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# Bounds On Treatment Effects On Transitions

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# Bounds On Treatment Effects On Transitions*

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**Abstract**

This paper considers identification of treatment effects on conditional transition probabilities. We show that even under random assignment only the instantaneous average treatment effect is point identified. Because treated and control units drop out at different rates, randomization only ensures the comparability of treatment and controls at the time of randomization, so that long run average treatment effects are not point identified. Instead we derive informative bounds on these average treatment effects. Our bounds do not impose (semi)parametric restrictions, as e.g. proportional hazards. We also explore various assumptions such as monotone treatment response, common shocks and positively correlated outcomes that tighten the bounds.

Keywords: Partial identification, duration model, randomized experiment, treatment effect
JEL classification: C14, C41

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# 1 Introduction

We consider the effect of an intervention if the outcome is a transition from an initial to a destination state. The population of interest is a cohort of units that are in the initial state at the time zero. Treatment is assigned to a subset of the population either at the time zero or at some later time. Initially we assume that the treatment assignment is random. One main point of this paper is that even if the treatment assignment is random, only certain average effects of the treatment are point identified. This is because the random assignment of treatment only ensures comparability of the treatment and control groups at the time of randomization. At later points in time treated units with characteristics that interact with the treatment to increase/decrease the transition probability relative to similar control units leave the initial state sooner/later than comparable control units, so that these characteristics are under/over represented among the remaining treated relative to the remaining controls and this confounds the effect of the treatment.

The confounding of the treatment effect by selective dropout is usually referred to as dynamic selection. Existing strategies that deal with dynamic selection rely heavily on parametric and semi-parametric models. An example is the approach of Abbring and Van den Berg (2003) who use the Mixed Proportional Hazard (MPH) model (their analysis is generalized to a multistate model in Abbring, 2008). In this model the instantaneous transition or hazard rate is written as the product of a time effect, the effect of the intervention and an unobservable individual effect. As shown by Elbers and Ridder (1982) the MPH model is nonparametrically identified, so that if the multiplicative structure is maintained, identification does not rely on arbitrary functional form or distributional assumptions beyond the assumed multiplicative specification. A second example is the approach of Heckman and Navarro (2007) who start from a threshold crossing model for transition probabilities. Again they establish semi-parametric identification, although their model requires the presence of additional covariates besides the treatment indicator that are independent of unobservable errors and have large support.

In this paper we ask what can be identified if the identifying assumptions of the semi-parametric models do not hold. We show that, because of dynamic selection, even under random assignment we cannot point identify most average treatment effects of interest. However, we derive sharp bounds on non-point-identified treatment effects, and show under what conditions they are informative. Our bounds are general, since beyond random assignment, we make no assumptions on functional form and additional covariates, and we allow for arbitrary heterogenous treatment effects as well as arbitrary unobserved heterogeneity. The bounds can also be applied if the treatment assignment is unconfounded by creating bounds conditional on the covariates (or the propensity score) that are averaged over the distribution of these covariates (or propensity score).

Besides these general bounds we derive bounds under additional (weak) assumptions like monotone treatment response and positively correlated outcomes. We relate these assumptions to the assumptions made in the MPH model and to assumptions often made in discrete duration models and structural models. The additional assumptions often tighten the bounds considerably. We also discuss how to apply our various identification results to construct asymptotically valid confidence intervals for the respective treatment effects.

There are many applications in which we are interested in the effect of an intervention on transition probabilities/rates. The Cox (1972) partial likelihood estimator is routinely used to estimate the effect of an intervention on the survival rate of subjects. Transition models are used in several fields. Van den Berg (2001) surveys the models used and their applications. These models also have been used to study the effect of interventions on transitions. Examples are Ridder (1986), Card and Sullivan (1988), Bonnal et al. (2007), Gritz (1993), Ham and LaLonde (1996), Abbring and Van den Berg (2003), and Heckman and Navarro (2007). A survey of models for dynamic treatment effects can be found in Abbring and Heckman (2007).

An alternative to the effect of a treatment on the transition rate is its effect on the cdf of the time to transition or its inverse, the quantile function. This avoids the problem of dynamic selection. From the effect on the cdf we can recover the effect on the average duration, but we cannot obtain the effect on the conditional transition probabilities, so that the effect on the cdf is not informative on the evolution of the treatment effect over time. This is a limitation since there are good reasons why we should be interested in the effect of an intervention on the conditional transition probability or the transition/hazard rate. One important reason is the close link between the hazard rate and economic theory (Van den Berg (2001)). Economic theory often predicts how the hazard rate changes over time. For example, in the application to a job bonus experiment considered in this paper, labor supply and search models predict that being eligible for a bonus if a job is found, increases the hazard rate from unemployment to employment. According to these models there is a positive effect only during the eligibility period, and the effect increases shortly before the end of the eligibility period. The timing of this increase depends on the arrival rate of job offers and is an indication of the control that the unemployed has over his/her re-employment time. Any such control has important policy implications. This can only be analyzed by considering how the effect on the hazard rate changes over time.

The evolution of the treatment effect over time is of key interest in different fields. For instance, consider two medical treatments that have the same effect on the average survival time. However, for one treatment the effect does not change over time while for the other the survival rate is initially low, e.g., due to side effects of the treatment, while after that initial period the survival rate is much higher. As another example, research on the effects of active labor market policies often documents a large negative lock-in effect and a later positive effect once the program has been completed, see e.g. the survey by Kluve et al. (2007).

We apply our bounds and confidence intervals to data from a job bonus experiment previously analyzed by Meyer (1996) among others. As discussed above economic theory has specific predictions for the dynamic effect of a re-employment bonus with a finite eligibility period. Meyer (1996) estimates these dynamic effects using an MPH model. We study what can be identified if we rely solely on random assignment and some additional (weak) assumptions.

In section 2 we define the treatment effects that are relevant if the outcome is a transition. Section 3 discusses their point or set identification in the case that the treatment is randomly assigned. This requires us to be precise on what we mean by random assignment in this setting. In section 4 we explore additional assumptions that tighten the bounds. In section 5 we derive the confidence intervals. Section 6 illustrates the bounds for the job bonus experiment. Section 7 concludes.

# 2 Setup

## 2.1 Motivating example

In this paper we consider identification of the effect of a treatment on the conditional transition probability, usually referred to as the transition rate or the hazard rate. Effects on transition rates are important in many applications. The job-bonus experiment considered in the application in this paper is one example. The experiment paid re-employment bonuses to unemployed individuals in the randomized treatment group who found employment within the first 11 weeks of unemployment. The fact that the bonus is only paid during the first 11 weeks has several interesting implications. Standard labor supply and search models predict that being eligible for the bonus should increase the transition rate from unemployment to employment during the 11 week eligibility period, but should have no effect after the end of the eligibility period. Another prediction is that the transition rate should increase shortly before the end of the eligibility period, as the unemployed run out of time to collect the bonus. These theoretical predictions can only be studied by examining how the effect of the job-bonus varies with time in unemployment, that is by studying the effect on the transition rate during the eligibility period, shortly before the end of the eligibility period and after the end of the eligibility period. Effects on the transition rate are also relevant in many other applications, including evaluations of medical treatments and active labor market policies.

The job-bonus experiment includes random treatment assignment, which ensures comparability of the treatment group and the control group at the time of randomization. At later time points some unemployed individuals have found a job, and this creates dynamic selection, that even under the initial random assignment might confound the comparability of the treatment and control groups. This is most easily seen if the fraction that has found a job differs between the two groups, and if those who have found a job have more favourable characteristics than those who remain unemployed. Under these conditions the remaining individuals in the treatment group will be negatively (positively) selected if the fraction remaining in unemployment is lower (higher) in the treatment group than in the control group. Moreover, even if the fraction still unemployed is the same in the treatment group and the control group we might still face a selection problem. In the job-bonus experiment, it could, for instance, be the case that individuals that respond to the bonus come from different parts of the ability distribution compared to those who find a job without the bonus. The implication of this is that the ability distribution differs between the treatment and the control groups, even if the fraction that has found a job is the same in the two groups. All this constitutes the dynamic selection problem that is addressed in this paper.

Previous studies that deal with the dynamic selection problem have mostly used parametric and semi-parametric models. For instance, Meyer (1996) uses a proportional hazard (PH) model to study how the effect of the job-bonus experiment considered in this paper varies before and after the 11 week eligibility period. A more general alternative to the PH model is to use a Mixed Proportional Hazard (MPH) model. In this model the instantaneous transition or hazard rate is written as the product of a time effect, the effect of the intervention and an unobservable individual effect. This model, however, imposes a multiplicative structure, a homogeneous treatment effect as well as other restrictions. In this paper we instead consider what can be identified if we rely solely on random assignment and do not impose the parametric restrictions that are implicit in the MPH model and other parametric and semi-parametric models.

## 2.2 Average treatment effect on transitions

We discuss the definition and identification of treatment effects on transition rates in discrete time with transitions occurring at times $t = 1, 2, \ldots$.[1] We assume that treatment is assigned at the beginning of the first period and that each unit is either always treated or always non-treated. In section 3.1 we generalize these results by allowing the treatment to start in any time period. Let the potential outcome $Y_t^1$ be the indicator of a transition in period $t$ if treated and similarly $Y_t^0$ is the potential outcome if non-treated.

In any definition of the causal effect of a treatment on the transition rate we must account for the dynamic selection that was discussed in the previous subsection. If we do not specify a model for the transition rate we need to find another way to maintain the comparability of the treatment and control groups over time. The approach that we take in this paper is to consider average transition rates where the average is taken over the same population for both treated and controls (or in general for different treatment arms). We initially propose to average over the subpopulation of individuals who would have survived until time $t$ if treated. This is the analogue of the average effect on the treated considered in the static treatment effect literature. This leads to the following definition

**Definition 1** *The causal effect on the transition probability of the treated survivors in $t$ is the Average Treatment Effect on Treated Survivors (ATETS) defined by*

$$\text{ATETS}_t = \mathbb{E}\left(Y_t^1 | Y_{t-1}^1 = 0, \ldots, Y_1^1 = 0\right) - \mathbb{E}\left(Y_t^0 | Y_{t-1}^1 = 0, \ldots, Y_1^1 = 0\right).$$

The differential selection only starts after the first period and the $\text{ATETS}_t$ controls for that by comparing the transition rates for individuals with a common survival experience.[2]

Note that we are only concerned with the comparability of the treatment and control groups over the spell, i.e. with the different levels of dynamic selection in the two groups. If we keep the treatment and control groups comparable over time, there is still the question of how to interpret the time path of the average treatment effect over the spell. In this paper we do not try to decompose this path into the average treatment effect for a population of unchanging composition and a selection effect relative to this population. We do not define the treatment effect for this population of unchanging composition, but rather for a population with a composition that changes over time due to dynamic selection. The dynamic selection is made equal in the treatment and control groups, so that the treatment effect is not confounded by dynamic selection. Again this is analogous to the difference between the Average Treatment Effect and the Average Treatment Effect on the Treated in the case of a static treatment effect where the latter is defined for the population selected for treatment and the treatment effect is for this selective population.

[1] The definition of causal effects in continuous time adds technical problems (see e.g. Gill and Robins (2001)) that would distract from the conceptual issues.

[2] In Appendix C we also consider the average effect for the subpopulation of individuals who would have survived until $t$ under both treatment and no treatment.

