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Econometric Analysis of Games with Multiple Equilibria*

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Abstract

This article reviews the recent literature on the econometric analysis of games where multiple solutions are possible. Multiplicity does not necessarily preclude the estimation of a particular model (and in certain cases even improves its identification), but ignoring it can lead to misspecifications. The survey starts with a general characterization of structural models that highlights how multiplicity affects the classical paradigm. Because the information structure is an important guide to identification and estimation strategies, I discuss games of complete and incomplete information separately. Whereas many of the techniques discussed in the article can be transported across different information environments, some of them are specific to particular models. I also survey models of social interactions in a different section. I close with a brief discussion of post-estimation issues and research prospects.

KEYWORDS: Identification, multiplicity, games, social interactions.

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1 Introduction

In this article I review the recent literature on the econometric analysis of games where multiple solutions are possible. Equilibrium models are a defining ingredient of Economics. Game theoretic models in particular have occupied a prominent role in various subfields of the discipline for many decades. When taking these models to data, a sample of games represented by markets, neighborhoods, or economies, is endowed with an interdependent payoff structure that depends on observable and unobservable variables (to the econometrician and potentially to the players) and participants choose actions. One pervasive feature in many of these models is the existence of multiple solutions for various payoff configurations, and this is an aspect that carries over to estimable versions of such systems.

Whereas the existence of more than one solution for a given realization of the payoff structure does not preclude the estimation of a particular model (and in certain cases even improves their identification), ignoring its possible occurrence opens the door to potentially severe misspecifications and non-robustness in the analysis of substantive questions. Fortunately a lot has been learnt in the recent past about the econometric properties of such models. The tools available benefit from advances in identification analysis, estimation techniques, and computational capabilities. I cover some of these in the ensuing pages.

Given space limitations, the survey is by no means exhaustive. I nevertheless try to cover some of the main developments thus far. Because most of the literature has so far concentrated on parametric models, this is also my focus here. As in many other contexts, which parametric and functional restrictions are imposed deserves careful deliberation and some of the parametric and functional restrictions in the models I present can be relaxed (e.g., the linearity of the parametric payoff function and the distributional assumptions in Theorem 2 of Tamer (2003) or the analysis of social interactions models in Brock and Durlauf (2007)). Once point or partial identification is established, estimation typically proceeds by applying well understood methods such as Maximum Likelihood and Method of Moments (many times with the assistance of simulations) in the case of point-identified models, or by carrying out recently developed methods for partially identified models. A thorough discussion of partially identified models would require much more space and I leave that for other surveys covering those methods in more generality and detail (see, e.g., Tamer (2010)). I nevertheless do discuss estimation and computation aspects that are somewhat peculiar to the environments described in this survey at various points.

In all the games analyzed here, given a set of payoffs for the economic agents involved, a solution concept defines the (possibly multiple) outcomes that are consistent with the economic environment. The solution concepts I use below essentially consist of mutual best responses (plus consistent beliefs when information is asymmetrically available) and I refer to those as "equilibria" or "solutions" indiscriminately (hopefully without much confusion to the reader). In the following sections, the solution concepts are Nash equilibrium for complete information games, Bayes-Nash or Markov Perfect equilibrium for games of incomplete information, and rational expectations equilibrium as defined in the social interactions literature for those types of models. Whereas these are commonly assumed solution concepts, others exist. Aradillas-Lopez and Tamer (2008) for example consider rationalizable strategies and network formation games rely on pairwise stability or similar concepts. Multiplicity is often an issue for these alternative definitions and many of the ideas discussed below (e.g., bounds) can be used when those concepts are adopted instead.

One important ingredient guiding identification and estimation strategies in these models is the information environment of a game. Whether a game is one of complete or incomplete (i.e., private) information may affect the econometric analysis in a substantive manner. Many of the techniques discussed below can be transported across these different information environments, but some of them are specific to particular models. In the next sections I discuss identification and estimation in games of complete and incomplete information separately. I also separately survey models of social interactions, where multiplicity occur as well. For certain specifications (though not always), these models coincide with games of incomplete information and the discussion of that class of games carries forward. Because the number of players in a given economy is typically large in this class of models though, additional estimation strategies handling multiplicity may be employed. I end the article with a discussion of post-estimation issues and research prospects.

2 Preliminary Foundations

Following early analyses of the problem such as Jovanovic (1989), I start by casting my discussion in terms of the perspective adopted by Koopmans and co-workers in the mid-20^{th} century. I do so because it makes it apparent how multiplicity interferes with usual econometric methodologies and lays many of the models on a common ground, making it easier to identify discrepancies and commonalities across models. Koopmans and colleagues recognized that the analysis of economic phenomena often requires that we go beyond the mere statistical description of the observable probability distributions of interest and pursue primitive parameters, or policy invariant features, of an economic model. In their own words,

> [i]n many fields the objective of the investigator's inquisitiveness is not just a "population" in the sense of a distribution of observable variables, but a physical structure projected behind this distribution, by which the latter is thought to be generated. (...) [T]he structure concept is based on the investigator's ideas as to the "explanation" or "formation" of the phenomena studied, briefly, on his theory of these phenomena, whether they are classified as physical in the literal sense, biological, psychological, sociological, economic or otherwise. (Koopmans and Reiersol (1950), p.1)

The above considerations embody the spirit of what became known as the Cowles Commission research program. The research philosophy typically identified with the Cowles Commission prescribed (1) defining an economic model, (2) specifying its probabilistic features to take the economic model to data and (3) using statistical methods to estimate policy invariant parameters and test relevant hypotheses about those parameters. The first step is a prerequisite for the study of any counterfactual intervention in the economy. The economic model essentially postulates how the outcomes and other variables in the model relate to each other. Some of these variables are observed by the econometrician and others are latent or unobserved by the researcher. When the distribution of observed variables is consistent with only one parameter configuration, the model is point identified. If that is not the case the model is partially or set identified and the set of parameters consistent with the observable data may still be informative about the analyst's research question.

One implicit assumption in the above analysis is that the economic model predicts only one value of the observed variables for a given realization of latent variables. This is not necessarily the case of many models of interest (e.g. models of strategic interactions). A simple but intriguing early example is that of the entry game depicted in Bresnahan and Reiss (1991) (see also Bjorn and Vuong (1984)). The population comprises i.i.d. copies of a game between two players. In the Bresnahan and Reiss paper the players are firms (e.g., small business in different geographical markets) deciding on whether to enter a particular market. In Bjorn and Vuong (1984), the players are husband and wife and the action is whether to participate in the labor force or not. This model is also related to the precursor works on dummy endogenous variables, although it should be noted that the multi-agent, fully simultaneous nature of the problem here brings in important differences.

The researcher observes different markets and in each of these markets there are two firms. It is typically assumed that firms can be labeled. In many applications, this label is natural and refers to the actual identity of the firm (i.e., Delta or United Airlines) or, in the case of a household, the roles of husband or wife. Here I will assume that firms are labeled by $i = 1, 2$. A firm's decision depends on its profit, which in turn depends on whether the other firm also entered the market or not. Let $y_i \in \{0, 1\}$ denote whether firm i enter ($y_i = 1$) or not ($y_i = 0$) and assume that profits are given by $\mathbf{x}_i^\top \beta_i + \Delta_i y_j + u_i, i = 1, 2, j \neq i$ where $\mathbf{x}_i$ are observable variables that affect firm i's profit and u_i is an unobserved (to the econometrician) variable affecting the profit for firm i. The distribution of $\mathbf{u} \equiv (u_1, u_2)$ is denoted by F. In Ciliberto and Tamer (2009)'s analysis of the airline industry, for example, the market is a particular route, the firms are different airlines (e.g., American, Delta, United Airlines) and $\mathbf{x}_i$ comprises market- and firm-specific variables affecting demand (e.g., population size, income) and costs for the firms (e.g., various distance measures between the route endpoints and to nearest hub airport of the firm). The linear functional form $\mathbf{x}_i^\top \beta_i$ can be relaxed, but additive separability between observed and latent variables in the payoff structure is typically necessary for inference. Because I assume that player identities or roles can be assigned, the parameters β_i and Δ_i are allowed to be label- or role-specific. If players' roles cannot be distinguished, symmetry in the coefficients would have to be imposed. (This is the case in the social interactions models reviewed in Section 5.) Here I suppose that $\mathbf{x}_i$ includes at least one variable that impacts only firm i. This is a common and powerful exclusion restriction used in the literature, though it may not always be necessary in the analysis of games.

I will use this simple payoff structure with varying information structures throughout this survey to highlight some of the more important aspects in the econometric analysis of games with possibly multiple equilibria. For now, I assume there is complete information for the players: realizations of $\mathbf{x}$ and $\mathbf{u}$ are observed by all agents in the game. The payoff matrix for this game is given by:

Table 1: 2-by-2 Game

Player 2

		0	1
Player 1	0	(0,0)	$(0, \mathbf{x}_2^\top \beta_2 + u_2)$
	1	$(\mathbf{x}_1^\top \beta_1 + u_1, 0)$	$(\mathbf{x}_1^\top \beta_1 + \Delta_1 + u_1, \mathbf{x}_2^\top \beta_2 + \Delta_2 + u_2)$

Much of the literature has focused on pure Nash equilibria as the preferred solution concept (see Berry and Tamer (2006) for a discussion of equilibria in mixed strategies though). If observable outcomes y correspond to Nash Equilibria of the above game, the following econometric model is obtained:[1]

$$\begin{aligned} y_i^* &= \mathbf{x}_i^\top \beta_i + y_j \Delta_i + u_i \\ y_i &= \mathbf{1}_{y_i^* \geq 0} \end{aligned} \tag{1}$$

Pictorially, if $\Delta_i < 0$ this can be analyzed using the following picture:

FIGURE 1 HERE

[1]See Aradillas-Lopez and Tamer (2008) for examples where other solutions concepts are used.

In the central region of the plane, the model predicts two possible solutions. What is missing here is an "equilibrium selection mechanism" and in that sense the model is "incomplete". For payoff realizations within the region of indeterminacy (i.e., the central region), one could imagine circumstances where $(0, 1)$ is always selected or scenarios where $(1, 0)$ is always selected. Also possible are intermediary cases where with some probability (potentially dependent on the realizations of $\mathbf{x}$ and $\mathbf{u}$ and the parameters in the model $\theta \equiv (\beta_1, \beta_2, \Delta_1, \Delta_2)$), say $\lambda(\mathbf{x}, \mathbf{u}, \theta) \in [0, 1]$, one of the two equilibria is selected whenever payoffs fall in the multiplicity region. Different selection probabilities (i.e., $\lambda(\mathbf{x}, \mathbf{u}, \theta)$ in the example) will induce different distributions over the observable outcomes y_i.

