rank is equal to $8 = MK^2$.

$[z_3, d_{i3}, d_{j3}]$	[0, 0, 0]	[0, 1, 0]	[0, 0, 1]	[0, 1, 1]	[1, 0, 0]	[1, 1, 0]	[1, 0, 1]	[1, 1, 1]
$[d_{j1}, z_5, d_{i5}]$								
[0, 0, 0]	0.0006	0.0029	0.0030	0.0106	0.0012	0.0037	0.0041	0.0159
[0, 0, 1]	0.0009	0.0044	0.0045	0.0167	0.0017	0.0056	0.0062	0.0248
[0, 1, 0]	0.0007	0.0033	0.0034	0.0126	0.0013	0.0041	0.0047	0.0187
[0, 1, 1]	0.0012	0.0059	0.0061	0.0233	0.0022	0.0075	0.0082	0.0346
[1, 0, 0]	0.0018	0.0087	0.0089	0.0324	0.0036	0.0112	0.0128	0.0505
[1, 0, 1]	0.0026	0.0131	0.0134	0.0506	0.0052	0.0170	0.0191	0.0789
[1, 1, 0]	0.0020	0.0099	0.0102	0.0383	0.0039	0.0126	0.0146	0.0599
[1, 1, 1]	0.0034	0.0175	0.0180	0.0706	0.0066	0.0228	0.0255	0.1100

4 Concluding Remarks

This article studies testable implications of multiple equilibria in discrete games with incomplete information where players' private signals are allowed to be correlated. In static games, independence across games whose equilibrium selection is correlated can be used to overcome difficulties in testing for multiple equilibria. In dynamic games with serially correlated discrete unobserved heterogeneity, the distribution of an observed history (sequence) of choices and states is a finite mixture over equilibria and unobserved heterogeneity. Our testable implication exploits the fact that the number of mixture components is a *known* function of the horizon of the history (length of the sequence), as well as the cardinality of equilibria and unobserved heterogeneity support. In both static and dynamic cases, the testable implications are conducive to formal tests using existing statistical tools.

Further connections between these strategies might allow us to relax the independence across games used in the testable implications of static games if private types are correlated across games through DUH. It is possible to employ the ideas in the dynamic settings we considered, in which the DUH is serially correlated. Exploiting the two jointly might allow for even more powerful detection of multiplicity, but would also require new testing protocols for which further research is needed.

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