## 3 Bounds on average treatment effects on transitions

We now consider identification of the $\text{ATETS}_t$ under random treatment assignment. Let $D$ be the indicator of treatment status. In our setting we have the following random assignment assumption

**Assumption 1** *Random assignment of treatment* $D \perp \{Y_t^1,\, Y_t^0,\, t = 1, 2, \ldots\}$

For illustrative purposes we first consider the two period case where the transition occurs in period 1, period 2 or after period 2. The main results of this paper can be easily understood in this setting. For every member of the population we have a vector of potential outcomes $Y_1^1, Y_1^0, Y_2^1, Y_2^0$, and the treatment indicator $D$. Let $Y_t$ be the observed indicator of a transition in period $t$. The observed outcome is related to the potential outcomes by the observation rule

$$Y_t = DY_t^1 + (1 - D)Y_t^0. \tag{1}$$

Under this assumption we can relate the observed and potential transition probabilities:

$$\mathbb{E}(Y_1|D = d) = \mathbb{E}(Y_1^d), \tag{2}$$

$$\mathbb{E}(Y_2|Y_1 = 0, D = d) = \mathbb{E}(Y_2^d|Y_1^d = 0). \tag{3}$$

We distinguish between instantaneous or short-run effects and dynamic or long-run effects. The instantaneous effect is the ATETS in the first period of treatment:

$$\text{ATETS}_1 = \mathbb{E}(Y_1^1) - \mathbb{E}(Y_1^0).$$

Under Assumption 1 it follows from equation (2) that we can point identify the instantaneous treatment effect, namely

$$\text{ATETS}_1 = \mathbb{E}(Y_1^1) - \mathbb{E}(Y_1^0) = \mathbb{E}(Y_1|D = 1) - \mathbb{E}(Y_1|D = 0).$$

With two periods the dynamic treatment effect is:

$$\text{ATETS}_2 = \mathbb{E}(Y_2^1|Y_1^1 = 0) - \mathbb{E}(Y_2^0|Y_1^1 = 0).$$

This is the average treatment effect in the second period of treatment for those who survive under treatment in the first period. Because all that can be deduced from the data is in equations (2) and (3), which hold under Assumption 1, $\text{ATETS}_2$ is, in general, not point identified. However, the observed transition probabilities place restrictions on the potential ones, and these are used to derive sharp bounds on $\text{ATETS}_2$. The bounds are sharp in the sense that there exist feasible joint distributions of the potential outcomes which are consistent with the upper bound and the lower bound.

Let us consider the intuition behind the derivation of the bounds. Note that the average effect of interest, $\text{ATETS}_2$, is a function of the distribution of the potential outcomes $Y_2^1$ and $Y_2^0$ for the subpopulation with $Y_1^1 = 0$. We have argued that we cannot point identify $\text{ATETS}_2$ using the observed data, but (2) and (3) provide partial information about the distribution of the potential outcomes $Y_2^1$ and $Y_2^0$. It is this partial information that we use to derive sharp bounds. Essentially, we obtain bounds on $\text{ATETS}_2$ by finding average values of $Y_2^1$ and $Y_2^0$ for the subpopulation with $Y_1^1 = 0$ that give the maximum and the minimum value of $\text{ATETS}_2$ under the restrictions given by the partial information in (2) and (3). Intuitively, one can think about this as a maximization and a minimization problem under a set of restrictions. Our derivations show that these maximization and minimization problems have closed-form solutions, and this gives our bounds.

For the case of arbitrary $t$ we need to introduce some additional notation. We use the notation $\overline{Y}_{t-1} = (Y_1, \ldots, Y_{t-1})$ and write 0 for the vector of zeros.

Our main result is presented in Theorem 1, which provides closed form expressions for the sharp bounds on $\text{ATETS}_t$.[3] Note that these bounds require no assumptions beyond random assignment. They allow, for instance, for arbitrary heterogeneity in treatment response. We explicitly show that the bounds are sharp. The bounds exist if $\Pr\left(\overline{Y}_{t-1} = 0 \mid D = 1\right) > 0$, because if this probability is 0 the subpopulation for which $\text{ATETS}_t$ is defined has no members.

**Theorem 1 (Bounds on ATETS)** *Suppose that Assumption 1 holds. If* $\Pr\left(\overline{Y}_{t-1} = 0 \mid D = 1\right) = 0$ *then* $\text{ATETS}_t$ *is not defined. If* $\Pr\left(\overline{Y}_{t-1} = 0 \mid D = 1\right) > 0$*, then we have the following sharp bounds*

$$\text{LB}_t \leq \text{ATETS}_t \leq \text{UB}_t,$$

*where*

$$\begin{aligned} \text{LB}_t \equiv{} & \Pr(Y_t = 1 \mid \overline{Y}_{t-1} = 0, D = 1) \\ & - \min\left\{1, \frac{1 - [1 - \Pr(Y_t = 1 \mid \overline{Y}_{t-1} = 0, D = 0)] \Pr\left(\overline{Y}_{t-1} = 0 \mid D = 0\right)}{\Pr(\overline{Y}_{t-1} = 0 \mid D = 1)}\right\}, \\ \text{UB}_t \equiv{} & \Pr(Y_t = 1 \mid \overline{Y}_{t-1} = 0, D = 1) \\ & - \max\left\{0, \frac{\Pr(Y_t = 1 \mid \overline{Y}_{t-1} = 0, D = 0) \Pr\left(\overline{Y}_{t-1} = 0 \mid D = 0\right) - 1}{\Pr(\overline{Y}_{t-1} = 0 \mid D = 1)} + 1\right\}. \end{aligned}$$

**Proof** See Appendix A.[4]

Next, consider the intuition behind these bounds using the job-bonus experiment as an illustration. Both the upper and the lower bound are increasing in the observed transition probability from unemployment to employment in the treatment group in period $t$, $\Pr(Y_t = 1 \mid \overline{Y}_{t-1} = 0, D = 1)$. This follows directly from the fact that we consider the average effect for treated individuals that remain in unemployment until time $t$. The bounds also depend on the observed transition probability in the control group, $\Pr(Y_t = 1 \mid \overline{Y}_{t-1} = 0, D = 0)$, but this relationship is more complicated than the relationship between the bounds and $\Pr(Y_t = 1 \mid \overline{Y}_{t-1} = 0, D = 1)$. In general we have that both the upper and the lower bound are decreasing in $\Pr(Y_t = 1 \mid \overline{Y}_{t-1} = 0, D = 0)$. The reason for this is that a high transition rate among the unemployed individuals in the control group allows for a larger counterfactual

[3] The bounds are for the case that the destination state is non-absorbing, as is the case in the bonus experiment. This implies that treated survivors could have experienced two consecutive transitions if they had not been treated. This is impossible if the destination state is indeed absorbing, e.g. death. The data have the treatment status and the time to the first transition, i.e. in the data we do not observe multiple transitions, but they could occur under the counterfactual treatment regime.

[4] In the working paper version (Vikström, Ridder, and Weidner (2015)) we provide a simplified proof for the two period case that was discussed above.

outcome under no treatment. Another important determinant of the bounds is the fraction in the treatment group that remains in unemployment until time $t$, $\Pr(\overline{Y}_{t-1} = 0 \,|\, D = 1)$. If this survival probability is small, there is more selection in the group of treated that remains in unemployment, i.e. more pronounced dynamic selection, leading to a larger difference between the upper and the lower bound.

From Theorem 1 we also have several other implications. Corollary 1 shows that if there is no dynamic selection, i.e., if $\Pr(\overline{Y}_{t-1} = 0 \,|\, D = 0) = 1$ and $\Pr(\overline{Y}_{t-1} = 0 \,|\, D = 1) = 1$, then the dynamic treatment effect $\text{ATETS}_t$ is point identified. If everyone survives the first $t-1$ periods we have under random treatment assignment in period 1 two groups of equal composition even in period $t$.

**Corollary 1 (Point identification)** $\text{ATETS}_t$ *is point identified if and only if both* $\Pr(\overline{Y}_{t-1} = 0 \,|\, D = 0) = 1$ *and* $\Pr(\overline{Y}_{t-1} = 0 \,|\, D = 1) = 1$.

The information in the bounds depends on the width of the implied interval. The best case is that none of the zero or one restrictions (imposed by the max and min in $\text{LB}_t$ and $\text{UB}_t$ above) is binding, and in that case the width of the bounds is

$$\text{UB}_t - \text{LB}_t = \frac{2 - \Pr(\overline{Y}_{t-1} = 0 \,|\, D = 1) - \Pr(\overline{Y}_{t-1} = 0 \,|\, D = 0)}{\Pr(\overline{Y}_{t-1} = 0 \,|\, D = 1)}.$$

This expression shows that the width of the bound is decreasing in $\Pr(\overline{Y}_{t-1} = 0 \,|\, D = 1)$ and $\Pr(\overline{Y}_{t-1} = 0 \,|\, D = 0)$. In the job-bonus application this implies that the width of the bound is directly related to the probability that unemployed individuals in the treatment group and in the control group remain in unemployment until time $t$.

### 3.1 Arbitrary time to treatment

So far we have considered the case with treatment assignment only at the beginning of the first period. We now consider a more general case in which the treatment could start in any time period. We assume that any treated unit remains treated in the subsequent periods. We denote the treatment indicator in period $t$ by $d_t$ and the treatment history up to and including period $t$ by $\overline{d}_t$. Let the potential outcome $Y_t^{\overline{d}_t}$ be an indicator of a transition in period $t$ if the treatment history up to and including $t$ is $\overline{d}_t$.

Let $\overline{d}_{1t}$ and $\overline{d}_{0t}$ be two specific treatment histories. We consider the average transition rate at $t$ for the subpopulation of individuals who would have survived until time $t$ under $\overline{d}_{1t}$:

$$\begin{aligned} &\text{ATETS}_t^{\overline{d}_{1t},\overline{d}_{0t}} \\ &\quad = \mathbb{E}\left(Y_t^{\overline{d}_{1t}} \,\middle|\, Y_{t-1}^{\overline{d}_{1,t-1}} = 0, \ldots, Y_1^{\overline{d}_{11}} = 0\right) - \mathbb{E}\left(Y_t^{\overline{d}_{0t}} \,\middle|\, Y_{t-1}^{\overline{d}_{1,t-1}} = 0, \ldots, Y_1^{\overline{d}_{11}} = 0\right). \end{aligned} \tag{4}$$

With treatment assignments in all periods we need a slightly different randomization assumption. Let $D_t$ be the an indicator of treatment in period $t$.[5] Since we assume that any treated unit remains treated in subsequent periods, $D_t = 1$ implies that the unit remains treated in the subsequent periods.

The relevant random assignment assumption is

[5]Note that under this definition a unit with $D_t = 1$ could either be treated or non-treated before $t$.

**Assumption 2** ***Sequential randomization among survivors*** *For all $t$ and $\overline{d}_s, s \geq t$, with the first $t-1$ components equal to 0,*

$$D_t \perp \left\{ Y_s^{\overline{d}_s}, s = t, t+1, \ldots \right\} \Big| D_{t-1} = 0,\ Y_{t-1}^0 = \cdots = Y_1^0 = 0.$$

This assumption implies that treatment is assigned randomly among survivors that have not been treated before.

The interpretation of the $\text{ATETS}_t^{\overline{d}_{1t}, \overline{d}_{0t}}$ depends on the treatments $\overline{d}_{0t}, \overline{d}_{1t}$. As before, we distinguish between instantaneous or short-run effects and dynamic or long-run effects. With two periods in which the treatment can start, the two instantaneous treatment effects are $\text{ATETS}_1^{1,0}$, which was discussed in section 3, and

$$\begin{aligned} \text{ATETS}_2^{01,00} &= \mathbb{E}(Y_2^{01} \mid Y_1^0 = 0) - \mathbb{E}(Y_2^{00} \mid Y_1^0 = 0) \\ &= \mathbb{E}(Y_2 \mid Y_1 = 0, D_1 = 0, D_2 = 1) - \mathbb{E}(Y_2 \mid Y_1 = 0, D_1 = 0, D_2 = 0). \end{aligned} \tag{5}$$

Here, the second equality follows from the sequential randomization assumption. The instantaneous treatment effects for $t > 2$ are identified using similar reasoning.