One could (and in many examples below does) include the equilibrium selection mechanism into the structure. It is nevertheless important to bear in mind that modelling the equilibrium selection process requires extra assumptions. This opens up an additional avenue for misspecification. Moreover, an estimated equilibrium selection mechanism is more likely to be policy-sensitive. This is because "in a game with multiple equilibria, anything that tends to focus the payers' attention on one equilibrium may make them all expect it and hence fulfill it, like a self-fulfilling prophecy. (...) [T]he question of which equilibrium would be focused on and played by real individuals in a given situation can be answered only with reference to the psychology of human perception and the cultural background of the players." (Myerson (1991), pp. 108, 113) Consequently, a question that permeates much of the literature on empirical games with possibly many equilibria is whether one can be agnostic about the equilibrium selection rule, economizing on identifying assumptions, and still be able recover the parameters of interest (i.e., θ in this example) or functions of these parameters. Of course, many authors adopt a less pessimistic view than Myerson's and point out that certain equilibria, such as Pareto efficient, risk-dominant or pure strategy equilibria, are more salient and likely to be played than others. In the husband and wife example, for instance, important papers in the intra-household allocation literature support the view that Pareto inefficient equilibria should not be selected. Inference methods that incorporate the equilibrium selection mechanism may be informative about when and how certain equilibria are likely to be played.

In his 1989 paper, Jovanovic stresses that multiplicity may or may not preclude point identification. (It can be checked that without further restrictions the above example is not point identified.) Certainly, even when the model is not point-identified, it may still be partially (i.e., set) identifiable and informative about many issues of interest. What is perhaps more intriguing is that multiplicity can, in some instances, help establish point identification or make set-identified models more informative partly because it introduces additional variation in the data (see, e.g. the discussion in Section 3.1 of Manski (1993)).

3 Games of Complete Information

In Section 2, I presented a two-player game where information is assumed to be complete: both players know the realized values of all observable and latent variables (i.e., there is no private information). Bresnahan and Reiss (1991) show that multiplicity (of pure strategy Nash equilibria) will happen with positive probability in similar models of simultaneous moves with more than two actions and two players if the support of the latent variables $\mathbf{u}$ is large enough (e.g., $u_i \in (-\infty, \infty)$) (see Proposition 1 in Bresnahan and Reiss (1991)). To keep matters simple, I keep with the example above with two actions (0 or 1) and two players. In what follows I discuss various approaches to inference in games of complete information.

3.1 Pooling Multiple Equilibria Outcomes

If $\Delta_i < 0, i = 1, 2$, the entry of a firm in the market affects the other firm's profits negatively and all equilibria in pure strategies for a given realization of $\mathbf{u}$ involve a unique number $y_1 + y_2$ of players choosing 1. Let $\mathbf{1}(N, \mathbf{x}, \mathbf{u}; \theta)$ be 1 if the number of players choosing 1 is N for $\mathbf{x}, \mathbf{u}$ and θ. In this case, the likelihood conditional on $\mathbf{x}$ of observing N players choosing 1 is given by

$$\mathbb{P}(N|\mathbf{x}; \theta) = \int \mathbf{1}(N, \mathbf{x}, \mathbf{u}; \theta) \mathbb{P}(d\mathbf{u}|\mathbf{x}).$$

For the illustrative example in the previous section, we get

$$\begin{aligned}
\mathbb{P}_\theta(N=0|\mathbf{x}) &= F_{u_1,u_2}(-\mathbf{x}_1^\top\beta_1, -\mathbf{x}_2^\top\beta_2) \\
\mathbb{P}_\theta(N=2|\mathbf{x}) &= 1 - F_{u_1}(-\mathbf{x}_1^\top\beta_1 - \Delta_1) - F_{u_1}(-\mathbf{x}_2^\top\beta_2 - \Delta_2) \\
&\quad + F_{u_1,u_2}(-\mathbf{x}_1^\top\beta_1 - \Delta_1, -\mathbf{x}_2^\top\beta_2 - \Delta_2) \\
\mathbb{P}_\theta(N=1|\mathbf{x}) &= 1 - \mathbb{P}_\theta(N=0|\mathbf{x}) - \mathbb{P}_\theta(N=2|\mathbf{x})
\end{aligned}$$

where $\mathbf{u}$ is assumed to be independent of $\mathbf{x}$ and F_{u_1,u_2} is its known cdf. Given the assumption of independence between $\mathbf{x}$ and $\mathbf{u}$ and known cdf F_{u_1,u_2}, point identification can be ascertained using arguments similar to those employed for parametric discrete choice models. Once point-identification of θ is demonstrated, the model can be estimated via Maximum Likelihood, under the assumption that $\Delta_i < 0$, with a random sample of games. This strategy is pursued, for example, in Bresnahan and Reiss (1990) to identify and estimate a model of firm entry in automobile retail markets and Berry (1992) for an entry model in the airline industry. It also highlights the point that multiplicity is not necessarily an impediment to identification. This happens because certain quantities (i.e. the number of entrants) are invariant across equilibria when more than one solution is possible.

More generally, the key insight is that in this model certain outcomes can only occur as unique equilibria. When $\Delta_i < 0, i = 1, 2$, this happens for $\mathbf{y} = (0, 0)$ and $\mathbf{y} = (1, 1)$. This provides an avenue for identification and estimation of the model. In a coordination game, where $\Delta_i > 0, i = 1, 2$, multiple (pure strategy) Nash equilibria occur whenever $\mathbf{u} = (u_1, u_2) \in [-\mathbf{x}_1^\top\beta_1 - \Delta_1, -\mathbf{x}_1^\top\beta_1] \times [-\mathbf{x}_2^\top\beta_2 - \Delta_2, -\mathbf{x}_2^\top\beta_2]$. In this case, both $\mathbf{y} = (0, 0)$ and $\mathbf{y} = (1, 1)$ are possible equilibria. ($\mathbf{y} = (0, 0)$ is a unique equilibrium when $u_i < -\mathbf{x}_i^\top\beta_i - \Delta_i, i = 1, 2$ and $\mathbf{y} = (1, 1)$ is a unique equilibrium when $u_i > -\mathbf{x}_i^\top\beta_i, i = 1, 2$.) The number of players choosing 1 is no longer the same across equilibria, but one could nonetheless mimic the previous strategy and consider the probability of events $\{(0, 1)\}, \{(1, 0)\}$ and $\{(1, 1), (0, 0)\}$, where one pools together any two outcomes that are both equilibria for some given $\mathbf{x}$ and $\mathbf{u}$. Once this is done, singleton events correspond to outcomes that only occur as unique equilibria regardless of the value taken by $\mathbf{x}$ and $\mathbf{u}$.

The idea of identifying certain (non-exhaustive) outcomes with the occurrence of multiple equilibria is pursued by Honoré and de Paula (2010) for the identification of a complete information timing game. In that paper, outcomes $\mathbf{y}$ are duration variables chosen by individuals to model circumstances where timing decisions by one person affect the payoff of another individual (e.g., joint migration, joint retirement, technology adoption). The paper shows that multiplicity occurs only when duration spells terminate simultaneously for both players (i.e., $y_1 = y_2$) and sequential spell terminations (i.e., $y_1 \neq y_2$) occur only as unique equilibria. Identification and estimation can then be achieved by considering those events separately using standard arguments in the duration literature.

Unfortunately, this strategy is not always feasible in more general games as most of the possible outcomes (if not all) might arise as equilibria alongside many other possible solutions for a set of $\mathbf{x}$ and $\mathbf{u}$ with high probability. This would limit the ability of this insight to achieve point-identification (though it might still allow for set-identification). Take for instance the simple example in Jovanovic (1989). This example corresponds to the two-action, two-player example from Section 2 with $\beta_1 = \beta_2 = 0$, $\Delta \equiv \Delta_1 = \Delta_2 \in [0, 1)$ and the pdf for $\mathbf{u}$ is $f_{\mathbf{u}}(u_1, u_2) = \mathbf{1}_{(u_1,u_2)\in[-1,0]^2}$ (i.e., the latent variables are independently and uniformly distributed on $[-1, 0]$). In this example, $\mathbf{y} = (0, 0)$ is a Nash equilibrium for any realization of $\mathbf{u}$. In addition, $\mathbf{y} = (1, 1)$ is also a Nash equilibrium for $(u_1, u_2) \in [-\Delta, 0]^2$. $(0, 1)$ or $(1, 0)$ are never equilibria. Notice that $(0, 0)$ arises as a unique equilibrium when $(u_1, u_2) \notin [-\Delta, 0]^2$, but also occur as equilibria alongside $(1, 1)$ otherwise. Observation of $\mathbf{y} = (0, 0)$ does not allow one to determine whether (u_1, u_2) were drawn in the region of multiplicity (i.e., $[-\Delta, 0]^2$) or not. Pooling the two outcomes as in the discussion above is hopeless for (either point or set) identification (and consequently estimation) since $\mathbb{P}_\theta(\{(0, 0), (1, 1)\}) = 1$. This is not an isolated example or a mere consequence of the lack of covariates: Bresnahan and Reiss (1991) show that this problem is generally present for a wide class of discrete action games (see Proposition 3 in that paper). In this case, pooling outcomes that are equilibria for the same realizations of observed and latent covariates provides no identification leverage.

3.2 Large Support and Identification

An alternative strategy for point-identification of the parameters of interest in the presence of multiplicity is provided by Tamer (2003), adapting similar ideas from the individual discrete choice literature (e.g., Manski (1988) and Heckman (1990)). Consider models where $\Delta_1 \times \Delta_2 < 0$ for example. Figure 2 depicts the possible equilibria on the space of latent covariates (u_1, u_2). In contrast with the case where $\Delta_1 \times \Delta_2 > 0$, any equilibrium is unique. Nonetheless, there are no equilibria in pure strategies in the central region of the picture. Tamer assumes that any outcome is possible in that area. One can rationalize this either because players are thought to choose one of the outcomes in ways that are not prescribed by the solution concept adopted here (Nash in pure strategies) or because the (unique) equilibrium in mixed strategy places positive probability on every outcome $\mathbf{y}$ (though the mixed strategy equilibrium probabilities are not incorporated into the econometric model). Either way, we end up with an "incomplete" econometric model where the outcome of the game in the central region is not resolved within the economic model. In this case, the strategy from the previous subsection would be inconclusive as in Jovanovic's example. This is because it would bundle together all possible outcomes given that they are all allowed in the central area of Figure 2.