Regarding the dynamic treatment effects, we consider the effect in period $t$ of a treatment that starts in period 1 relative to a treatment that starts in a later period before period $t$ or after period $t$. We only discuss this case here, but the bounds for the case that treatment starts between periods 1 and $t$ can be derived in the same way. The relevant Average Treatment Effect on Survivors is $\text{ATETS}_t^{1,0}$ where 1 and 0 stand for $t$ vectors of 1 and 0, i.e. treatment in all periods and control in all periods, and is defined by

$$\text{ATETS}_t^{1,0} = \mathbb{E}\left[ Y_t^1 \,\middle|\, \overline{Y}_{t-1}^1 = 0 \right] - \mathbb{E}\left[ Y_t^0 \,\middle|\, \overline{Y}_{t-1}^1 = 0 \right].$$

Theorem 2 provides closed form expressions for the sharp bounds on $\text{ATETS}_t^{1,0}$ under sequential random assignment among survivors.

**Theorem 2 (Bounds on ATETS with arbitrary time to treatment)** *Suppose that Assumption 2 holds.* $\Pr\left(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1\right) = 0$ *then* $\text{ATETS}_t^{1,0}$ *is not defined.*

*If* $\Pr\left(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1\right) > 0$, *then we have the following sharp bounds*

$$\text{LB}_t^{1,0} \leq \text{ATETS}_t^{1,0} \leq \text{UB}_t^{1,0},$$

*where*

$$\begin{aligned} \text{LB}_t^{1,0} \equiv{} & \Pr(Y_t = 1 | \overline{Y}_{t-1} = 0, \overline{D}_t = 1) \\ & - \min\left\{ 1, \frac{1 - [1 - \Pr(Y_t = 1 | \overline{Y}_{t-1} = 0, \overline{D}_t = 0)] \Pr\left(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0\right)}{\Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1)} \right\}, \\ \text{UB}_t^{1,0} \equiv{} & \Pr(Y_t = 1 | \overline{Y}_{t-1} = 0, \overline{D}_t = 1) \\ & - \max\left\{ 0, \frac{\Pr(Y_t = 1 | \overline{Y}_{t-1} = 0, \overline{D}_t = 0) \Pr\left(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0\right) - 1}{\Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1)} + 1 \right\}. \end{aligned}$$

**Proof** See Appendix A.

Note the similarity to the bounds in Theorem 1 with only treatment assignment at the beginning of the first period. The only difference is that in the case of two time periods only units non-treated in both period 1 and 2 are used to identify the counterfactual outcome under no treatment. Under sequential randomization among survivors the non-treated in both periods are comparable to the those entering treatment in period 2, and this means that we can adjust for assignments in period 2 by censoring those that become treated in the second period. The same argument applies for the treatment effect in $t$.

# 4 Bounds on treatment effects on transitions under additional assumptions

## 4.1 Monotone Treatment Response, Common Shocks, and Positively Correlated Outcomes

The sharp bounds in the previous section did not impose any assumptions beyond random assignment. In this section, we explore the identifying power of additional assumptions. The assumptions that we make are implicit in parametric models as the MPH model, and also in the discrete duration models and structural models presented in this section. Consider the following discrete duration model for the control and treated outcomes

$$\begin{aligned} Y_{it}^0 &= I(\alpha_t + V_i - \varepsilon_{it} \geq 0), \\ Y_{it}^1 &= I(\alpha_t + \gamma_{it} + V_i - \varepsilon_{it} \geq 0). \end{aligned} \tag{6}$$

This discrete duration model has a composite error that is the sum of unobserved heterogeneity $V_i$ and a random shock $\varepsilon_{it}$. The model restricts the joint distribution of the potential outcomes. A less restrictive model has different random shocks $\varepsilon_{it}, \tilde{\varepsilon}_{it}$ that are independent, but even in this case the potential outcomes are positively correlated through their dependence on $V_i$. In the sequel we consider assumptions on the joint distribution of potential outcomes in different treatment arms, that are in line with the assumptions implicit in this model, but do not assume that the potential outcomes are exactly as in this model. These assumptions will be used in combination with a weaker version of the constant treatment effect assumption. In the above model the treatment has a positive effect on the survival time for all individuals and time periods if $\gamma_{it} \leq 0$ for all $i, t$. This is essentially the Monotone Treatment Response (MTR) assumption introduced by Manski (1997) and Manski and Pepper (2000). Since the assumptions introduced in this section do not rely on a particular discrete duration model they are consistent with nonproportional structural hazard models suggested by Van den Berg (2001).

To define MTR we denote the event of survival under both $\overline{d}_0(t)$ and $\overline{d}_1(t)$ by $S_t$.

**Assumption 3 (Monotone Treatment Response (MTR))** *For treatment paths $\overline{d}_{0t}, \overline{d}_{1t}$ we have that for all $i$ either*

$$\Pr\left(Y_{it}^{\overline{d}_{1t}} = 1 \middle| S_{i,t-1}\right) \geq \Pr\left(Y_{it}^{\overline{d}_{0t}} = 1 \middle| S_{i,t-1}\right),$$

*for all $t$, or*

$$\Pr\left(Y_{it}^{\overline{d}_{1t}} = 1 \middle| S_{i,t-1}\right) \leq \Pr\left(Y_{it}^{\overline{d}_{0t}} = 1 \middle| S_{i,t-1}\right),$$

*for all t.*

For $t = 1$ Assumption 3 implies that for all $i$

$$\Pr(Y_{i1}^{1} = 1) \geq \Pr(Y_{i1}^{0} = 1),$$

or

$$\Pr(Y_{i1}^{1} = 1) \leq \Pr(Y_{i1}^{0} = 1).$$

Note that it is assumed that the effect is either positive or negative for all $t$. This assumption can be relaxed at the expense of more complicated bounds.

Assumption 3 refers to the individual transition probability and not to the transition indicators. These individual transition probabilities are defined with respect to the distribution of the individual idiosyncratic shocks $\varepsilon_{it}$ in (6). The population transition probabilities that appear in the definition of the ATETS and in Theorem 1 are individual transition probabilities averaged over the distribution of the individual heterogeneity among the treated survivors.

Note that Assumption 3 does not imply a specific direction of the effects, it merely implies that the effects either are positive or negative for all individuals. For the job-bonus experiment considered in this paper this assumption rules out that the bonus increases the transition rate for some unemployed individuals and decreases the transition rate for others.

The next assumption restricts the joint distribution of potential outcomes in the treatment arms. The assumption essentially imposes that the outcomes in all treatment arms involve the same random shocks. Consider the discrete duration model in (6). If $\gamma_{it} \leq 0$ then the treated have a larger survival probability in $t$. Therefore the event that $i$ survives in $t$ if not treated, i.e. $Y_{it}^{0} = 0$, is equivalent to $\varepsilon_{it} \geq \alpha_t + V_i$, so that this event implies that $\varepsilon_{it} \geq \alpha_t + \gamma_{it} + V_i \geq 0$, i.e. $Y_{it}^{1} = 0$. Note that we assume that the random shock $\varepsilon_{it}$ is invariant under a change in treatment status. This is stronger than the assumption that the *distribution* of the random shocks is the same whether $i$ is treated or not. The latter assumption allows for independent random shocks $\varepsilon_{it}, \tilde{\varepsilon}_{it}$ in the model above, if we assume that they have the same distribution. In a structural model the random shocks are often invariant, as is illustrated in a simple job search model below.

**Assumption 4 (Common Shocks (CS))** *For all $i$, $t$ and treatment paths $\bar{d}_0(t)$ and $\bar{d}_1(t)$*

$$\Pr(Y_{it}^{\bar{d}_{1t}} = 0|S_{i,t-1}) \geq \Pr(Y_{it}^{\bar{d}_{0t}} = 0|S_{i,t-1}) \quad \Rightarrow \quad \Pr(Y_{it}^{\bar{d}_{1t}} = 0|S_{i,t-1}, Y_{it}^{\bar{d}_{0t}} = 0) = 1, \tag{7}$$

*and*

$$\Pr(Y_{it}^{\bar{d}_{1t}} = 0|S_{i,t-1}) \leq \Pr(Y_{it}^{\bar{d}_{0t}} = 0|S_{i,t-1}) \quad \Rightarrow \quad \Pr(Y_{it}^{\bar{d}_{0t}} = 0|S_{i,t-1}, Y_{it}^{\bar{d}_{1t}} = 0) = 1. \tag{8}$$

Because the right-hand side of (7) is equivalent to $\Pr(Y_{it}^{\bar{d}_{1t}} = 1|S_{i,t-1}, Y_{it}^{\bar{d}_{0t}} = 0) = 0$, it is also equivalent to $\Pr(Y_{it}^{\bar{d}_{1t}} = 1, Y_{it}^{\bar{d}_{0t}} = 0|S_{i,t-1}) = 0$, which in turn is equivalent to $\Pr(Y_{it}^{\bar{d}_{1t}} > Y_{it}^{\bar{d}_{0t}}|S_{i,t-1}) = 0$.

Assumption 4 is satisfied in structural models. Consider for instance a non-stationary job search model for an unemployed individual as in Van den Berg (1990) or Meyer (1996). The treatment is a re-employment bonus as discussed in Section 5 below. In each period a job offer is obtained with probability $p(t, V_i)$. Let $Y_{of,it}$ be the indicator of an offer in period $t$ and $Y_{of,it} = I(\varepsilon_{of,it} \in A(t, V_i))$ with $A(t, V_i)$ a set. If the job offer is not under control of $i$, the arrival process is the same under treatment and control. The reservation wage is denoted by $\xi^1_{it}$ for the treated and $\xi^0_{it}$ for the controls. In general (see Meyer (1996)) $\xi^1(t, V_i) \leq \xi^0(t, V_i)$, so that if $H$ is the wage offer c.d.f. we have the acceptance probabilities $1 - H(\xi^1(t, V_i)) \geq 1 - H(\xi^0(t, V_i))$. The acceptance indicators are $Y^0_{ac,it} = I(\varepsilon_{w,it} \geq \xi^0(t, V_i))$ and $Y^1_{ac,it} = I(\varepsilon_{w,it} \geq \xi^1(t, V_i))$ with $\varepsilon_{w,it}$ the wage offer. Because $Y^0_{it} = Y_{of,it} Y^0_{ac,it}$ and $Y^1_{it} = Y_{of,it} Y^1_{ac,it}$, we see that

$$Y^1_{it} = 0 \quad \Rightarrow \quad Y^0_{it} = 0.$$

Note that the dimension of $V_i$ is arbitrary and that we have two random shocks that have a structural interpretation and are invariant under a change in treatment status.

In the job-bonus application the intuition behind this assumption is that CS implies that all random events leading to a job offer and employment are the same irrespective if a specific unemployed individual is randomized to the treatment group or to the control group. This almost always holds as the examples suggest.

The third assumption is on the relation between counterfactual outcomes over time. We introduce the assumption for the two periods case. If we compare the transition probability $\Pr(Y^{00}_2 = 1 | Y^1_1 = 0, Y^0_1 = 0)$ to $\Pr(Y^{00}_2 = 1 | Y^1_1 = 1, Y^0_1 = 0)$, i.e. the probability of a transition in period 2 if no treatment was received in periods 1 and 2 given survival with or without treatment in period 1 to the same probability given survival without but not with treatment in period 1, then it is reasonable to assume that the former probability is not larger than the latter. Individuals with $Y^1_1 = 0, Y^0_1 = 0$ have characteristics that make them not leave the initial state as opposed to individuals with $Y^1_1 = 1, Y^0_1 = 0$ that have characteristics that make them leave the initial state if treated in period 1. If the variables that affect the transition out of the initial state are positively correlated between periods, then

$$\Pr(Y^{00}_2 = 1 | Y^1_1 = 0, Y^0_1 = 0) \leq \Pr(Y^{00}_2 = 1 | Y^1_1 = 1, Y^0_1 = 0). \tag{9}$$

As before we motivate the assumption in a discrete duration model similar to (6) but without the CS assumption

$$\begin{aligned} Y^0_{it} &= I(\alpha_t + V_i - \varepsilon_{it} \geq 0), \\ Y^1_{it} &= I(\alpha_t + \gamma_{it} + V_i - \tilde{\varepsilon}_{it} \geq 0). \end{aligned}$$

where the superscripts $0, 1$ are $t$ vectors.

Let $k = 1, \ldots, t-1$ be the period of a transition if treated. The conditioning events are $Y^0_{is} = 0, s = 1, \ldots, t-1$ and $Y^1_{is} = 0, s = 1, \ldots, t-1$ if no transition, or

$$\begin{aligned} &V_i - \varepsilon_{is} < -\alpha_s, && s = 1, \ldots, t-1, \\ &V_i - \tilde{\varepsilon}_{is} < -\alpha_s - \gamma_{is}, && s = 1, \ldots, t-1, \end{aligned}$$

and $Y^0_{is} = 0$, for $s = 1, \ldots, t-1$, $Y^1_{is} = 0$, for $s = 1, \ldots, k-1$, and $Y^1_{ik} = 1$ if a transition in $k$ if treated, or

$$\begin{aligned} &V_i - \varepsilon_{is} < -\alpha_s, && s = 1, \ldots, t-1, \\ &V_i - \tilde{\varepsilon}_{is} < -\alpha_s - \gamma_{is}, && s = 1, \ldots, k-1, \\ &V_i - \tilde{\varepsilon}_{ik} \geq -\alpha_k - \gamma_{ik}. && \end{aligned}$$

For example, for $t = 2$ and $k = 1$ the conditioning events are if no transition

$$V_i - \varepsilon_{i1} < -\alpha_1, \qquad\qquad V_i - \tilde{\varepsilon}_{i1} < -\alpha_1 - \gamma_{i1},$$

and if a transition in 1 if treated

$$V_i - \varepsilon_{i1} < -\alpha_1, \qquad\qquad V_i - \tilde{\varepsilon}_{i1} \geq -\alpha_1 - \gamma_{i1}.$$

In the conditioning event the inequality on $V_i - \tilde{\varepsilon}_{it}$ flips, so that if $V_i - \tilde{\varepsilon}_{i1}$ is positively related with $V_i - \varepsilon_{i2}$, then (9) holds.

In the general case we have by the same reasoning

$$\begin{aligned}
&\Pr(Y_t^0 = 1 | Y_k^1 = 1, Y_{k-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0) \\
&\quad \geq \Pr(Y_t^0 = 1 | Y_k^1 = 0, Y_{k-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0) \\
&\qquad \geq \Pr(Y_t^0 = 1 | Y_{t-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0).
\end{aligned}$$

An analogous argument can be made for $\Pr(Y_t^1 = 1 | Y_{t-1}^1 = 0, \dots, Y_1^1 = 0, Y_k^0 = 1, Y_{k-1}^0 = 0, \dots, Y_0^1 = 0)$. These arguments lead to the following assumption

**Assumption 5 (Positively Correlated Outcomes (PCO))** *For all $k = 1, \dots, t-1$ we have*

$$\begin{aligned}
&\Pr(Y_t^0 = 1 | Y_k^1 = 1, Y_{k-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0) \\
&\quad \geq \Pr(Y_t^0 = 1 | Y_{t-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0),
\end{aligned}$$

*and*

$$\begin{aligned}
&\Pr(Y_t^1 = 1 | Y_{t-1}^1 = 0, \dots, Y_1^1 = 0, Y_k^0 = 1, Y_{k-1}^0 = 0, \dots, Y_0^1 = 0) \\
&\quad \geq \Pr(Y_t^1 = 1 | Y_{t-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0),
\end{aligned}$$

*and*

$$\begin{aligned}
&\Pr(Y_t^0 = 1 | Y_{t-1}^1 = 0, \dots, Y_1^1 = 0, Y_k^0 = 1, Y_{k-1}^0 = 0, \dots, Y_0^1 = 0) \\
&\quad \geq \Pr(Y_t^0 = 1 | Y_{t-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0),
\end{aligned}$$

*and*

$$\begin{aligned}
&\Pr(Y_t^1 = 1 | Y_k^1 = 1, Y_{k-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0) \\
&\quad \geq \Pr(Y_t^0 = 1 | Y_{t-1}^1 = 0, \dots, Y_1^1 = 0, Y_{t-1}^0 = 0, \dots, Y_1^0 = 0).
\end{aligned}$$

The motivating example shows that PCO does not imply nor is implied by MTR or CS. The CS assumption is on the contemporaneous correlation of random shocks while PCO relates to a (positive) relation of the combined random error over time. Since the latter in general contains an important individual effect, positive correlation is not a strong assumption.

For the job-bonus application PCO has several implications. As an illustration, consider two groups consisting of unemployed who find and unemployed who do not find employment in the first period if non-treated. In this case PCO implies that in the second period, the transition rate under treatment on average is weakly larger in the former group compared to the latter. This holds if the ranking of the unemployed individuals in terms of the characteristics that determine job offers, such as ability, experience and job search effort, remains the same during the entire unemployment spell.

## 4.2 Bounds under the additional assumptions

We now obtain bounds on ATETS for arbitrary $t$ when we compare a treatment started in period 1 to no treatment in all periods. Bounds under MTR and CS are given in Theorem 3 and Theorem 4 provides bounds under PCO. Bounds under all three additional assumptions are in Theorem 5.

**Theorem 3 (Bounds on ATETS under MTR and CS for $t$ periods)** *Let the Assumptions 2, 3, and 4 hold. If* $\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1\right)=0$ *then* $\text{ATETS}_t^{1,0}$ *is not defined. If* $\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1\right)>0$*, then we have the following sharp bounds*

$$\text{LB}_t^{1,0} \leq \text{ATETS}_t^{1,0} \leq \text{UB}_t^{1,0},$$

*where*

$$\begin{aligned}
\text{LB}_t^{1,0} = {} & \Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=1) \\
& - \min\left\{1,\ 1+\frac{\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}\right. \\
& \qquad \left. - \frac{\min\left\{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1),\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)\right\}}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}\right\}, \\
\text{UB}_t^{1,0} = {} & \Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=1) \\
& - \max\left\{0\,,\frac{\left[\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)-1\right]\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}\right. \\
& \qquad \left. + \frac{\min\left\{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1),\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)\right\}}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}\right\}.
\end{aligned}$$

**Proof** See Appendix A.

Assumption 3 states that the treatment effect is either non-negative or non-positive for all i. Since in period 1 we can estimate the ATETS directly because there is no dynamic selection yet, the possibility that MTR holds with a non-positive effect, can be excluded if the ATETS in period 1 is non-negative. If we make the stronger assumption that the effect has the same sign for all i and for all t then a non-negative ATETS in period 1 excludes non-positive MTR in all periods. In that case the ATETS is non-negative in all time periods and this improves the lower bound on the ATETS, but has no effect on the upper bound that is between 0 and 1. The lower bound on the ATETS if non-negative MTR holds is[6]:

$$\begin{aligned}
\text{LB}_t^{1,0} &= \max\left\{0,\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=1)\right. \\
&\quad \left.-\frac{\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}\right\}.
\end{aligned}$$

[6]In the same way, if the ATETS in period 1 is non-positive, the possibility that MTR holds with a non-negative effect can be excluded,affecting the upper bound in an obvious way.

If MTR can change sign between periods we would require prior knowledge of the sign in each time period to improve on the bounds in Theorem 3.

**Theorem 4 (Bounds on ATETS under PCO for $t$ periods)** *Let Assumptions 2 and 5 hold. If* $\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1\right)=0$ *then* $\text{ATETS}_t^{1,0}$ *is not defined. If* $\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1\right)>0$ *and* $\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1>0$ *for all* $s=1,\ldots,t-1$*, then we have the following sharp bounds*

$$\text{LB}_t^{1,0}\leq \text{ATETS}_t^{1,0}\leq \text{UB}_t^{1,0},$$

*where*

$$\begin{aligned}
\text{LB}_t^{1,0}&=\Pr(Y_t=1|\overline{D}_t=1,\overline{Y}_{t-1}=0)-1+\frac{1-\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}\\
&\quad\times\prod_{s=1}^{t-1}[\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1],\\
\text{UB}_t^{1,0}&=\Pr(Y_t=1|\overline{D}_t=1,\overline{Y}_{t-1}=0)\\
&-\max\left\{0,\frac{(\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)-1)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)}{\prod_{s=1}^{t-1}[\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1]}+1\right\}.
\end{aligned}$$

*If* $\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1\right)>0$ *and* $\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1\leq 0$ *for some* $s\leq t$*, then we have the sharp bounds*

$$\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=1)-1\leq \text{ATETS}_t^{1,0}\leq \Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=1).$$

**Proof** See Appendix A.

**Theorem 5 (Bounds on ATETS under MTR, CS and PCO for $t$ periods)** *Let the Assumptions 2-5 hold. If* $\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1\right)=0$ *then* $\text{ATETS}_t^{1,0}$ *is not defined. If* $\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1\right)>0$*, then we have the following sharp bounds*

$$\text{LB}_t^{1,0}\leq \text{ATETS}_t^{1,0}\leq \text{UB}_t^{1,0},$$

*where*

$$\begin{aligned}
\text{LB}_t^{1,0}&=\Pr(Y_t=1|\overline{D}_t=1,\overline{Y}_{t-1}=0)-1+\frac{1-\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}\\
&\qquad\times\min\left\{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1),\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)\right\},\\
\text{UB}_t^{1,0}&=\Pr(Y_t=1|\overline{D}_t=1,\overline{Y}_{t-1}=0)\\
&\qquad-\max\left\{0,\frac{(\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)-1)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)}{\min\left\{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1),\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)\right\}}+1\right\}.
\end{aligned}$$

**Proof** See Appendix A.

## 5 Inference

Initially, for a given time period $t$, we consider inference on $\theta_0 = \text{ATETS}_t$ based on the identification result in Theorem 1. We assume that $\Pr(\overline{Y}_{t-1} = 0|D = 1) > 0$. The bounds in the theorem can then be expressed as

$$\max(a_1, a_2) =: \ell \leq \theta_0 \leq u := \min(a_3, a_4), \tag{10}$$

with

$$\begin{aligned}
a_1 &= a_3 - 1, \\
a_2 &= a_3 - \frac{1 - [1 - \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0)]\Pr(\overline{Y}_{t-1} = 0|D = 0)}{\Pr(\overline{Y}_{t-1} = 0|D = 1)}, \\
a_3 &= \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 1), \\
a_4 &= a_3 - 1 + \frac{1 - \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0)\Pr(\overline{Y}_{t-1} = 0|D = 0)}{\Pr(\overline{Y}_{t-1} = 0|D = 1)}.
\end{aligned}$$

If we observe an iid sample $\{(Y_{i1}, Y_{i2}, \ldots, Y_{it}, D_i), i \in 1, \ldots, n\}$, then the sample analog of $a = (a_1, a_2, a_3, a_4)'$ can easily be constructed, for example

$$\widehat{a}_3 = \frac{\frac{1}{n}\sum_{i=1}^n \mathbb{1}(Y_{it} = 1, Y_{i1} = 0, Y_{i2} = 0, \ldots, Y_{i,t-1} = 0, D_i = 0)}{\frac{1}{n}\sum_{i=1}^n \mathbb{1}(Y_{i1} = 0, Y_{i2} = 0, \ldots, Y_{i,t-1} = 0, D_i = 0)}, \qquad \widehat{a}_1 = \widehat{a}_3 - 1,$$

and analogously for $\widehat{a}_2$ and $\widehat{a}_4$. It is easy to show that as the sample size $n$ goes to infinity

$$\sqrt{n}(\widehat{a} - a) \Rightarrow \mathcal{N}(0, \Sigma_a), \tag{11}$$

and we can construct a consistent estimator $\widehat{\Sigma}_a$ of the $4 \times 4$ matrix $\Sigma_a$ (for example, we use bootstrapping to calculate $\widehat{\Sigma}_a$ in our application in Section 6). In the following we assume that $\Sigma_{a,kk} > 0$ for all $k = 1, 2, 3, 4$.[7]

The identification results in Theorem 2 on $\theta_0 = \text{ATETS}_t^{1,0}$ can also be expressed in the form (10) for suitable $a = (a_1, a_2, a_3, a_4)'$ that can be estimated such that (11) holds asymptotically. The identified set for $\theta_0 = \text{ATETS}_t^{1,0}$ in Theorem 3 can similarly be expressed as $\max(a_1, \min(a_2, a_3)) \leq \theta_0 \leq \min(a_4, \max(a_5, a_6))$, with appropriate definition of $a = (a_1, a_2, a_3, a_4, a_5, a_6)'$, whose estimator is again jointly normally distributed asymptotically, and the inference discussion below can be easily generalized to this case. Similarly with Theorem 4 and 5.