FIGURE 2 HERE

Theorem 2 in Tamer's paper nevertheless demonstrates the point-identification of the parameter vector θ using exclusion restrictions and large support conditions on the observable covariates. Without loss of generality, assume that $\Delta_1 > 0$ and $\Delta_2 < 0$. In this case, given that games are identically and independently distributed and assuming that $\mathbf{u}$ is independent of other covariates and has a known c.d.f., the following result holds:

Theorem 2 (Tamer 2003) *Assume that for $i = 1$ or $i = 2$, there exists a regressor x_{ik} with $\beta_k \neq 0$ such that $x_{ik} \notin \mathbf{x}_j, j \neq i$ and such that the distribution of $x_{ik}|\mathbf{x}_{-ik}$ has everywhere positive density, where $\mathbf{x}_{-ik} = (x_{i1}, \ldots, x_{ik-1}, x_{ik+1}, \ldots, x_{iK})$ and K is the dimension of $\beta_i, i = 1, 2$. Then the parameter vector $\theta = (\beta_1, \beta_2, \Delta_1, \Delta_2)$ is identified if the matrices $\mathbb{E}[\mathbf{x}_1\mathbf{x}_1^\top]$ and $\mathbb{E}[\mathbf{x}_2\mathbf{x}_2^\top]$ are nonsingular.*

Here I only sketch the idea behind this result. Without loss of generality, assume that $\beta_k > 0$ (an analogous argument can be made for $\beta_k < 0$). Because of the large support condition, one can find extreme values of x_{ik} such that only unique equilibria in pure strategies are realized (because $\mathbb{P}(u_i > -\Delta - \mathbf{x}_i^\top \beta_i) \to 1$ and $\mathbb{P}(u_i > -\mathbf{x}_i^\top \beta_i) \to 1$ as $x_{ik} \to \infty$). On the other $\beta_j \neq \beta_j', j \neq i$, the full rank condition on $\mathbb{E}[\mathbf{x}_j \mathbf{x}_j^\top]$ guarantees that $\mathbf{x}_j^\top \beta_j \neq \mathbf{x}_j^\top \beta_j'$ with positive probability. This then implies that in the limit (as $x_{ik} \to \infty$), $\mathbb{P}_{\beta_j}(y_j = 1|\mathbf{x}) \neq \mathbb{P}_{\beta_j'}(y_j = 1|\mathbf{x})$ and the model identifies the parameter β_j. Similar arguments can be employed to demonstrate the identifiability of the remaining parameters. This point-identification strategy can be extended to other contexts. Grieco (2012), for example, extends it to a game of incomplete information where some variables are publicly observed by the participants but not by the econometrician. I also note that the result uses the excluded regressor x_{ik} in conjunction with the large support assumption. This is nonetheless not necessary in more general contexts, for example when the payoff is not a linear function of $\mathbf{x}$ for every player (see for example the generalization in Bajari, Hong, and Ryan (2010)).

The essence of this result is to find values of the regressor for which the actions of all but one player are dictated by dominant strategies (regardless of the realizations of the latent unobservable variables). This turns the problem into one of a discrete choice by the single agent that does not play dominant strategies (given the covariates). The existence of a regressor value that induces some of the players to always choose one action may be hard to ascertain in many empirical contexts, but one may nevertheless get reasonably close to that ideal.

Ciliberto and Tamer (2009), for example, estimate a firm entry model for the airline industry where the distance between two airports affects differentially the decision of traditional carriers and discount airlines to operate a route between those two airports. They find that distance carries a very negative effect on the payoff for Southwest Airlines and a mildly positive effect on the payoff for traditional carriers. As they point out, "[t]his is consistent with anecdotal evidence that Southwest serves shorter markets than the larger national carriers" (p.1818). One could then think of markets where the distance between airports is large as inducing a small probability of entry by Southwest and other discount airlines but still providing variation in entry decisions by other, larger, carriers.

When the regressor values guaranteeing a dominant strategy for a subset of players cannot be obtained in an empirical application, other identification ideas need to be employed. One feasible identification plan relies on bounds, focusing on a (possibly non-singleton) set of parameters that are consistent with the model (see next subsection). In that case, if the probability that certain players adopt dominant strategies gets larger, one approaches the conditions for the point-identification strategy above and the identified set of parameters gets smaller.

Because the point-identification result above relies on extreme values of the covariates, even with the ideal conditions for Tamer's result satisfied, this "identification at infinity" strategy will have important consequences for inference as pointed out in Khan and Tamer (2010), leading to asymptotic convergence rates that are slower than parametric rates as the sample size (i.e., number of games) gets larger (see also the discussion in Bajari, Hahn, Hong, and Ridder (2011)).[2]

Bajari, Hong, and Ryan (2010) incorporate the equilibrium selection mechanism into the problem and demonstrate how large support conditions can help establish necessary conditions for point-identification of the selection process. In expanding the model to include the equilibrium selection, they generalize the paper by Bjorn and Vuong (1984) where equilibria are selected with a nondegenerate probability which is estimated along the other payoff relevant parameters. Bajari, Hong and Ryan point out that "[e]stimating the selection mechanism allows the researcher to simulate the model, which is central to performing counterfactuals". Some caution is nonetheless warranted in justifying the policy-invariance of the equilibrium selection mechanism in relation to the particular counterfactual of interest as I indicate earlier (and also later) in this survey.

To maintain consistency with the discussion above, I assume that there are always equilibria in pure strategy and remain in the two-player, two-action environment introduced

[2]The statement of Theorem 3 in Tamer (2003) is inaccurate. An additional term in the asymptotic variance is missing, potentially invalidating the efficiency claim in the result, as noted in Hahn and Tamer (2004).

earlier. Bajari, Hong and Ryan nevertheless do allow for equilibria in mixed strategies (see also Berry and Tamer (2006)), which are guaranteed to exist under more general conditions. Let $\mathcal{E}(\mathbf{x}, \mathbf{u}, \theta) \subset \{0,1\}^2$ be the set of equilibrium profiles given particular realizations of $\mathbf{x}$ and $\mathbf{u}$ and value of θ. Denote by $\lambda(\mathbf{y}|\mathcal{E}(\mathbf{x}, \mathbf{u}, \theta), \mathbf{x}, \mathbf{u}, \theta)$ the probability that $\mathbf{y}$ is selected when the equilibrium set is $\mathcal{E}(\mathbf{x}, \mathbf{u}, \theta)$ given covariates at $\mathbf{x}$ and $\mathbf{u}$ and parameter θ. The distribution of latent variables F is known. When $\mathbf{y}$ is the unique equilibrium for the realizations of $\mathbf{x}$ and $\mathbf{u}$ and parameter vector θ, $\lambda(\mathbf{y}|\mathcal{E}(\mathbf{x}, \mathbf{u}, \theta), \mathbf{x}, \mathbf{u}, \theta) = 1$. When it is not an equilibrium, $\lambda(\mathbf{y}|\mathcal{E}(\mathbf{x}, \mathbf{u}, \theta), \mathbf{x}, \mathbf{u}, \theta) = 0$. Finally, when there are other equilibria, $\lambda(\mathbf{y}|\mathcal{E}(\mathbf{x}, \mathbf{u}, \theta), \mathbf{x}, \mathbf{u}, \theta) \in [0,1]$. The conditional probabilities of actions are then

$$\mathbb{P}(\mathbf{y}|\mathbf{x}; (\theta, F_{\mathbf{u}})) = \int \lambda(\mathbf{y}|\mathcal{E}(\mathbf{x}, \mathbf{u}, \theta), \mathbf{x}, \mathbf{u}, \theta) \mathbf{1}_{\mathbf{y} \in \mathcal{E}(\mathbf{x}, \mathbf{u}, \theta)} f_{\mathbf{u}}(\mathbf{u}) d\mathbf{u} \tag{2}$$

where $f_{\mathbf{u}}(\cdot)$ is the pdf for $\mathbf{u}$, which is assumed to be independent of $\mathbf{x}$.

Given the specification summarised in (2), it is not clear whether the model identifies the equilibrium selection mechanism $\lambda(\cdot)$. As a matter of fact, unless further restrictions are imposed, it does not. To take an extreme, but very simple illustration, consider again the example in Jovanovic (1989) with no covariates. In this case, let λ denote the probability that $(1,1)$ is selected whenever there are multiple equilibria (i.e., $(u_1, u_2) \in [-\Delta, 0]^2$). Then,

$$\mathbb{P}(\mathbf{y} = (1,1)) = \lambda \Delta^2 \text{ and } \mathbb{P}(\mathbf{y} = (0,0)) = 1 - \mathbb{P}(\mathbf{y} = (1,1)).$$

It is not possible to pin down λ and Δ from the distribution of outcomes. One of the issues here is that there are more unknowns than equations. To reduce the degrees of freedom in the problem, Bajari, Hong and Ryan impose additional structure. The first thing to note is that the number of equations can be increased if the support of covariates $\mathbf{x}$ is relatively large, generating additional conditional moments. To keep the number of parameters under control, Bajari and co-authors assume that selection probabilities depend only on utility indices through a relatively low-dimensional "sufficient statistic". In essence this allows them to parameterize the equilibrium selection mechanism as $\lambda(\cdot) = \lambda(\cdot; \gamma)$ for some parameter γ of relatively low dimension compared to the cardinality of covariate support. Once this is guaranteed and exclusion restrictions like those in Tamer (2003)'s theorem above are imposed, one can rely on there being at least as many conditional moments as there are parameters (i.e., θ and γ), which is *necessary* for identification (see their Theorem 3).

Of course, one should be careful not to introduce misspecifications while incorporating the equilibrium selection mechanism into the econometric model (especially a parametric one). Because the number of equilibria and the selection probability of an equilibrium may depend on all variables observed by the players and the parameter θ, the selection mechanism should typically allow for this possibility. Any misspecification will likely contaminate the estimation of θ. Furthermore, it should be noted that the results in Bajari, Hong, and Ryan (2010) provide necessary but not sufficient conditions for the point-identification of the selection mechanism. Finally, as highlighted earlier, one ought to be careful not to extrapolate the equilibrium selection mechanism in counterfactual experiments mimicking policies that might affect the way in which equilibria are selected.

If point-identification can be established, the authors suggest estimating the parameters of interest using the conditional probability in (2) by the Method of Simulated Moments (MSM) in order to handle the integration over the latent variables $\mathbf{u}$. (As pointed out in the text, the Simulated Maximum Likelihood (SML) method could also be applied here.) Here the estimation procedure requires the determination of $\mathcal{E}(\mathbf{x}, \mathbf{u}, \theta)$ for given realizations of $\mathbf{x}$, $\mathbf{u}$ and values of θ. Computing the whole equilibrium set becomes prohibitive already with a relatively small number of players and actions. Bajari, Hong and Ryan also suggest strategies to partly accommodate the computational issues.

With a random sample of $g = 1, \ldots, G$ games, one can then estimate the parameters of interest using moments such as

$$\mathbb{E}\left[\left(\mathbf{1}_{\mathbf{y}_t = a} - \mathbb{P}(\mathbf{y} = a | \mathbf{x}; \theta, \gamma, F)\right) h(\mathbf{x})\right] = 0$$

where $h(\cdot)$ are appropriately chosen weight functions of the observable covariates and a are admissible action profiles. The sample analog of the above moment for the MSM estimator is given by

$$\sum_{g=1}^{G} \left(\mathbf{1}_{\mathbf{y}_g = a} - \widehat{\mathbb{P}}(\mathbf{y} = a | \mathbf{x}; \theta, \gamma, F)\right) h(\mathbf{x})$$

where $\widehat{\mathbb{P}}$ is a computer simulated estimate of $\mathbb{P}$.

3.3 Bounds

Typically the covariates' support is not rich enough and the previous point-identification strategy will not suffice. Another avenue for inference in games with possibly multiple equilibria is to rely on partial identification and use bounds for estimation. Take again for instance the example in Jovanovic (1989) and note that $\mathbb{P}(\mathbf{y} = (1,1))$ is at most Δ^2 (if $(1,1)$ is always selected when $(u_1, u_2) \in [-\Delta, 0]^2$). Hence, because $\Delta < 1$ we have that $\sqrt{\mathbb{P}(\mathbf{y} = (1,1))} \leq \Delta < 1$. This is the set of all parameters Δ which are consistent with the observable distribution of outcomes: for every Δ within $[\sqrt{\mathbb{P}(\mathbf{y} = (1,1))}, 1)$ there is an equilibrium selection probability that delivers the same distribution of observables. Since $\mathbb{P}(\mathbf{y} = (1,1))$ can be consistently estimated, one can estimate the identified set by $[\sqrt{\widehat{\mathbb{P}}(\mathbf{y} = (1,1))}, 1)$.