### 5.1 Connection to the Moment Inequality Literature

The inference problem for $\theta_0$ that is summarized by (10) and (11) is asymptotically equivalent to an inference problem on a finite number of moment inequalities that is well-studied in the literature, for example in Chernozhukov, Hong, and Tamer (2007), Romano and Shaikh (2008), Rosen (2008), Andrews and Guggenberger (2009), Andrews and Soares (2010), and Andrews and Barwick (2012). To make this connection explicit we define

[7]Since $\widehat{a}_1$ and $\widehat{a}_3$ are perfectly correlated we have $\Sigma_a v = 0$ for the vector $v = (1, -1, 0, 0)'$, implying that $\text{rank}(\Sigma_a) \leq 3$, but this rank deficiency turns out not to be important for our purposes.

$$m(\theta) := \begin{pmatrix} \Sigma_{a,11}^{-1/2}(a_1 - \theta) \\ \Sigma_{a,22}^{-1/2}(a_2 - \theta) \\ \Sigma_{a,33}^{-1/2}(\theta - a_3) \\ \Sigma_{a,44}^{-1/2}(\theta - a_4) \end{pmatrix}, \qquad \widehat{m}(\theta) := \begin{pmatrix} \widehat{\Sigma}_{a,11}^{-1/2}(\widehat{a}_1 - \theta) \\ \widehat{\Sigma}_{a,22}^{-1/2}(\widehat{a}_2 - \theta) \\ \widehat{\Sigma}_{a,33}^{-1/2}(\theta - \widehat{a}_3) \\ \widehat{\Sigma}_{a,44}^{-1/2}(\theta - \widehat{a}_4) \end{pmatrix}.$$

The bounds (10) can then equivalently be expressed as $m(\theta_0) \leq 0$, which is analogous to imposing four moment inequalities.[8] For convenience we have normalized $m(\theta)$ such that each component of $\sqrt{n}\,[\widehat{m}(\theta) - m(\theta)]$ has asymptotic variance equal to one. Using (11) we obtain $\sqrt{n}\,[\widehat{m}(\theta) - m(\theta)] \Rightarrow \mathcal{N}(0, \Sigma_m)$, where $\Sigma_m = A \Sigma_a A$, with $A = \text{diag}(\Sigma_{a,11}^{-1/2}, \Sigma_{a,22}^{-1/2}, -\Sigma_{a,33}^{-1/2}, -\Sigma_{a,44}^{-1/2})$. An estimator $\widehat{\Sigma}_m$ can be constructed analogously.

All the papers on moment inequalities cited above start from choosing an objective function (or criterion function, or test statistics), whose sample version we denote by $\widehat{Q}(\theta)$, and then construct a confidence set for $\theta_0$ as

$$\widehat{\Theta}(C_{1-\alpha}) = \{\theta \in \mathbb{R} : n\widehat{Q}(\theta) \leq C_{1-\alpha}\}, \tag{12}$$

where $C_{1-\alpha} \geq 0$ is a critical value that is chosen such that confidence $1 - \alpha$ is achieved asymptotically, i.e. $\lim_{n\to\infty} \Pr(\theta_0 \in \widehat{\Theta}(C_{1-\alpha})) \geq 1 - \alpha$.[9] Various objective functions have been considered in the literature. For example, the objective function considered in Chernozhukov, Hong, and Tamer (2007) reads in our notation $\widehat{Q}(\theta) = \|[\widehat{m}(\theta)]_+\|^2$, where $\|.\|$ refers to the Euclidian norm, and $[\widehat{m}(\theta)]_+ := \max(0, \widehat{m}(\theta))$, applied componentwise to the vector $\widehat{m}(\theta)$.

## 5.2 Construction of Confidence Intervals

Our specific inference problem is easier than the general inference problem for moment inequalities, because in our case the parameter $\theta_0$ is just a scalar, and the total number of inequalities is relatively small. Our goal in the following is therefore to outline a concrete method of how to construct a confidence interval in that special case.

We choose the objective function $\widehat{Q}(\theta) = \|[\widehat{m}(\theta)]_+\|_\infty^2$, where $\|.\|_\infty$ is the infinity norm,[10] i.e. we have $\widehat{Q}(\theta) = \max\{0, \widehat{m}_1(\theta), \widehat{m}_2(\theta), \widehat{m}_3(\theta), \widehat{m}_4(\theta)\}^2$. This objective function is convenient for our purposes, because the confidence set defined above then takes the intuitive

[8] $m(\theta)$ is not actually a moment function, but has a slightly more complicated structure (e.g. $a_3$ is a conditional probability that can be expressed as the ratio between two moments). This, however, does not matter for the asymptotic analysis since the estimator $\widehat{m}(\theta)$ has the same first order asymptotic properties as it would have in the moment inequality case. We can therefore fully draw on the insights of the existing literature.

[9] As discussed in e.g. Andrews and Soares (2010), it is important that the coverage probability is asymptotically bounded by $1 - \alpha$ uniformly over $\theta_0$ and over the distribution of the observables. We have only formulated the pointwise condition here to keep the presentation simple.

[10] This is special case of the "test function" $S_3(m, \Sigma)$ introduced in equation (3.6) of Andrews and Soares (2010), with $p_1 = 1$ and $v = 0$ in their notation.

form

$$\widehat{\Theta}(C_{1-\alpha})$$
$$= \left[\max\left(\widehat{a}_1 - \frac{c_{1-\alpha}\widehat{\Sigma}^{1/2}_{a,11}}{\sqrt{n}}, \widehat{a}_2 - \frac{c_{1-\alpha}\widehat{\Sigma}^{1/2}_{a,22}}{\sqrt{n}}\right), \min\left(\widehat{a}_3 + \frac{c_{1-\alpha}\widehat{\Sigma}^{1/2}_{a,33}}{\sqrt{n}}, \widehat{a}_4 + \frac{c_{1-\alpha}\widehat{\Sigma}^{1/2}_{a,44}}{\sqrt{n}}\right)\right], \tag{13}$$

where $c_{1-\alpha} := \sqrt{C_{1-\alpha}}$. This confidence interval can be constructed very easily.

**Most Robust Critical Value**

The critical value $c_{1-\alpha}$ still needs to be chosen. The problem with choosing the critical value in moment inequality problems is that this choice depends on the unknown slackness vector $m(\theta_0)$, which indicates whether each inequality $m_k(\theta_0) \leq 0$ is binding, close to binding, or far from binding. It is known, however, that the largest ("worst case") critical value needs to be chosen if $m(\theta_0) = 0$, i.e. if all moment inequalities are binding at the true parameter. To find this critical value one can use the fact that in this worst case $n\widehat{Q}(\theta)$ is asymptotically distributed as $\|[Z]_+\|^2_\infty$, where $Z \sim \mathcal{N}(0, \Sigma_m)$ is a random four vector. Using the estimator $\widehat{\Sigma}_m$ one can simulate this distribution. However, it can easily be shown that the $1-\alpha$ quantile of $\|[Z]_+\|_\infty$ is always smaller or equal to the following conservative critical value

$$c_{1-\alpha} = \Phi^{-1}\left(1 - \frac{\alpha}{4}\right), \tag{14}$$

where $\Phi^{-1}$ is the quantile function (the inverse cdf) of the standard normal distribution. The factor 1/4 that appears here reflects the fact that we have four moment inequalities. Combining equations (13) and (14) provides a confidence interval that is uniformly valid, i.e. whose asymptotic size is bounded by $\alpha$, independent of what the true values of $a_1$, $a_2$, $a_3$ and $a_4$ are.

**Critical Value for the Case $\ell \ll u$**

The critical values based on the "worst case" where all inequalities are binding ($m(\theta_0) = 0$) can be very conservative if one or multiple inequalities are far from binding ($m_k(\theta_0) \ll 0$).[11] Furthermore, for the inference on $\theta_0 = \text{ATETS}_t$ based on Theorem 1, with $a$'s as given above, it can easily be shown that if $\Pr(\overline{Y}_{t-1} = 0|D = 1) > 0$ and $\Pr(\overline{Y}_{t-1} = 0|D = 0) < 1$, then we have $\max(a_1, a_2) =: \ell < u := \min(a_3, a_4)$, implying that $m(\theta_0) = 0$ is impossible. However, what matters for the coverage rate of the confidence interval for a finite sample is not whether $\ell < u$, but whether the difference $u - \ell$ is large relative to the standard deviations $\Sigma^{1/2}_{a,kk}$ of the $\widehat{a}_k$, $k = 1, 2, 3, 4$. This is what we mean by $\ell \ll u$ in the subsection title above.

To formalize this one can consider a pretest of the hypothesis $H_0 : \ell = u$, against the alternative $H_a : \ell < u$, with pretest size $\alpha_n^{\text{pre}}$ chosen to be very small, e.g. $\alpha_n^{\text{pre}} = 0.001 \ll \alpha$.[12]

[11] In addition, the formula (14) only provides an upper bound for the optimal critical value at $m(\theta_0) = 0$, but this second issue is often not very severe. For example, for $\alpha = 0.05$ and $\Sigma_m = \mathbb{I}_4$ one finds by simulation that the 0.95 quantile of $\|[Z]_+\|_\infty$, with $Z \sim \mathcal{N}(0, \Sigma_m)$, is $c_{0.95} = 2.234$, while the much easier to computer conservative critical value in (14) is $\Phi^{-1}(0.9875) = 2.241$.

[12] Theoretically one can assume $\alpha_n^{\text{pre}} \to 0$ as $n \to \infty$ to avoid asymptotic size distortions due to the pretest.

If the pretest is not rejected, then the critical value (14) should be chosen. If the pretest is rejected, then the two problems of choosing a suitable lower and upper bound for the confidence interval $\widehat{\Theta}$ completely decouple, because with high confidence we know that for any $\theta$ only one of those bounds can be binding at the same time, implying that at most two of the moment inequalities $m(\theta_0) \leq 0$ can be binding. In this latter case we can therefore choose the less conservative critical value

$$c_{1-\alpha} = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right), \tag{15}$$

when computing the confidence interval (13).

**Critical Value for the Case $a_1 \ll a_2 \ll u$**

Analogous to the discussion of (14), the critical value (15) is again potentially conservative because it is based on the case where two of the inequalities $m(\theta_0) \leq 0$ (for either the lower or the upper bound, respectively) are jointly binding.[13] For example, if we find that $a_1 \ll a_2 \ll u$ (by which we again mean that the null hypotheses $H_0 : a_1 = a_2$, vs. $H_a : a_1 < a_2$, and $H_0 : a_2 = u$, vs. $H_a : a_2 < u$, are rejected with very high confidence), then a natural confidence interval to report is

$$\widehat{\Theta} = \left[\widehat{a}_2 - \frac{\Phi^{-1}(1-\alpha)\,\widehat{\Sigma}_{a,22}^{1/2}}{\sqrt{n}}, \min\left(\widehat{a}_3 + \frac{\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\widehat{\Sigma}_{a,33}^{1/2}}{\sqrt{n}}, \widehat{a}_4 + \frac{\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\widehat{\Sigma}_{a,44}^{1/2}}{\sqrt{n}}\right)\right].$$

Note that the lower bound of $\widehat{\Theta}$ now corresponds to inverting a standard one-sided t-test. Analogous confidence intervals can obviously be constructed in other cases, e.g. $\ell \ll a_3 \ll a_4$ or $a_2 \ll a_1 \ll a_4 \ll a_3$, etc.