This insight appears in Jovanovic's discussion of this particular example and is explored in more detail by Tamer (2003). Consider the entry game in Section 2 for $\Delta_i < 0, i = 1, 2$. We were able to handle this case before using the fact that, whenever multiple equilibria occur, the number of entrants is the same. In more general cases (e.g. when there are more players and payoffs are heterogeneous), this is not always the case even if we restrict ourselves to pure strategy equilibria as noted in Ciliberto and Tamer (2009). We use this simple example to illustrate the construction of bounds for the parameters of interest.

Remember that in this case $(0,0)$ and $(1,1)$ always occur as unique (pure) strategy equilibria. Note also that

$$\mathbb{P}(\mathbf{y} = (0,1)|\mathbf{x}) \leq 1 - \mathbb{P}(\mathbf{y} = (1,1)|\mathbf{x}) - \mathbb{P}(\mathbf{y} = (0,0)|\mathbf{x})$$

and

$$\begin{aligned} \mathbb{P}(\mathbf{y} = (0,1)|\mathbf{x}) \quad \geq \quad & 1 - \mathbb{P}(\mathbf{y} = (1,1)|\mathbf{x}) - \mathbb{P}(\mathbf{y} = (0,0)|\mathbf{x}) \\ & -\mathbb{P}\left((u_1, u_2) \in \times_{i=1,2}[-\mathbf{x}_i^\top \beta_i, -\mathbf{x}_i^\top \beta_i - \Delta]|\mathbf{x}\right) \end{aligned}$$

where, as before

$$\begin{aligned}
\mathbb{P}(\mathbf{y}=(0,0)|\mathbf{x}) &= F_{u_1,u_2}(-\mathbf{x}_1^\top\beta_1, -\mathbf{x}_2^\top\beta_2) \text{ and} \\
\mathbb{P}(\mathbf{y}=(1,1)|\mathbf{x}) &= 1 - F_{u_1}(-\mathbf{x}_1^\top\beta_1 - \Delta_1) - F_{u_1}(-\mathbf{x}_2^\top\beta_2 - \Delta_2) \\
&\quad + F_{u_1,u_2}(-\mathbf{x}_1^\top\beta_1 - \Delta_1, -\mathbf{x}_2^\top\beta_2 - \Delta_2).
\end{aligned}$$

The upper bound on $\mathbb{P}(\mathbf{y}=(0,1)|\mathbf{x})$ is attained if $(0,1)$ is always selected when (u_1, u_2) falls in the region of multiplicity. Conversely, the lower bound is attained when $(0,1)$ is never selected in that case. Hence, we subtract the probability that (u_1, u_2) falls in that region from the previous probability. Similar bounds hold for $\mathbb{P}(\mathbf{y}=(1,0)|\mathbf{x})$. The bounds can be modified to allow for mixed strategy equilibria as done in Berry and Tamer (2006).

These inequalities define a region of the parameter space where the true parameter vector resides. When the covariates support is not as rich as outlined in the previous section to guarantee point-identification, this set may (and typically will) be larger than a singleton. Inference may nonetheless be carried out using methods developed in the recent literature on estimation of partially identified models. Papers on the topic that pay special attention to games with possibly multiple equilibria include Andrews, Berry, and Jia (2004), Ciliberto and Tamer (2009), Beresteanu, Molchanov, and Molinari (2009), Pakes, Porter, Ho, and Ishii (2011), Galichon and Henry (2011), Moon and Schorfheide (2012) and Chesher and Rosen (2012). Other papers, more general in treatment, could also be cited and many of the ideas can be applied to games under different information structures. Pakes, Porter, Ho, and Ishii (2011), for instance, handle games where there is uncertainty which is symmetric across players and information is complete as well as games of incomplete information (i.e., some information about payoffs is private information to players). A thorough treatment of the techniques in that literature would nonetheless be beyond the scope of this survey and require much more space. For a recent introduction to that literature, see the survey by Tamer (2010) and references therein.

3.4 Additional Topics

I close this section with a short discussion. First, much of the work dealing with multiplicity has focused on discrete games. Many of the issues encountered in discrete games will nevertheless also appear when the action space is continuous. These models are particularly relevant in the study of pricing games among firms for example. If more than one solution for a given realization of the payoff structure is possible, observed outcomes will be a mixture over equilibria having the equilibrium selection mechanism as the mixing distribution. Most of the ideas above can in principle be applied in those settings (see for instance the discussion in Pakes, Porter, Ho, and Ishii (2011)), but more work in the future may reveal further advantages or disadvantages of environments with continuous action spaces.

One notable reference in Econometrics focusing on continuous action spaces and multiple equilibria is Echenique and Komunjer (2009). They analyse a coordination game with continuous actions and use equilibrium results typically found in the literature of supermodular games. In this case, Tarski's fixed point theorem can be employed to establish maximal and minimal equilibria. Certain features of the structure under study (e.g., monotone comparative statics) can then be econometrically tested for using these extremal equilibria (see also de Paula (2009) and Lazzati (2012) for econometric multi-agent models using Tarski's fixed point theorem and Molinari and Rosen (2008) for a discussion of identification in supermodular games with discrete actions). A similar environment (with payoff complementarities and discrete action space) also appears for the empirical analyses of Ackerberg and Gowrinsankaran (2006) (where an equilibrium selection mechanism with support on the best and worst equilibria is incorporated incorporated into the econometric model) and of Jia (2008) (where one of the extremal equilibria is assumed to be selected). Jia (2008) also exploits a constructive version of Tarski's theorem for computation.

4 Games of Incomplete Information

The previous section presented ideas for the analysis of games of complete information. There, individuals are assumed to know the payoff realizations of every other player. This is nevertheless not always a good assumption. In the case of firms contemplating entry into a particular market, at least some part of the cost structure is private information to each firm. It turns out that games where information is incomplete and agents possess at least partially private information about payoffs provide different avenues for identification and estimation of the structure of interest when there are multiple equilibria.

I start with the same payoff structure as in the example from Table 1, but assume that u_i is known only to person i, where $i = 1, 2$. As in the previous section, I assume that roles or labels can be assigned to players. Given realizations of $\mathbf{x}$ and $\mathbf{u}$, agents now decide on their actions based on the payoffs they expect, given the distribution of "types" for the other individual involved in the game (i.e. the private information components of their payoffs, u_i). As before, I consider only pure strategy equilibria for simplicity. Given i's opponent's strategy, her best response dictates that

$$y_i = 1 \text{ if } \mathbf{x}_i^\top \beta_i + \mathbb{P}(y_j = 1|\mathbf{x}, u_i)\Delta_i + u_i \geq 0, \quad j \neq i \tag{3}$$

and 0 otherwise. The term "$\mathbb{P}(y_j = 1|\mathbf{x}, u_i)$" accommodates i's beliefs about the other person's type given her information set (comprising $\mathbf{x}$ and u_i) and the other person's strategy about choosing 0 or 1. A Bayes-Nash equilibrium (in pure strategies) is characterized by mutual best reponses of players 1 and 2 and consistent beliefs.

When information is incomplete, the expression above highlights the fact that the joint distribution of the latent variables (u_1, u_2) plays an important role. Unless these variables are independent, $\mathbb{P}(y_j = 1|\mathbf{x}, u_i)$ will be a non-trivial function of u_i, $i = 1, 2$. When u_1 and u_2 are not independent, player i's type (i.e., u_i) is informative about her opponents type (i.e., u_j), and in equilibrium, will likely affect what she expects her opponent to play. If, on the other hand, $u_1 \perp\!\!\!\perp u_2|\mathbf{x}$, i's expectations about j's action will not depend on u_i since her type brings no hint about u_j. Because, conditional on $\mathbf{x}$, the equilibrium y_i depends solely on u_i, this assumption also implies that y_1 and y_2 are conditionally independent given $\mathbf{x}$ for a particular equilibrium. Unless otherwise noted and except for the last segment in this section, I will assume that the private shocks are (conditionally) independent within a game. In this case, we can write $\mathbb{P}(y_j = 1|\mathbf{x}, u_i) = p_j(\mathbf{x})$ and equilibrium choice probabilities should solve the following system of equations:

$$p_i(\mathbf{x}) = 1 - F_{u_i|\mathbf{x}}(-\mathbf{x}_i^\top \beta_i - p_j(\mathbf{x})\Delta_i|\mathbf{x}), \quad i = 1, 2, i \neq j. \tag{4}$$

This system can easily be seen to admit multiple solutions. Assume for example a symmetric payoff structure such that $\Delta_1 = \Delta_2 = 10.5$, u_1 and u_2 follow independent logistic distributions and $\mathbf{x}_1^\top \beta_1$ and $\mathbf{x}_2^\top \beta_2$ are drawn such that $\mathbf{x}_1^\top \beta_1 = \mathbf{x}_2^\top \beta_2 = -3.5$. Then, $p_1(\mathbf{x}) = p_2(\mathbf{x}) = 0.047$ and $p_1(\mathbf{x}) = p_2(\mathbf{x}) = 0.204$ are symmetric solutions to the system above. (A third solution where both players choose 1 with probability close to 1 also exists.)

In general the constituents of this model are not point-identified. To see this, denote by $\mathcal{E}(\mathbf{x}, \theta, F)$ the set of equilibria for a given realization of $\mathbf{x}$ and primitives (θ, F). Note that the equilibrium set now depends not on $\mathbf{u}$ (which are only privately observed) but on its distribution. Because for now I retain the assumption that $u_1 \perp\!\!\!\perp u_2|\mathbf{x}$, recall that in equilibrium y_1 and y_2 are conditionally independent, and an equilibrium is hence characterized by the marginal distributions of y_i given $\mathbf{x}$, $i = 1, 2$. Let $(p_1^k(\mathbf{x}), p_2^k(\mathbf{x}))_{k=1}^{|\mathcal{E}(\mathbf{x},\theta,F)|}$ be an ordering of the equilibrium set $\mathcal{E}(\mathbf{x}, \theta, F)$. Then, the distribution of outcomes conditional on covariates $\mathbf{x}$ is given by

$$\mathbb{P}(y_1, y_2|\mathbf{x}) = \sum_{k=1}^{|\mathcal{E}(\mathbf{x},\theta,F)|} \lambda(k|\mathcal{E}(\mathbf{x}, \theta, F), \mathbf{x}, \theta, F) p_1^k(\mathbf{x})^{y_1}(1 - p_1^k(\mathbf{x}))^{1-y_1} p_2^k(\mathbf{x})^{y_2}(1 - p_2^k(\mathbf{x}))^{1-y_2},$$

where $\lambda(\cdot)$ is the equilibrium selection mechanism and places probability $\lambda(k|\mathcal{E}(\mathbf{x}, \theta, F), \mathbf{x}, \theta, F)$ on the selection of equilibrium k. This is a mixture of independent random variables and results such as those in Hall and Zhou (2002) can be used to demonstrate that the component distribution $(p_1^k(\mathbf{x}), p_2^k(\mathbf{x}))_{k=1}^{|\mathcal{E}(\mathbf{x},\theta,F)|}$ and mixing probabilities are not point-identified by the model if $|\mathcal{E}(\mathbf{x}, \theta, F)|$ is strictly greater than 2. Similar results can be obtained for more players (using Hall, Neeman, Pakyari, and Elmore (2005), for example): if the number of equilibria on the support of the selection mechanism is large relative to the number of players, the structure is not point-identified by the model (see for instance the Supplementary Appendix in de Paula and Tang (2012)).