The different critical values and corresponding confidence intervals discussed above correspond to cases where different subsets of the inequalities $m(\theta_0) \leq 0$ can be simultaneously binding, i.e. to a moment selection problem. A much more general discussion of moment selection is given e.g. in Andrews and Soares (2010). Different confidence intervals than those discussed here, e.g. based on different objective functions $\widehat{Q}(\theta)$, can of course also be considered.

It should be noted that pretesting is not required if we use the approach in Hahn and Ridder (2014) who obtain a confidence interval by inverting the Likelihood Ratio test for the composite null and composite alternative test. Their current results do not cover the case considered here and we did not attempt the non-trivial extension to the case considered here.

# 6 Application to the Illinois bonus experiment

## 6.1 The re-employment bonus experiment

Between mid-1984 and mid-1985, the Illinois Department of Employment Security conducted a randomized social experiment.[14] The goal of the experiment was to explore, whether re-

[13] It is also conservative, because the information in the correlation matrix $\Sigma_m$ is not used to construct (15). It corresponds to the the most extreme case where both lower bound estimators $\widehat{a}_1$ and $\widehat{a}_2$ (or both upper bound estimators $\widehat{a}_3$ and $\widehat{a}_4$) are perfectly negatively correlated.

[14] A complete description of the experiment and a summary of its results can be found in Woodbury and Spiegelman (1987).

employment bonuses paid to Unemployment Insurance (UI) beneficiaries (treatment 1) or their employers (treatment 2) reduced the length of unemployment spells.

Both treatments consisted of a $500 re-employment bonus, which was about four times the average weekly unemployment insurance benefit. In the experiment, newly unemployed UI claimants were randomly divided into three groups:

1. The *Claimant Bonus Group*. The members of this group were instructed that they would qualify for a cash bonus of $500 if they found a job (of at least 30 hours) within 11 weeks and, if they held that job for at least 4 months. A total of 4186 individuals were selected for this group, and 3527 (84%) agreed to participate.
2. The *Employer Bonus Group*. The members of this group were told that their next employer would qualify for a cash bonus of $500 if they, the claimants, found a job (of at least 30 hours) within 11 weeks and, if they held that job for at least four months. A total of 3963 were selected for this group and 2586 (65%) agreed to participate.
3. The *Control Group*, i.e. all claimants not assigned to one of the treatment groups. This group consisted of 3952 individuals. The individuals assigned to the control group were excluded from participation in the experiment. In fact, they did not know that the experiment took place.

The descriptive statistics in Table 2 in Woodbury and Spiegelman (1987) confirm that the randomization resulted in three similar groups.

### 6.2 Results of previous studies

Woodbury and Spiegelman (1987) concluded from a direct comparison of the control group and the two treatment groups that the claimant bonus group had a significantly shorter average unemployment duration. The average unemployment duration was also shorter for the employer bonus group, but the difference was not significantly different from zero. In Illinois UI benefits end after 26 weeks and since administrative data were used, all unemployment durations are censored at 26 weeks. Woodbury and Spiegelman ignore the censoring and take as outcome variable the number of weeks of insured unemployment.

Meyer (1996) analyzed the same data but focused on the treatment effects on conditional transition probabilities which allows him to properly account for censoring. Meyer focuses on the conditional transitions rates because both labor supply and search theory imply specific dynamic treatment effects. The bonus is only given to an unemployed individual if (s)he finds a job within 11 weeks and retains it for four months. The cash bonus is the same for all unemployed. Theory predicts that (i) the transition rate during the eligibility period (first 11 weeks) will be higher in the two treatment groups compared with the control group, and (ii) that the transition rate in the treatment groups will rise just before the end of the eligibility period, as the unemployed run out of time to collect the bonus.

To test these predictions, Meyer (1996) estimates a proportional hazard (PH) model with a flexible specification of the baseline hazard. He uses the treatment indicator as an explanatory variable. Since there was partial compliance with treatment his estimator can be interpreted as a intention to treat (ITT) estimator.[15] In his analysis Meyer controls for

[15]The partial compliance is addressed in detail by Bijwaard and Ridder (2005). They introduce a new method to handle the selective compliance in the treatment group. If there is full compliance in the control group, their two-stage linear rank estimator is able to handle the selective compliance in the treatment group even for censored durations. In order to achieve this they assume a MPH structure for the transition rate.

age, the logarithm of base period earnings, ethnicity , gender and the logarithm of the size of the UI benefits. He finds a significantly positive effect of the claimant bonus and a positive but insignificant effect of the employer bonus. A more detailed analysis of the effects for the claimant group reveals a positive effect on the transition rate during the first 11 weeks in unemployment, an increased effect during week 9 and 10, and no significant effect on the transition rate after week 11 as predicted by labor supply and search theory.

### 6.3 Estimates of bounds

In his study Meyer (1996) relies on the proportionality of the hazard rate to investigate his hypotheses. We now ask what can be said if the assumptions of the MPH model do not hold, that is what can be identified if we rely solely on random assignment and the additional assumptions. As Meyer we consider the ITT effect, i.e. we do not correct for partial compliance. We divide the 24 month observation period into 12 subperiods: week 1-2, week 3-4, ... , week 23-24. The reason for this is that there is a pronounced even-odd week effect in the data, with higher transition rate during odd weeks. With these subperiods the predictions we wish to test are: (i) a positive treatment effect during periods 1-5, i.e.

$$\text{ATETS}_t^{1,0} > 0 \ , \quad t = 1, \ldots, 5,$$

(ii) no effect after the bonus offer has expired in periods 6-12, i.e.

$$\text{ATETS}_t^{1,0} = 0 \ , \quad t = 6, \ldots, 12,$$

and (iii) a larger effect of the bonus offer at the end of the eligibility period in period 5, i.e.

$$\text{ATETS}_5^{1,0} > \text{ATETS}_4^{1,0}.$$

Note that in this experiment the treatment assignment is in period 1, so that in $\text{ATETS}_t^{1,0}$ the superscripts 1 and 0 are $t$ vectors with components equal to 1 and 0.

We report both the bounds that are obtained by simply replacing the population moments with their sample analogs, as well as the confidence intervals based on the approach described in section 5.[16] Table 1 presents the upper and the lower bound and the confidence interval on $\text{ATETS}_t^{1,0}$ for the claimant group assuming only random assignment. We find that the instantaneous treatment effect on the transition probability (week 1-2) is point identified and indicates a positive effect of the re-employment bonus. The transition probability is about 2 percentage points higher in the treatment group compared to the control group. This estimate is statistically significant. From week 3-4 and onwards the bounds are quite wide. In fact, without further assumptions we cannot rule out that the bonus actually has a negative impact on the conditional transition probability after week 3. However, the bounds are nevertheless informative on the average treatment effect in all time periods.

Table 1 also shows that the confidence intervals are marginally wider than the actual bounds. That is the uncertainty arising from the dynamic selection is far greater than the uncertainty due to sampling variation.

Their estimates indicate that the ITT estimates by Meyer (1996) underestimate the true treatment effect.

[16]The covariance matrix $\Sigma_a$ is estimated using the bootstrap with 399 replications. Constructing confidence intervals furthermore requires moment selection, e.g. for the bounds under just random assignment we find that with very high confidence only one inequality is binding for the lower as well as the upper bound. Details are available from the authors upon request.

Next, Table 1 presents bounds under the additional assumptions in Section 4. As expected, if we impose additional assumptions the bounds are considerably narrower. Under MTR and CS we can rule out very large negative and very large positive dynamic treatment effects. Imposing MTR, CS as well as PCO further tightens the bounds. If these assumptions hold simultaneously we can, if we disregard sampling variation, rule out that the bonus offer has a negative effect on the transition rate out of unemployment up to week 20. This conclusions changes slightly when sampling variation is taken into account.

Let us return to the three hypotheses suggested by labor supply and search theory, and consider our most restrictive bounds under MTR, CS and PCO. We find that there is a positive effect of the bonus offer on the conditional transition rate up to week 11. This confirms the first hypothesis. The upper bound increases in time period 5 (weeks 9-10), but the lower bound does not increase enough, so that both an increase and no change (and even a small decrease) in the transition probability out of unemployment are consistent with the data. Now consider the third hypothesis that there is no effect on the transition rate after week 11. Again the bounds do not rule out that there is a positive effect on the conditional transition probability after week 11. These results illustrate that the evidence for the second and third hypotheses presented by a number of authors rely on the imposed structure, e.g. proportionality of the hazard or the restrictions implied by a particular discrete-time duration model.

We next examine heterogenous effects. To this end we split our sample by gender, race and pre-unemployment income and estimate our bounds for each subgroup. We provide results for bounds without additional assumptions and bounds under MTR, CS and PCO. The other bounds are available upon request. If we focus on the bounds under MTR, CS and PCO, Table 2 indicates several interesting differences between males and females. For males we find significant effects in the beginning of the unemployment spell (weeks 1-2) and shortly before the bonus expires (weeks 7-10). For females on the other hand we only find significant effects in weeks 1-4, but no effects before week 11. This indicates that females quickly responds to the bonus offer, whereas a large part of the effects for males occur shortly before the end of the subsidy. Table 3 in Appendix B also reveals some differences between blacks and non-blacks. For both groups we find significant effects during the first 11 weeks of unemployment, but for non-blacks the bonus offer also increases the transition rates after the bonus offers has expired (e.g. during weeks 15-16). Finally, Tables 4 in Appendix B reveals no significant differences between how workers with low and high income react to the bonus offer.

# 7 Conclusions

In this paper, we have derived bounds on treatment effects on conditional transition probabilities under (sequential) randomization. The partial identification problem arises since random assignment only ensures comparability of the treatment and control groups at the time of randomization. In the literature this problem is often refereed to as the dynamic selection problem. For that reason only instantaneous or short-run effects are point identified, whereas dynamic or long-run effects in general are not point identified. Our weakest bounds impose no assumptions beyond (sequential) random assignment, so that they are not sensitive to arbitrary functional form assumptions, require no additional covariates and allow arbitrary heterogenous treatment effects as well as arbitrary unobserved heterogeneity. These non-parametric bounds offer an alternative to semi-parametric methods. They tend to be wide and therefore we have also derived more informative bounds under additional assumptions that often hold in semi-parametric reduced form and structural models.

An analysis of data from the Illinois re-employment bonus experiment shows that our bounds are informative about average treatment effects. It also demonstrates that previous results on the evolution of the average treatment effect require that assumptions as the proportionality of the hazard rate or those embodied in a particular (semi-)parametric discrete-time hazard model hold.

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# Tables

Table 1: Bounds on $ATETS^{1,0}$ for the Illinois job bonus experiment

| | No assumption bounds [A] | | | | MTR+CS [B] | | | |
|---|---|---|---|---|---|---|---|---|
| | Lower-CI | LB | UB | Upper-CI | Lower-CI | LB | UB | Upper-CI |
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| Week | | | | | | | | |
| 1-2 | 0.012 | 0.023 | 0.023 | 0.034 | 0.012 | 0.023 | 0.023 | 0.034 |
| 3-4 | -0.145 | -0.137 | 0.094 | 0.102 | 0.000 | 0.011 | 0.038 | 0.050 |
| 5-6 | -0.259 | -0.251 | 0.074 | 0.082 | -0.007 | 0.004 | 0.046 | 0.056 |
| 7-8 | -0.346 | -0.339 | 0.078 | 0.086 | 0.004 | 0.013 | 0.063 | 0.073 |
| 9-10 | -0.452 | -0.444 | 0.069 | 0.077 | 0.000 | 0.008 | 0.069 | 0.079 |
| 11-12 | -0.552 | -0.544 | 0.062 | 0.070 | 0.000 | 0.008 | 0.062 | 0.072 |
| 13-14 | -0.655 | -0.648 | 0.056 | 0.064 | -0.010 | -0.002 | 0.056 | 0.064 |
| 15-16 | -0.750 | -0.743 | 0.051 | 0.058 | -0.004 | 0.003 | 0.051 | 0.058 |
| 17-18 | -0.844 | -0.836 | 0.049 | 0.057 | -0.007 | 0.000 | 0.049 | 0.057 |
| 19-20 | -0.943 | -0.936 | 0.049 | 0.057 | -0.011 | -0.004 | 0.049 | 0.056 |
| 21-22 | -0.994 | -0.953 | 0.047 | 0.056 | -0.028 | -0.021 | 0.047 | 0.055 |
| 23-24 | -0.989 | -0.944 | 0.056 | 0.064 | -0.011 | -0.002 | 0.056 | 0.064 |
| | PCO [C] | | | | MTR+CS+PCO [D] | | | |
| | Lower-CI | LB | UB | Upper-CI | Lower-CI | LB | UB | Upper-CI |
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| Week | | | | | | | | |
| 1-2 | 0.012 | 0.023 | 0.023 | 0.034 | 0.012 | 0.023 | 0.023 | 0.034 |
| 3-4 | -0.131 | -0.123 | 0.094 | 0.102 | 0.002 | 0.014 | 0.038 | 0.049 |
| 5-6 | -0.209 | -0.202 | 0.074 | 0.082 | -0.004 | 0.007 | 0.046 | 0.055 |
| 7-8 | -0.256 | -0.247 | 0.078 | 0.087 | 0.008 | 0.016 | 0.063 | 0.072 |
| 9-10 | -0.306 | -0.299 | 0.069 | 0.077 | 0.004 | 0.012 | 0.069 | 0.078 |
| 11-12 | -0.348 | -0.340 | 0.062 | 0.070 | 0.004 | 0.012 | 0.062 | 0.071 |
| 13-14 | -0.388 | -0.379 | 0.056 | 0.064 | -0.004 | 0.003 | 0.056 | 0.064 |
| 15-16 | -0.419 | -0.411 | 0.051 | 0.058 | 0.000 | 0.007 | 0.051 | 0.059 |
| 17-18 | -0.445 | -0.438 | 0.049 | 0.057 | -0.003 | 0.005 | 0.049 | 0.058 |
| 19-20 | -0.472 | -0.464 | 0.049 | 0.057 | -0.006 | 0.001 | 0.049 | 0.057 |
| 21-22 | -0.504 | -0.496 | 0.047 | 0.063 | -0.022 | -0.014 | 0.047 | 0.055 |
| 23-24 | -0.523 | -0.513 | 0.056 | 0.073 | -0.006 | 0.003 | 0.056 | 0.065 |

Notes: CI is 95% confidence intervals. Variances and covariances used to obtain the CI are estimated using bootstrap (399 replications).

Table 2: Bounds on $ATETS^{1,0}$ for the Illinois job bonus experiment. Heterogenous effects for males and females

| | **Panel A: Males** | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | No assumption bounds | | | | MTR+CS+PCO | | | |
| | Lower-CI | LB | UB | Upper-CI | Lower-CI | LB | UB | Upper-CI |
| | (1) | (2) | (3) | (4) | (1) | (2) | (3) | (4) |
| Week | | | | | | | | |
| 1-2 | -0.004 | 0.016 | 0.016 | 0.037 | 0.002 | 0.016 | 0.016 | 0.030 |
| 3-4 | -0.152 | -0.141 | 0.094 | 0.105 | -0.004 | 0.009 | 0.026 | 0.039 |
| 5-6 | -0.269 | -0.259 | 0.075 | 0.084 | -0.010 | 0.003 | 0.030 | 0.043 |
| 7-8 | -0.349 | -0.338 | 0.085 | 0.096 | 0.009 | 0.024 | 0.054 | 0.069 |
| 9-10 | -0.464 | -0.453 | 0.076 | 0.087 | 0.000 | 0.014 | 0.070 | 0.084 |
| 11-12 | -0.573 | -0.562 | 0.069 | 0.080 | 0.005 | 0.015 | 0.069 | 0.081 |
| 13-14 | -0.688 | -0.676 | 0.065 | 0.076 | -0.004 | 0.006 | 0.065 | 0.077 |
| 15-16 | -0.793 | -0.782 | 0.054 | 0.064 | 0.004 | 0.014 | 0.054 | 0.064 |
| 17-18 | -0.899 | -0.887 | 0.056 | 0.067 | -0.008 | 0.003 | 0.056 | 0.066 |
| 19-20 | -0.994 | -0.941 | 0.059 | 0.071 | -0.004 | 0.008 | 0.059 | 0.071 |
| 21-22 | -1.006 | -0.948 | 0.052 | 0.063 | -0.028 | -0.017 | 0.052 | 0.066 |
| 23-24 | -1.006 | -0.941 | 0.059 | 0.071 | -0.010 | 0.002 | 0.059 | 0.074 |
| | PCO [C] | | | | MTR+CS+PCO [D] | | | |
| | **Panel B: Females** | | | | | | | |
| | No assumption bounds | | | | MTR+CS+PCO | | | |
| | (1) | (2) | (3) | (4) | (1) | (2) | (3) | (4) |
| Week | | | | | | | | |
| 1-2 | 0.008 | 0.031 | 0.031 | 0.054 | 0.014 | 0.031 | 0.031 | 0.047 |
| 3-4 | -0.143 | -0.131 | 0.093 | 0.105 | 0.003 | 0.019 | 0.053 | 0.069 |
| 5-6 | -0.251 | -0.239 | 0.074 | 0.085 | 0.000 | 0.012 | 0.066 | 0.080 |
| 7-8 | -0.348 | -0.337 | 0.068 | 0.079 | -0.006 | 0.005 | 0.068 | 0.082 |
| 9-10 | -0.441 | -0.430 | 0.060 | 0.071 | -0.003 | 0.009 | 0.060 | 0.073 |
| 11-12 | -0.528 | -0.517 | 0.053 | 0.064 | -0.002 | 0.008 | 0.053 | 0.066 |
| 13-14 | -0.616 | -0.606 | 0.045 | 0.055 | -0.011 | 0.000 | 0.045 | 0.055 |
| 15-16 | -0.698 | -0.686 | 0.046 | 0.057 | -0.012 | 0.000 | 0.046 | 0.059 |
| 17-18 | -0.775 | -0.764 | 0.041 | 0.052 | -0.008 | 0.007 | 0.041 | 0.055 |
| 19-20 | -0.861 | -0.851 | 0.036 | 0.047 | -0.016 | -0.006 | 0.036 | 0.047 |
| 21-22 | -0.949 | -0.936 | 0.041 | 0.054 | -0.022 | -0.011 | 0.041 | 0.055 |
| 23-24 | -1.020 | -0.948 | 0.052 | 0.066 | -0.009 | 0.004 | 0.052 | 0.068 |

Notes: CI is 95% confidence intervals. Variances and covariances used to obtain the CI are estimated using bootstrap (399 replications).

# Appendix A: Proofs

### Proof of Theorem 1

We use the following notation for the distribution of the potential outcomes. For $d = 0, 1$

$$p_t^d(1|0,0) =: \Pr(Y_t^d = 1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0),$$
$$p_t^d(1|0,\neq 0) =: \Pr(Y_t^d = 1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 \neq 0),$$
$$p_t^d(1|\neq 0,0) =: \Pr(Y_t^d = 1|\overline{Y}_{t-1}^1 \neq 0, \overline{Y}_{t-1}^0 = 0),$$

and for the joint distribution of $\overline{Y}_{t-1}^1, \overline{Y}_{t-1}^0$

$$p_{t-1}(0,0) =: \Pr(\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0),$$
$$p_{t-1}(0,\neq 0) =: \Pr(\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 \neq 0),$$
$$p_{t-1}(\neq 0,0) =: \Pr(\overline{Y}_{t-1}^1 \neq 0, \overline{Y}_{t-1}^0 = 0),$$

We derive bounds on $\text{ATETS}_t^{1,0}$ defined by

$$\mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0\right] - \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \tag{A.1}$$

with the data providing the observed transition probabilities $\Pr(Y_t = y_t|\overline{Y}_{t-1} = 0, D = 1)$ and $\Pr(Y_t = y_t|\overline{Y}_{t-1} = 0, D = 0)$.

Under Assumption 1

$$\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0] = \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 1),$$

so that if $\Pr(\overline{Y}_{t-1}^1 = 0|D = 1) = \Pr(\overline{Y}_{t-1} = 0|D = 1) > 0$ then $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0]$ is point-identified, and if $\Pr(\overline{Y}_{t-1}^1 = 0|D = 1) = \Pr(\overline{Y}_{t-1} = 0|D = 1) = 0$ then $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0], \mathbb{E}[Y_t^0|\overline{Y}_{t-1}^1 = 0]$ and $\text{ATETS}_t^{1,0}$ are not defined. Note that the point identification of this mean is similar to the point identification of the treated mean in the ATET in static settings.

Next, we have for the counterfactual transition probability

$$\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] = \frac{p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)}{p_{t-1}(0,0) + p_{t-1}(0,\neq 0)}. \tag{A.2}$$

By Assumption 1

$$\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0) = \Pr(Y_t^0 = 1, \overline{Y}_{t-1}^0 = 0|D = 0) = \Pr(Y_t^0 = 1, \overline{Y}_{t-1}^0 = 0).$$

By the law of total probability

$$\Pr(Y_t^0 = 1, \overline{Y}_{t-1}^0 = 0) = \Pr(\overline{Y}_{t-1}^1 = 0, Y_t^0 = 1, \overline{Y}_{t-1}^0 = 0) + \Pr(\overline{Y}_{t-1}^1 \neq 0, Y_t^0 = 1, \overline{Y}_{t-1}^0 = 0) =$$
$$p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0).$$

Therefore,

$$\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0) = p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0)$$

Solving for $p_t^0(1|0,0)$ gives

$$p_t^0(1|0,0) = \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0) - p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0)}{p_{t-1}(0,0)}.$$

and upon substitution

$$\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] = \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0)}{p_{t-1}(0,0) + p_{t-1}(0,\neq 0)} - \frac{p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0) - p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)}{p_{t-1}(0,0) + p_{t-1}(0,\neq 0)}.$$

The expression on the right-hand side is decreasing in $p_t^0(1|\neq 0,0)$ and increasing in $p_t^0(1|0,\neq 0)$. The lower bound is obtained by setting $p_t^0(1|\neq 0,0)$ at 1 and $p_t^0(1|0,\neq 0)$ at 0 and the upper bound by setting $p_t^0(1|\neq 0,0)$ at 0 and $p_t^0(1|0,\neq 0)$ at 1 so that

$$\frac{\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0)\Pr(\overline{Y}_{t-1} = 0|D = 0) - p_{t-1}(\neq 0,0)}{p_{t-1}(0,0) + p_{t-1}(0,\neq 0)}$$

$$\leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq$$

$$\frac{\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0)\Pr(\overline{Y}_{t-1} = 0|D = 0) + p_{t-1}(0,\neq 0)}{p_{t-1}(0,0) + p_{t-1}(0,\neq 0)}.$$

where we note that

$$\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0)\Pr(\overline{Y}_{t-1} = 0|D = 0) = \Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0) = 0$$

if $\Pr(\overline{Y}_{t-1} = 0|D = 0) = 0$.