As I did in the previous section, I now discuss different approaches to inference in games of incomplete information like the one I just outlined.

4.1 Degenerate Equilibrium Selection Mechanism

The previous discussion underscores the benefits of further restrictions on the equilibrium selection mechanism for the econometric analysis of incomplete information games with possibly many equilibria. One common strategy is to assume that

$$\lambda(k|\mathcal{E}(\mathbf{x}, \theta, F), \mathbf{x}, \theta, F) = \mathbf{1}_{k=\mathcal{K}}$$

for some $\mathcal{K} \in \mathcal{E}(\mathbf{x}, \theta, F)$. In words, whenever primitives and covariates coincide for two games, thus inducing an identical equilibrium set, the same equilibrium is played in these two games. One can nevertheless be agnostic about which equilibrium is selected (i.e., which element of $\mathcal{E}(\mathbf{x}, \theta, F)$ is selected).

When is it realistic to assume that the same equilibrium is played across games? As Mailath (1998) points out, "[t]he evolution of conventions and social norms is an instance of players learning to play an equilibrium". If an equilibrium is established as a mode of behavior by past play, 'custom' or culture, this equilibrium becomes a focal point for those involved. When observed games are drawn from a population which is culturally or geographically close, sharing similar norms and conventions, one would expect this assumption to be adequate.

If a single equilibrium occurs in the data for every given payoff configuration, this restriction essentially allows one to treat the model in standard ways for identification and inference. Once identification is guaranteed, one possible strategy for estimation is the two-step estimator proposed in Bajari, Hong, Krainer, and Nekipelov (2010). The strategy builds on the earlier work by Hotz and Miller (1993) for the estimation of empirical (individual) dynamic discrete choice models and later on by Aguirregabiria and Mira (2007) and Bajari, Benkard, and Levin (2007) for dynamic games (where information is incomplete). It also resembles estimators proposed in the social interactions literature and covered below. I describe the procedure using the example from Table 1 (still with conditionally independent latent variables u_1 and u_2). Bajari, Hong, Krainer, and Nekipelov (2010) present the method in more generality.

STEP 1. In a first step, the (reduced form) conditional choice probabilities for the various outcomes in the action space are estimated nonparametrically: $\widehat{\mathbb{P}}(y_i|\mathbf{x}), i = 1, 2$. This can be done using sieves or kernel methods.

STEP 2. In the second step, estimates for the structural parameters are obtained from the conditional choice probability estimates. If a unique equilibrium is played in the data for same realizations of the payoff matrix, $p_i(\mathbf{x})$ and the expected payoff for player i of choosing 1 using equilibrium beliefs (i.e., $\mathbf{x}_i^\top \beta_i + p_j(\mathbf{x})\Delta_i$) are in one-to-one correspondence. In the example this correspondence is simply obtained as

$$\mathbf{x}_i^\top \beta_i + p_j(\mathbf{x})\Delta_i = -F_{u_i|\mathbf{x}}^{-1}(1 - p_i(\mathbf{x})|\mathbf{x}).$$

Given a sample of G iid games, the estimation then proceeds to minimize the following least squares function with respect to θ:

$$\sum_{g=1}^{G} \left[-F_{u_i|\mathbf{x}}^{-1}(1 - \widehat{p}_i(\mathbf{x}_g)|\mathbf{x}_g) - \mathbf{x}_{ig}^\top \beta_i - \widehat{p}_j(\mathbf{x}_g)\Delta \right]^2 h(\mathbf{x}_g)$$

where $h(\cdot)$ is again an appropriately chosen weight function. Under adequate conditions the estimator for θ converges at parametric rates as pointed out in the paper.

The assumption of a unique equilibrium in the data is crucial to travel from the conditional choice probabilities to the second-step estimator as it guarantees (together with other additional restrictions) that $p_i(\mathbf{x})$ and the expected payoff for player i of choosing 1 using equilibrium beliefs (i.e., $\mathbf{x}_i^\top \beta_i + p_j(\mathbf{x})\Delta_i$) are in one-to-one relationship. Because of this the above procedure bypasses the calculation of all equilibria for each possible value of θ much as Hotz and Miller (1993) avoids to computation of a dynamic program in the estimation of individual dynamic discrete choice models. Of course, as long as there is a unique selected equilibrium in the data, other estimators which rely on uniqueness such as Aradillas-Lopez (2010) can also be used. The assumption is also employed for example in Kasy (2012), which proposes a procedure to perform inference on the number of feasible equilibria in a game: even if the equilibrium selection mechanism is assumed to be degenerate, once the primitives (i.e., $(\beta_1, \beta_2, \Delta_1, \Delta_2)$ in my example, but more general non-parametric functions in Kasy (2012)) are estimated, one can infer the number of equilibria for the estimated particular payoff structure at given realization of covariates.

I should note that, as discussed in Bajari, Hong, Krainer, and Nekipelov (2010), when covariates $\mathbf{x}$ have a continuous support, the requirement that the data come from a unique equilibrium imposes subtle additional restrictions as the implementation of the above estimation strategy typically demands smoothness of the first stage estimator with respect to the covariates. Figure 3 sketches the possible ways in which this prerequisite might fail. The horizontal axis represents a generic continuous covariate $x \in \mathbb{R}$ and the vertical axis schematically represents the equilibrium conditional choice probabilities $p(x)$. The graph sketches the potential equilibrium probabilities corresponding to each x. For $x > x'$, there are three potential equilibria whereas for $x < x'$, there is only one. The red line marks the equilibrium that is selected in the data and stands for a possible (degenerate) equilibrium selection mechanism. Because the number of equilibria $|\mathcal{E}(x, \theta, F)|$ may change with x as in the scheme below, small changes in the covariates may provoke abrupt changes in the cardinality of the equilibrium set. In the figure this happens around x'. At this bifurcation point, the equilibrium conditional choice probability selected $p(x)$ may vary non-smoothly as is the case for the equilibrium selection mechanism. An implicit condition then is that such bifurcations happen in ways that do not affect the estimation. It is also possible that minute changes in the covariates $\mathbf{x}$ may tip the selection of the equilibrium observed in the data in a discontinuous manner. For the scheme in the figure this happens at x''. Both points (x' and x'') present possible challenges to the smoothness requirement.

FIGURE 3

In the presence of non-smooth equilibrium selection mechanisms, the two-step estimator suggested above will perform poorly. Aguirregabiria and Mira (2008) produce some Monte Carlo evidence of the defficiencies of this estimator when there are discontinuities in the equilibrium selection mechanism and suggest an alternative strategy, combining a Pseudo Maximum Likelihood estimator with a Genetic Algorithm.

The assumption of a degenerate equilibrium selection mechanism is usually invoked as well in the estimation of dynamic incomplete information games. In these environments, an action profile affects not only the players payoffs, but also the evolution of the state variables. The prescribed equilibrium strategies in this case generalize (3) to account for future repercussions. Common estimation strategies for dynamic games such as Aguirregabiria and Mira (2007), Bajari, Benkard, and Levin (2007), Berry, Pakes, and Ostrovsky (2007) and Pesendorfer and Schmidt-Dengler (2008) adapt well-known procedures for the estimation of single-agent Markov decision problems such as Hotz and Miller (1993). In essence, these estimators will rely on the existence of a one-to-one mapping between value functions and conditional choice probabilities. For this to hold, they rely on there being a single equilibrium in the data for games with identical payoff realizations as well as other assumptions usually invoked to guarantee the existence of such mapping (e.g., additive separability of u_i).

4.2 Identifying Power of Multiplicity

Sweeting (2009) follows a different route and incorporates the equilibrium selection mechanism in the estimation (see references in the previous section for similar strategies in complete information models).

He studies coordination (or not) on the timing of commercial breaks among radio stations (i.e. players) within a geographical market (i.e. game). In terms of the application in his paper the intution is as follows. "If stations want to coordinate then there may be an equilibrium where stations cluster their commercials at time 1 and another equilibrium where they cluster their commercials at time 2. (...) If stations want to play commercials at different times then we would expect to observe excess dispersion within markets (market distributions less concentrated than the aggregate) rather than clustering." (p.713 in Sweeting (2009))

In his discussion, he focuses on symmetric (stable) pure strategy Bayesian Nash equilibria in a coordination game. To keep matters simple, I consider again the two-player game with payoff structure from Table 2. Assume that $\mathbf{x}_i^\top \beta_i = \alpha, i = 1, 2$ and $\Delta = \Delta_1 = \Delta_2 \geq 0$. As has been the case up to this point, private information is assumed to be independent across agents and from $\mathbf{x}$. Given the parametrization in his model, multiplicity always produces three symmetric equilibria, two of which are stable. Denote the choice probabilities in these two stable equilibria p^l and p^h, where $p_l < p_h$.

To "complete" the model, Sweeting assumes that the stable equilibria are selected with probability $\lambda_k, k = l, h$, whereas the unstable equilibrium is never selected. The outcome probability is then

$$\mathbb{P}(y_1, y_2|\mathbf{x}) = \sum_{k=l,h} \lambda_k \begin{pmatrix} 2 \\ y_1 + y_2 \end{pmatrix} (p^k)^{y_1+y_2}(1-p^k)^{2-y_1-y_2}, \quad \lambda_l + \lambda_h = 1. \tag{5}$$

Notice that with a unique equilibrium in the data, (5) corresponds to the distribution of a binomial random variable (with parameters 2 and p). Using simulations, Sweeting notes that "a mixture generates greater variance in the number of stations choosing a particular outcome than can be generated by a single binomial component" when $\Delta > 0$ (see p.723 in his paper). As he points out, this suggests that multiplicity provides additional information about the payoff structure of the game under analysis.

de Paula and Tang (2012) formalize and generalize this idea in many directions. For the basic insight, take expression (4) and compare two equilibria where $p_j^h(\mathbf{x}) > p_j^l(\mathbf{x})$. If $\Delta_i > 0$, it has to be the case that $p_i^h(\mathbf{x}) > p_i^l(\mathbf{x})$. (This is because $F_{u_i|\mathbf{x}}$ is increasing since it is a cdf.) On the other hand, if $\Delta_i < 0$ one must have $p_i^h(\mathbf{x}) < p_i^l(\mathbf{x})$. With two players, multiple equilibria are possible only if both Δ_1 and Δ_2 have the same sign (in contrast with the complete information case). Consequently, if Δ_1 and Δ_2 are positive, $p_1^h(\mathbf{x}) > p_1^l(\mathbf{x})$ if and only if $p_2^h(\mathbf{x}) > p_2^l(\mathbf{x})$. In other words, the equilibrium choice probabilities by player 2 are an increasing function of the equilibrium choice probabilities by player 1. Conversely, when both Δ_1 and Δ_2 are negative, the equilibrium choice probabilities by player 2 are a decreasing function of the equilibrium choice probabilities by player 1. Because of this monotonicity, the correlation of player actions across games will be positive when both Δ_1 and Δ_2 are positive and negative otherwise. Interestingly, because private types $u_i = 1, 2$ are assumed to be independent (conditionally on $\mathbf{x}$), if there is only one equilibrium in the data the actions will be uncorrelated and uninformative about $\Delta_i, i = 1, 2$. This gives the following specialization of Proposition 1 in de Paula and Tang (2012)):

Proposition 1 (de Paula and Tang (2012)) *Suppose $u_1 \perp\!\!\!\perp u_2|\mathbf{x}$. (i) For any given $\mathbf{x}$, multiple equilibria exist in the data-generating process if and only if $\mathbb{E}[y_1y_2|\mathbf{x}] \neq \mathbb{E}[y_1|\mathbf{x}]\mathbb{E}[y_2|\mathbf{x}]$; and (ii) if that is the case for some $\mathbf{x}$,*

$$sign\left(\mathbb{E}[y_1y_2|\mathbf{x}] - \mathbb{E}[y_1|\mathbf{x}]\mathbb{E}[y_2|\mathbf{x}]\right) = sign(\Delta_i), i = 1, 2$$

In the paper, the baseline utility $\mathbf{x}_i^\top\beta_i$ can be replaced by a generic function of $\mathbf{x}_i$ and the parameter Δ_i is also allowed to depend on $\mathbf{x}_i$, but I keep with the payoffs in Table 2 for simplicity and consistency. The result also holds for an arbitrary number of players and allows for asymmetric equilibria and situations where the number of equilibria is unknown. (Kline (2012) recently extends the idea of this result to environments with complete information.)

The subtle implication of the proposition is that multiplicity is informative about aspects of the model in ways that uniqueness is not. Of course, for the set of covariate values where there is only one equilibrium in the data the estimation can then be performed under various restrictions on utilities, $\Delta_i, i = 1, 2$ and $F_{\mathbf{u}|\mathbf{x}}$ (see Aradillas-Lopez (2010), Berry and Tamer (2006), and Bajari, Hong, Krainer, and Nekipelov (2010) for more details).

For games with more than two players, de Paula and Tang (2012) rely on this result to suggest a test for the hypothesis that there are multiple solutions in the data and, conditional on there being more than one equilibrium, on the sign of $\Delta_i, i = 1, 2$. To implement this test, they build on recent developments in the statistical literature on multiple comparisons (Romano and Wolf (2005)) (see the recent survey by Romano, Shaikh, and Wolf (2010)). Because the test proposed relies on conditional covariances, with discrete covariates it is implementable using well-known results in the multiple testing literature. Based on his model, Sweeting (2009) also suggests tests for multiple symmetric equilibria when the cardinality of the equilibrium set is known. The first is based on calculating the percentage of pairs of players whose actions are correlated. The other is a test of the null of a unique equilibrium against the alternative of exactly two equilibria using MLE.

Finally, I should note that the essential assumption of conditional independence of the latent variables $\mathbf{u}$ is also commonly found in dynamic games of incomplete information. Optimal decision rules in those settings involve not only equilibrium beliefs but continuation value functions that may change across equilibria. Nevertheless, the characterization of optimal policy rules in that context suggests that the existence of a unique equilibrium in the data can still be detected by the lack of correlation in actions across players of a given game as presented in the current paper much as in the static game. (The identification of $sign(\Delta_i)$ would require additional restrictions though.) Because most of the known methods for semi-parametric estimation of incomplete information (static or dynamic) rely on the existence of a single equilibrium in the data (see above), a formal test for the assumption of a unique equilibrium in the data-generating process can be quite useful.

4.3 Game Level Heterogeneity and Correlated Private Signals

To establish Proposition 1 in de Paula and Tang (2012), it is paramount that the latent variables be conditionally independent. Any association between u_1 and u_2 will lead to correlation in actions even under a unique equilibrium but also change the nature of equilibrium decision rules in important ways (i.e., $\mathbb{P}(y_j = 1|\mathbf{x}, u_i)$ in (3) is now a non-trivial function of u_i). Aradillas-Lopez (2010) suggests in a subsection an estimation procedure to handle cases allowing for correlated private values, but relies on the assumption that a single equilibrium is played in the data. Another example is Wan and Xu (2010), who nevertheless also require that a unique (monotone) Bayesian-Nash equilibrium be played in the data. As long as one is comfortable with the assumption of a single equilibrium in the data described previously, these methods can be used for estimation. To my knowledge, general results along the lines of de Paula and Tang (2012) that allow for both correlation of private values and multiplicity have not been proposed (though see the working paper version of that article for a discussion of some possible characterizations).

One empirically important form of association between latent variables that has nevertheless received some attention amounts to decomposing u_i into a public observed component ϵ_i and a privately observed component ν_i. Whereas we retain the assumption that ν_1 and ν_2 are (conditionally) independent, I assume for simplicity that the publicly observed errors take the form of a game-level shock $\epsilon_1 = \epsilon_2 = \epsilon$. In the empirical games literature, the presence of this "market" level shock is typically referred to as unobserved heterogeneity (even though ν_1 and ν_2 are themselves heterogeneous and unobserved within and across games).

The presence of ϵ prevents one from employing the results in de Paula and Tang (2012) much as the correlation in fully private u_1 and u_2 would. The practical solution is nevertheless to account for it by modelling the distribution of publicly observed latent shocks when a cross-section is employed or using a "market-invariant" fixed or (possibly correlated) random effect again under the assumption that a unique equilibrium occurs in the data for each particular market when panel data is available. In a dynamic context, Aguirregabiria and Mira (2007) for instance introduce game level shocks as a correlated random effect with finite support (in the tradition of Heckman and Singer (1984) for duration models). Grieco (2012) also models the distribution of publicly observed shocks in a static game estimated on a cross-section.

Alternatively Bajari, Hong, Krainer, and Nekipelov (2010) note that "[i]f a large panel data with a large time dimension for each market is available, both the nonparametric and semiparametric estimators can be implemented market by market to allow for a substantial amount of unobserved heterogeneity" (p.475). The requirement of a long panel helps circumvent the incidental parameters problem in this fairly nonlinear context. It should also be noted that, provided the equilibrium is the same for the different editions of the game within a particular "market", different equilibria are allowed across different markets. Similar ideas also appear in other papers of the literature such as Bajari, Benkard, and Levin (2007) or Pesendorfer and Schmidt-Dengler (2008). Even in the absence of long panels, Bajari, Hong, Krainer, and Nekipelov (2010) suggest a few interesting strategies such as the use of conditional likelihood methods when the ν errors are logistic or the use of Manski (1987)'s panel data rank estimator. These proposals are nevertheless not further developed in that paper and some caution might be warranted given the simultaneous equation nature of the problem (i.e., the presence of $p_j(\mathbf{x})$ among the regressors in (4) might have repercussions for the two step procedure since $p_j(\mathbf{x})$ is also affected by the game level shock ϵ).

5 Models of Social Interactions

Social interaction models have gained widespread attention since Manski (1993). Whereas that paper focused on linear social interaction systems where equilibrium is unique, Brock and Durlauf (2001) and Brock and Durlauf (2007) consider a model of social interactions with discrete choices where multiple equilibria are possible. In the model, the value of a particular choice depends on the distribution of actions among other individuals in the community. In what follows I adapt the expressions in Brock and Durlauf's papers to keep with the previous notation here. Normalizing the value of choosing zero to 0, the utility obtained from choosing 1 can be written as

$$g_i(\mathbb{P}(\mathbf{y}_{-i}|\mathbf{x}), \mathbf{x}, \theta) + u_i$$

(see page 238 in Brock and Durlauf (2001). The term $g_i(\mathbb{P}(\mathbf{y}_{-i}|\mathbf{x}), \mathbf{x}, \theta)$ encompasses what Brock and Durlauf call the private and the social utilities attached to this particular choice (or, given our normalization, the difference in those utilities of choosing 1 over 0). The social utility depends on the conditional probability that i places on the choices of others when making her choice. As before, u_i is a random utility component which is known to the individual but unknown to others and is identically and independently distributed across agents. Typically, u_i is taken to be logistically distributed.

The probability distribution that i puts on the choices of others is denoted by $\mathbb{P}(\mathbf{y}_{-i}|\mathbf{x})$ and is determined in equilibrium. Person i's choice is then given by

$$y_i = 1 \text{ if } g_i(\mathbb{P}(\mathbf{y}_{-i}|\mathbf{x}), \mathbf{x}, \theta) + u_i \geq 0. \tag{6}$$

As with incomplete information games, an equilibrium will consist of mutual best responses and self-consistent beliefs. Given the nonlinearities in the model, multiplicity is again a possibility. Note nevertheless that instead of comparing (equilibrium) expected utilities as in (3), the expression above compares utility functions that depend on the (equilibrium) distributions of actions. These two will differ for general parameterizations (possibly nonlinear in $\mathbb{P}$). This is the case, for example, when there is preference for conformity as discussed in Brock and Durlauf (2001) (p.239). This is an important distinction. In the main (linear) parameterization considered in that paper, they nevertheless coincide. This specification postulates that

$$g_i(\mathbb{P}(\mathbf{y}_{-i}|\mathbf{x}), \mathbf{x}, \theta) = \mathbf{x}_i^\top \beta + \Delta \mathbb{E}\left[\frac{\sum_{j\neq i} y_j}{N-1}\middle|\mathbf{x}\right]$$

where N is the group size as before and β and Δ are parameters to be estimated. Because

$$\mathbf{x}_i^\top \beta + \Delta \mathbb{E}\left[\frac{\sum_{j\neq i} y_j}{N-1}\middle|\mathbf{x}\right] = \mathbb{E}\left[\mathbf{x}_i^\top \beta + \Delta \frac{\sum_{j\neq i} y_j}{N-1}\middle|\mathbf{x}\right],$$

the (equilibrium) expected utility agrees with the utility at the expected (equilibrium) profile of actions. This is in particular the best response predicament in my example with two players and two actions under incomplete information when $\beta_1 = \beta_2$ and $\Delta_1 = \Delta_2$. A noteworthy difference between this setup and that in the previous section is that I do not assume that players' roles or labels (e.g., firm identity, husband or wife) can be assigned. This "anonimity" assumption is more natural in the social interactions systems where a large number of people is typically observed per group.

Because the equilibrium moment $\mathbb{E}\left[\sum_{j\neq i} y_j/(N-1)\middle|\mathbf{x}\right]$ is observable, it can be estimated by the average choice in this particular game and equilibrium (e.g., Brock and Durlauf (2007), p.58). When there is a small number of players (as is the case in Industrial Organization applications for example), the choice probabilities will not be reliably estimated by averaging choices within a game. In this case some combination of observations across games is unavoidable and I refer the reader to the discussion in the previous section. For most applications of social interaction models however, the number of agents within a game is much larger. Consequently, the average peer choice in a game will consistently estimate the average action within the game (see Proposition 6 in Brock and Durlauf (2001)). Because average peer choice will display little variability within a group, variation across games can then be exploited to estimate Δ and β.

I now briefly discuss estimation and additional topics in this class of models in two different subsections.

5.1 Estimation

In this context, two alternative estimators are suggested in Bisin, Moro, and Topa (2011) (building on Moro (2003)). The first procedure estimates the game via Maximum Likelihood imposing the self-consistency equilibrium conditions. Because here a parameter vector might induce multiple equilibria for a given data set, the estimation proceeds by selecting that equilibrium which maximizes the likelihood (for a given parameter vector) and subsequently maximizing the likelihood function over the parameter space. (This resembles the procedure suggested in Chen, Tamer, and Torgovitsky (2011) which suggests profiling the equilibrium selection mechanism in a sieve-Maximum Likelihood procedure.) The second procedure is a plug-in estimator where $\mathbb{E}\left[\sum_{j\neq i} y_j/(N-1)\middle|\mathbf{x}\right]$ is replaced by the average peer choice within the game and parameters are then estimated via Maximum Likelihood. The log-likelihood function in the case of my example is given by:

$$\sum_{g=1}^{G}\sum_{i\in g}\ln\left[1-F_u\left(-\mathbf{x}_i^\top\beta-\Delta\frac{\sum_{j\neq i,j\in g} y_j}{N-1}\right)\right],$$

where $i \in g$ indicates that person i belongs to neighborhood g. This estimator can be seen as a two-step estimator akin to the procedure outlined in the previous section for incomplete information games. As was the case in the previous section, this estimator avoids the calculation of all the equilibria for a given parameter value. In response to the computational difficulties that this entails, they advise using the two-step estimator as an initial guess in the direct estimator. Similar two-step procedures with (possibly) group-level unobservables are also suggested by Shang and Lee (2011). Bisin, Moro and Topa present Monte-Carlo evidence in a model with possibly many equilibria for certain parameter configurations that highlights the computational costs and statistical properties of the two estimators. Because the asymptotic approximations rely on $N \to \infty$, I must also point out that the econometric estimators in such large population games might present some delicate issues given the dependence in equilibrium outcomes within a game as the number of players grows. This is a topic of ongoing research (see, for example, Menzel (2010) and Song (2012)).

5.2 Additional Topics

As was the case in the previous section, group level unobserved heterogeneity is potentially important in many applications. Ignoring it essentially rules out an important channel of unobserved contextual effects (or correlated effects) in the terminology coined by Manski (1993). Section 4 of Brock and Durlauf (2007) also discusses a series of potential scenarios that would allow the model to identify (at least partially) the parameters of interest. Those include the use of panels, restrictions on the distribution of unobserved group shocks (i.e., large support, stochastic monotonicity, unimodality) as well as other features of the model (i.e., linearity in terms of group shocks). Some of these restrictions may also be useful in the incomplete information games described previously.

6 Discussion

I will finish this survey with a brief discussion on post-estimation analysis, potential areas of development and applications.

6.1 Counterfactuals and Post-Estimation

Once in possession of point- or set-estimates for the parameters of interest, one may be interested in the effect of counterfactual policies. This goal is after all behind the very development of the Cowles Comission agenda delineated in the beginning of this article. If the equilibrium selection mechanism is included in the structure as an estimation object, one has a complete model which can be simulated to generate counterfactual distributions after the introduction of alternative policies (see, e.g., Bajari, Hong, and Ryan (2010), p.1537).

As pointed out earlier, one must nevertheless be cautious about the policy-invariance of the estimated equilibrium selection mechanism. Quoting Berry, Pakes, and Ostrovsky (2007), "though our assumptions are sufficient to use the data to pick out the equilibrium that was played in the past, they do not allow us to pick out the equilibrium that would follow the introduction of a new policy. On the other hand, the (...) estimates should give the researcher the ability to examine what could happen after a policy change (say, by examining all possible post-policy-change equilibria)" (p.375). This is done for example in Ciliberto and Tamer (2009) where a range of possible counterfactual outcomes is provided for an intervention repealing a particular piece of legislation on the entry behavior of airlines in their empirical analysis. Of course, any counterfactual analysis (with or without an estimated equilibrium selection policy) would require the computation of all equilibria though only for the range of parameters estimated.

An additional motivation for including the equilibrium selection mechanism as an estimation object is to retrospectively learn about the process and covariates determining which equilibria come to be played in the data. Even when the game is estimated under the assumption that a unique equilibrium is played in the data, possession of estimated parameters allows one to go back and calculate all the potential equilibria for a particular parametric configuration. In their study of stock analyst's recommendations, for instance, Bajari, Hong, Krainer, and Nekipelov (2010) notice the existence of multiple equilibria before General Attorney Eliot Spitzer of New York launched a series of investigations on conflict of interest, one of these equilibria yielding much more optimistic ratings than those granted in the equilibrium post-Spitzer.

6.2 Potential Avenues for Future Research

In the previous sections I tried to present many of the tools used in the econometric analysis of games with multiple equilibria. There are nevertheless still much to be understood in these settings. One interesting avenue which appears in some of the papers cited here is the connection with panel data methods. There, just as under multiplicity the distribution of outcomes in game theoretic models is a mixture over equilibrium-specific outcome distributions, the observable distribution of outcomes in panel data models is a mixture over the distribution of individuals effects. Important idiosyncrasies such as the (typical) finiteness of the equilibrium set (which would correspond to a finite support for the individual effects) may help bring in interesting technical results in the panel literature to shed light into some of the properties of econometric game theoretic models. Examples of studies in that vein are Hahn and Moon (2010) and Bajari, Hahn, Hong, and Ridder (2011). Here a important caveat already mentioned earlier is that the cardinality of the equilibrium set, $|\mathcal{E}(\theta, F, \mathbf{x})|$, will depend on the covariates and parametric configurations whereas the support of the individual effects in the usual panel data model suffers no such restrictions. This might introduce important complications.

In a similar fashion, Grieco (2012) and Chen, Tamer, and Torgovitsky (2011) suggest treating the equilibrium selection mechanims as a (possibly infinite dimensional) nuisance parameter which is concentrated out in a profile sieve-MLE procedure aimed at estimating semiparametric partially identified models. Again in this case, the dependence of the the cardinality of the equilibrium set, $|\mathcal{E}(\theta, F, \mathbf{x})|$, on the covariates and parametric configurations might introduce subtle complications as the class of functions which contain the equilibrium set might have to vary with θ, F and $\mathbf{x}$ to accommodate this dependence.

6.3 Applications

Here I discussed some of the commonly analyzed models where multiplicity shows up prominently in the econometric analysis of games and the usual procedures to handle this feature. I left out other models where nonuniqueness is rampant but so far have been given less attention. One example are static models of network formation à la Jackson and Wolinsky (1996). Whereas the solution concepts used in that literature differ from those used in the non-cooperative games studied above, it is not not uncommon to encounter many possible stable networks for a given payoff configuration. Those difficulties are perhaps compounded in econometric analogs of network formation models because equilibrium existence results for heterogenous payoffs are less readily available in the theory literature and computational challenges in their analysis are notorious.

For space considerations, I have only briefly mentioned empirical applications and focused on more methodological aspects of the analysis. Many of the papers cited here nevertheless provide (or actually even focus) on empirical questions (e.g., Sweeting (2009) or Ciliberto and Tamer (2009)) and many other examples in the empirical IO or peer effects literatures can be enumerated. Recent applications in other areas of Economics are nevertheless available (e.g., Card and Giuliano (2011) or Todd and Wolpin (2012)) and are likely to become more common.

References

Ackerberg, D., and G. Gowrinsankaran (2006): "Quantifying Equilibrium Network Externalities in the ACH Banking Industry," *Rand Journal of Economics*, 37, 738–761.

Aguirregabiria, V., and P. Mira (2007): "Sequential Estimation of Dynamic Discrete Games," *Econometrica*, 75(1), 1–53.

——— (2008): "Structural Estimation of Games with Multiple Equilibria in the Data," *Working Paper*.

Andrews, D., S. Berry, and P. Jia (2004): "Confidence Regions for Parameters in Discrete Games with Multiple Equilibria, With an Application to Discount Chain Store Location," *Working Paper*.

Aradillas-Lopez, A. (2010): "Semi-parametric Estimation of Simultaneous Games with Incomplete Information," *Journal of Econometrics*, 157(2), 409–431.

Aradillas-Lopez, A., and E. Tamer (2008): "The Identification Power of Equilibrium in Simple Games," *Journal of Business and Economic Studies*, 26, 261.

Bajari, P., L. Benkard, and J. Levin (2007): "Estimating Dynamic Models of Imperfect Competition," *Econometrica*, 75, 1331–1370.

Bajari, P., J. Hahn, H. Hong, and G. Ridder (2011): "A Note on Semiparametric Estimation of Finite Mixtures of Discrete Choice Models with Application to Game Theoretic Models," *International Economic Review*, 52.

Bajari, P., H. Hong, J. Krainer, and D. Nekipelov (2010): "Estimating Static Games of Incomplete Information," *Journal of Business and Economic Studies*, 28(4).

Bajari, P., H. Hong, and S. Ryan (2010): *Econometrica*78, 1529–1568.

Beresteanu, A., I. Molchanov, and F. Molinari (2009): "Sharp Identification Regions in Models with Convex Predictions: Games, Individual Choice, and Incomplete Data," CeMMAP Working Paper # CWP27/09.

Berry, S., A. Pakes, and M. Ostrovsky (2007): "Simple Estimators for the Parameters of Dynamic Games, With Entry/Exit Examples," *Rand Journal of Economics*, 38, 373399.

Berry, S., and E. Tamer (2006): "Empirical Models of Oligopoly Entry," in *Advances in Economics and Econometrics, Theory and Applications: Ninth World Congress of the Econometric Society*, ed. by R. Blundell, W. Newey, and T. Persson. Cambridge: Cambridge University Press.

Berry, S. T. (1992): "Estimation of a Model of Entry in the Airline Industry," *Econometrica*, 60, 889–917.

Bisin, A., A. Moro, and G. Topa (2011): "Empirical Content of Models with Multiple Equilibria in Economies with Social Interactions," *NBER Working Paper*.

Bjorn, P., and Q. Vuong (1984): "Simultaneous Equations Models for Dummy Endogenous Variables: A Game Theoretic Formulation with an Application to Labor Force Participation," Caltech Working Paper 537.

Bresnahan, T., and P. Reiss (1990): "Entry in Monopoly Markets," *Review of Economic Studies*, 57, 57–81.

Bresnahan, T. F., and P. C. Reiss (1991): "Entry and Competition in Concentrated Markets," *Journal of Political Economy*, 99.

Brock, W., and S. Durlauf (2001): "Discrete Choice with Social Interactions," *Review of Economic Studies*, 62, 235–260.

Brock, W., and S. Durlauf (2007): "Identification of Binary Choice Models with Social Interactions," *Journal of Econometrics*, 140(1), 52–75.

Card, D., and P. Giuliano (2011): "Peer Effects with Multiple Equilibria in the Risky Behavior of Friends," NBER Working Paper.

Chen, X., E. Tamer, and A. Torgovitsky (2011): "Sensitivity Analysis in Semiparametric Likelihood Models," *Working Paper*.

Chesher, A., and A. Rosen (2012): "Simultaneous equations models for discrete outcomes: coherence, completeness, and identification," CeMMAP Working Paper.

Ciliberto, F., and E. Tamer (2009): "Market Structure and Multiple Equilibria in Airline Markets," *Econometrica*, 77, 1791–1828.

de Paula, A. (2009): "Inference in a Synchronization Game with Social Interactions," *Journal of Econometrics*, 148, 56–71.

de Paula, A., and X. Tang (2012): "Inference of Signs of Interaction Effects in Simultaneous Games with Incomplete Information," *Econometrica*, 80, 143–172.

Echenique, F., and I. Komunjer (2009): "Testing Models with Multiple Equilibria by Quantile Methods," *Econometrica*, 77, 1281–1298.

Galichon, A., and M. Henry (2011): "Set Identification in Models with Multiple Equilibria," *Review of Economic Studies*, 78, 1264–1298.

Grieco, P. (2012): "Discrete Choice Games with Flexible Information Structure: An Application to Local Grocery Markets," Working Paper, Pennsylvania State University.

Hahn, J., and R. Moon (2010): "Panel Data Models with Finite Number of Multiple Equilibria," *Econometric Theory*, 26, 863–881.

Hahn, J., and E. Tamer (2004): "Corrigendum on "Incomplete Bivariate Discrete Response Model with Multiple Equilibria"," Working Paper.

Hall, P., A. Neeman, R. Pakyari, and R. Elmore (2005): "Nonparametric Inference in Multivariate Mixtures," *Biometrika*, 92(3).

Heckman, J. J. (1990): "Varieties of Selection Bias," *American Economic Review, Papers and Proceedings*, 80, 313–318.

Heckman, J. J., and B. Singer (1984): "A Method for Minimizing the Impact of Distributional Assumptions in Econometric Models for Duration Data," *Econometrica*, 52, 271–320.

Honoré, B., and A. de Paula (2010): "Interdependent Durations," *Review of Economic Studies*, 77, 1138–1163.

Hotz, J., and R. Miller (1993): "Conditional Choice Probabilities and the Estimation of Dynamic Models," *Review of Economic Studies*, 60, 397–429.

Jackson, M., and A. Wolinsky (1996): "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory*, 71, 44–74.

Jia, P. (2008): "What Happens When Wal-Mart Comes to Town: An Empirical Analysis of the Discount Retailing Industry," *Econometrica*, pp. 1263–1316.

Jovanovic, B. (1989): "Observable Implications of Models with Multiple Equilibria," *Econometrica*, 57, 1431–1437.

Kasy, M. (2012): "Nonparametric Inference on the Number of Equilibria," *Working Paper*.

Khan, S., and E. Tamer (2010): "Irregular Identification, Support Conditions and Inverse Weight Estimation," *Econometrica*, 78, 2021 – 2042.

Kline, B. (2012): "Identificaton of Complete Information Games," *Working Paper*.

Koopmans, T., and O. Reiersol (1950): "The Identification of Structural Characteristics," *Annals of Mathematical Statistics*, 21, 165–181.

Lazzati, N. (2012): "Treatment Response with Social Interactions: Partial Identification via Monotone Comparative Statics," *Working Paper*.

Mailath, G. (1998): "Do People Play Nash Equilibrium? Lessons from Evolutionary Game Theory," *Journal of Economic Literature*, 36(3), 1347–1374.

Manski, C. (1987): "Semiparametric Analysis of Random Effects Linear Models From Binary Response Data," *Econometrica*, 55, 357–362.

——— (1988): "Identification of Binary Response Models," *Journal of the American Statistical Association*, 83, 729–738.

——— (1993): "Identification of Endogenous Social Effects: The Reflection Problem," *Review of Economic Studies*, 60, 531–542.

Menzel, K. (2010): "Inference for Large Games with Exchangeable Players," *Working Paper*.

Molinari, F., and A. Rosen (2008): "The Identification Power of Equilibrium in Games: The Supermodular Case," *Journal of Business and Economic Studies*, 26, 297–302.

Moon, R., and F. Schorfheide (2012): "Bayesian and Frequentist Inference in Partially Identifeid Models," *Econometrica*, 80, 755–782.

Moro, A. (2003): "The Effect of Statistical Discrimination on Black-White Wage Inequality: Estimating a Model with Multiple Equilibria," *International Economic Review*, 44, 457–500.

Myerson, R. (1991): *Game Theory: The Analysis of Conflict*. Cambridge: Harvard University Press.

Pakes, A., J. Porter, K. Ho, and J. Ishii (2011): "Moment Inequalities and Their Applications," *Working Paper*.

Pesendorfer, M., and P. Schmidt-Dengler (2008): "Asymptotic Least Square Estimator for Dynamic Games," *Review of Economic Studies*, 75, 90128.

Romano, J., A. Shaikh, and M. Wolf (2010): *Annual Review of Economics*2, 75–104.

Romano, J. P., and M. Wolf (2005): "Stepwise Multiple Testing as Formalized Data Snooping," *Econometrica*, 73(4), 1237–1282.

Shang, Q., and L.-f. Lee (2011): "Two-Step Estimation of Endogenous and Exogenous Group Effects," *Econometric Reviews*, 30(2), 173–207.

Song, K. (2012): "Econometric Inference on a Large Bayesian Game," *Working Paper*.

Sweeting, A. (2009): "The Strategic Timing of Radio Commercials: An Empirical Analysis Using Multiple Equilibria," *RAND Journal of Economics*, 40(4), 710–742.

Tamer, E. (2003): "Incomplete Simultaneous Discrete Response Model with Multiple Equilibria," *Review of Economic Studies*, 70, 147–167.

——— (2010): "Partial Identification in Econometrics," *Annual Reviews of Economics*, 2, 167–195.

Todd, P., and K. Wolpin (2012): "Estimating a Coordination Game within the Classroom," *Working Paper*.

Wan, Y., and H. Xu (2010): "Semiparametric Estimation of Binary Decision Games of Incomplete Information with Correlated Private Signals," Working Paper, Pennsylvania State University.

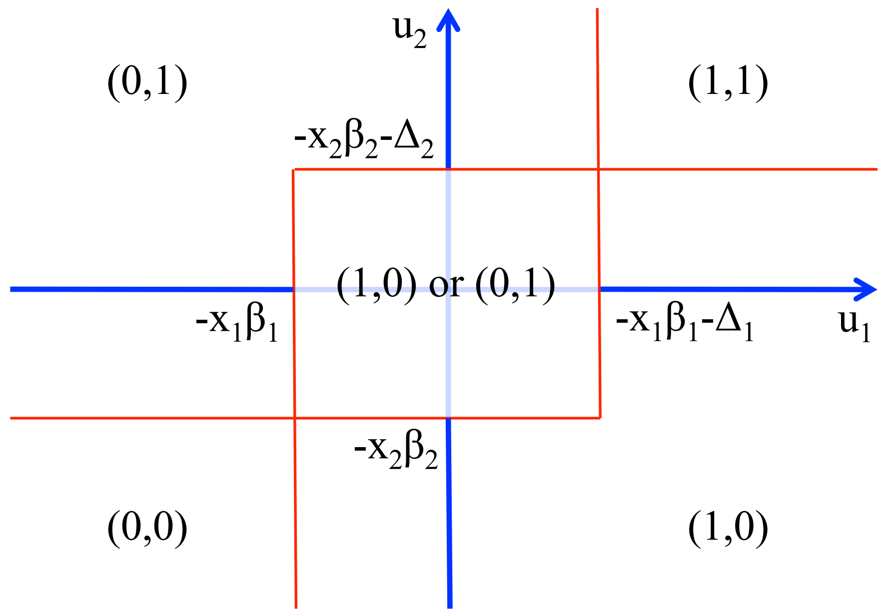

Figure 1

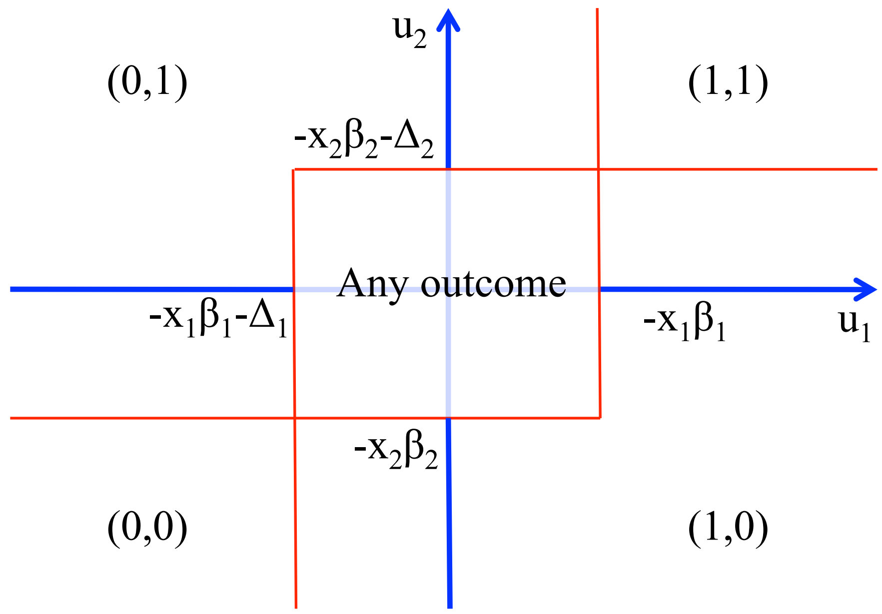

Figure 2

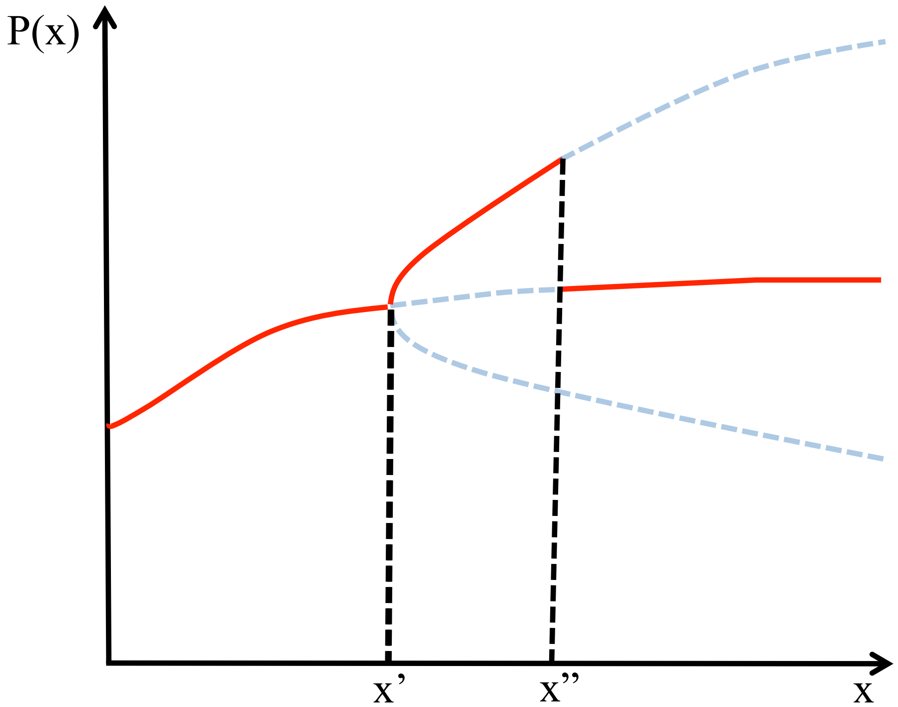

Figure 3