Because

$$\Pr(\overline{Y}_{t-1} = 0|D = 1) = p_{t-1}(0,0) + p_{t-1}(0,\neq 0)$$

and

$$\Pr(\overline{Y}_{t-1} = 0|D = 0) = p_{t-1}(0,0) + p_{t-1}(\neq 0,0)$$

we have

$$\frac{[\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0) - 1]\Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) + p_{t-1}(0,0)}{\Pr(\overline{Y}_{t-1} = 0|D = 1)} \tag{A.3}$$

$$\leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq$$

$$\frac{\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0)\Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - p_{t-1}(0,0)}{\Pr(\overline{Y}_{t-1} = 0|D = 1)} + 1.$$

The upper bound is decreasing and the lower bound is increasing in $p_{t-1}(0,0)$. By the Bonferroni inequality

$$p_{t-1}(0,0) \geq \max\left\{\Pr(\overline{Y}_{t-1}^1 = 0) + \Pr(\overline{Y}_{t-1}^0 = 0) - 1, 0\right\} =$$

$$\max\left\{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1, 0\right\}.$$

If

$$\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1 \leq 0$$

the lower bound on $p_{t-1}(0,0)$ is 0. In that case the lower bound in (A.3) is non-positive and the upper bound is greater than or equal to 1 so that

$$0 \leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq 1.$$

If $\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1 > 0$ we have upon substitution of the lower bound on $p_{t-1}(0,0)$ into (A.3) and because the probability $\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right]$ is bounded by zero and one

$$\max\left\{0, \frac{\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0)\Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}{\Pr(\overline{Y}_{t-1} = 0|D = 1)} + 1\right\}$$

$$\leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq \tag{A.4}$$

$$\min\left\{1, \frac{1 - [1 - \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, D = 0)]\Pr\left(\overline{Y}_{t-1} = 0|D = 0\right)}{\Pr(\overline{Y}_{t-1} = 0|D = 1)}\right\}.$$

Finally, we combine these bounds with the point-identified $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0]$ to obtain bounds on $\text{ATETS}_t$.

We now prove that the bounds are sharp. We will show that there is a joint distribution of the latent potential outcomes such that the counterfactual transition probability in (A.2) is equal to the upper bound in (A.4). There are two cases. First, consider the upper bound if

$$\frac{1 - [1 - \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0)]\Pr\left(\overline{Y}_{t-1} = 0|D = 0\right)}{\Pr(\overline{Y}_{t-1} = 0|D = 1)} \leq 1 \tag{A.5}$$

Because the denominator of the counterfactual transition probability and this upper bound are the same we have to choose the joint distribution of the latent potential outcomes such that

$$p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0) = 1 - \Pr(\overline{Y}_{t-1} = 0|D = 0) + \Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0) \tag{A.6}$$

and the restrictions

$$\begin{aligned} p_{t-1}(0,0) + p_{t-1}(0,\neq 0) &= \Pr(\overline{Y}_{t-1} = 0|D = 1) \\ p_{t-1}(0,0) + p_{t-1}(\neq 0,0) &= \Pr(\overline{Y}_{t-1} = 0|D = 0) \\ p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0) &= \Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0), \end{aligned} \tag{A.7}$$

hold. Now set $p_{t-1}(0,0)$ at its lower bound. If that bound is 0, then the counterfactual transition probability in (A.2) is equal to $p_t^0(1|0,\neq 0)$ that is unrestricted so that the $[0,1]$ interval gives sharp bounds. If the lower bound is[17] $\Pr(\overline{Y}_{t-1} = 0|D = 1) + \Pr(\overline{Y}_{t-1} = 0|D = 0) - 1 > 0$, then $p_{t-1}(0,\neq 0) = 1 - \Pr(\overline{Y}_{t-1} = 0|D = 0)$ and $p_{t-1}(\neq 0,0) = 1 - \Pr(\overline{Y}_{t-1} = 0|D = 1)$. Therefore to satisfy (A.6) and (A.7) the probabilities $p_t^0(1|0 \neq, 0)$ and $p_t^0(1|\neq 0,0)$ must satisfy

$$p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0) - p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0) = p_{t-1}(0,\neq 0) \tag{A.8}$$

[17]Note that this implies that the probability that $Y_{t-1}^1 = 0$ or $Y_{t-1}^0 = 0$ is 1.

with $p_{t-1}(0,\neq 0) = 1-\Pr(\overline{Y}_{t-1}=0|D=0)$ and $p_{t-1}(\neq 0,0) = 1-\Pr(\overline{Y}_{t-1}=0|D=1)$. The only values in the unit interval are $p_t^0(1|0,\neq 0)=1$ and $p_t^0(1|\neq 0,0)=0$. Therefore if the joint distribution of the potential outcomes is such that $p_{t-1}(0,0) = \Pr(\overline{Y}_{t-1}=0|D=1)+\Pr(\overline{Y}_{t-1}=0|D=0)-1$, $p_{t-1}(0,\neq 0) = 1-\Pr(\overline{Y}_{t-1}|D=0)$, $p_{t-1}(\neq 0,0) = 1-\Pr(\overline{Y}_{t-1}|D=1)$, $p_t^0(1|0,\neq 0)=1$, $p_t^0(1|\neq 0,0)=0$ and

$$p_t^0(1|0,0) = \frac{\Pr(Y_t=1,\overline{Y}_{t-1}=0|D=0)}{\Pr(\overline{Y}_{t-1}=0|D=1)+\Pr(\overline{Y}_{t-1}=0|D=0)-1}, \tag{A.9}$$

then the counterfactual transition probability is equal to its upper bound. Note that the right hand side of (A.9) is not greater than 1 if (A.5) holds.

Second, consider the upper bound if

$$\frac{1-[1-\Pr(Y_t=1|\overline{Y}_{t-1}=0,D=0)]\Pr\left(\overline{Y}_{t-1}=0|D=0\right)}{\Pr(\overline{Y}_{t-1}=0|D=1)} > 1. \tag{A.10}$$

Then $\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1=0\right]$ in (A.2) is at its upper bound 1 if the latent distribution satisfies

$$p_t^0(1|0,0)p_{t-1}(0,0)+p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0) = \Pr(\overline{Y}_{t-1}=0|D=1) \tag{A.11}$$

Now set $p_{t-1}(0,0)$ at its lower bound. As above, if that bound is 0, then the counterfactual transition probability is equal to $p_t^0(1|0,\neq 0)$ that is unrestricted so that the $[0,1]$ interval gives sharp bounds. If the lower bound on $p_{t-1}(0,0)$ is $\Pr(\overline{Y}_{t-1}=0|D=1)+\Pr(\overline{Y}_{t-1}=0|D=0)-1$, then $p_{t-1}(0,\neq 0) = 1-\Pr(\overline{Y}_{t-1}=0|D=0)$ and $p_{t-1}(\neq 0,0) = 1-\Pr(\overline{Y}_{t-1}=0|D=1)$. If we consider (A.11) for these values of $p_{t-1}(0,0), p_{t-1}(0,\neq 0)$, i.e., the equation

$$p_t^0(1|0,0)(\Pr(\overline{Y}_{t-1}=0|D=1)+\Pr(\overline{Y}_{t-1}=0|D=0)-1)+p_t^0(1|0,\neq 0)(1-\Pr(\overline{Y}_{t-1}=0|D=1)) =$$

$$\Pr(\overline{Y}_{t-1}=0|D=0)$$

then because if $p_t^0(1|0,0)=0$ then $p_t^0(1|0,\neq 0)>1$ and if $p_t^0(1|0,\neq 0)=0$ then $p_t^0(1|0,0)>1$ the only values of $p_t^0(1|0,0), p_t^0(1|0,\neq 0)$ in the unit interval that satisfy this equation are $p_t^0(1|0,0)=1$ and $p_t^0(1|0,\neq 0)=1$.

The final restriction in (A.7) is satisfied if

$$p_t^0(1|\neq 0,0) = 1-\frac{[1-\Pr(Y_t=1|\overline{Y}_{t-1}=0,D=0)]\Pr(\overline{Y}_{t-1}=0|D=0)}{1-\Pr(\overline{Y}_{t-1}=0|D=1)} =$$

$$\frac{1-\Pr(\overline{Y}_{t-1}=0|D=1)-\Pr(\overline{Y}_{t-1}=0|D=1)+\Pr(Y_t=1,\overline{Y}_{t-1}=0|D=0)}{1-\Pr(\overline{Y}_{t-1}=0|D=1)}$$

which is non-negative by (A.10) and obviously not greater than 1.

Note that this construction breaks down if $\Pr(\overline{Y}_{t-1}=0|D=1)=1$. In that case because

$$1-\Pr\left(\overline{Y}_{t-1}=0|D=0\right)+\Pr(Y_t=1,\overline{Y}_{t-1}=0|D=0)] \leq 1$$

we have that (A.10) cannot hold, so that no construction is needed.

We conclude that in all cases we can find a distribution of the latent potential outcomes so that the ATETS attains the upper bound in (A.4). The argument that the lower bound is sharp is analogous.

**Proof of Theorem 2**

We derive bounds on $\text{ATETS}_t^{1,0} = \mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0\right] - \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right]$, with the data providing the observed transition probabilities $\Pr(Y_t = y_t|\overline{Y}_{t-1} = 0, \overline{D}_t = 1)$ and $\Pr(Y_t = y_t|\overline{Y}_{t-1} = 0, \overline{D}_t = 0)$. Under Assumption 2 $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0] = \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 1)$,so that if $\Pr(\overline{Y}_{t-1}^1 = 0|\overline{D}_{t-1} = 1) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1) > 0$ then $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0]$ is point-identified, and if $\Pr(\overline{Y}_{t-1}^1 = 0|\overline{D}_{t-1} = 1) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1) = 0$ then $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0], \mathbb{E}[Y_t^0|\overline{Y}_{t-1}^1 = 0]$ and $\text{ATETS}_t^{1,0}$ are not defined.

Next, we have for the counterfactual transition probability

$$\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] = \frac{p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)}{p_{t-1}(0,0) + p_{t-1}(0,\neq 0)}. \tag{A.12}$$

By Assumption 2 and using similar reasoning as for the proof of Theorem 1 (see (A.3))

$$\frac{[\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0) - 1] \Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_t = 0\right) + p_{t-1}(0,0)}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_t = 1)} \tag{A.13}$$

$$\leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq$$

$$\frac{\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0) \Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_t = 0\right) - p_{t-1}(0,0)}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_t = 1)} + 1.$$

The upper bound is decreasing and the lower bound is increasing in $p_{t-1}(0,0)$. Next, by the Bonferroni inequality

$$p_{t-1}(0,0) \geq \max\left\{\Pr(\overline{Y}_{t-1}^1 = 0) + \Pr(\overline{Y}_{t-1}^0 = 0) - 1, 0\right\}.$$

Also with $Y_0 \equiv 0$

$$\Pr(\overline{Y}_{t-1}^1 = 0) = \prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) = \Pr(\overline{Y}_{t-1}|\overline{D}_{t-1} = 1)$$

and

$$\Pr(\overline{Y}_{t-1}^0 = 0) = \prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) = \Pr(\overline{Y}_{t-1}|\overline{D}_{t-1} = 0)$$

so that

$$p_{t-1}(0,0) \geq \max\left\{\Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0\right) - 1, 0\right\}.$$

If $\Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0\right) - 1 \leq 0$ the lower bound on $p_{t-1}(0,0)$ is 0. In that case the lower bound in (A.13) is non-positive and the upper bound is greater than or equal to 1 so that $0 \leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq 1$.

If $\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1\right)+\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)-1>0$ we have upon substitution of the lower bound on $p_{t-1}(0,0)$ into (A.13) and because the probability $\mathbb{E}\left[Y_t^0|\overline{Y}^1_{t-1}=0\right]$ is bounded by zero and one

$$\max\left\{0,\frac{\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_t=0\right)-1}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_t=1)}+1\right\}$$

$$\leq\mathbb{E}\left[Y_t^0|\overline{Y}^1_{t-1}=0\right]\leq \tag{A.14}$$

$$\min\left\{1,\frac{1-\left[1-\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)\right]\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_t=0\right)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_t=1)}\right\}.$$

Finally, we combine these bounds with the point-identified $\mathbb{E}[Y_t^1|\overline{Y}^1_{t-1}=0]$ to obtain the bounds. Sharpness follows using the same reasoning as for Theorem 1 (essentially by replacing $D$ with $\overline{D}_t$ or $\overline{D}_{t-1}$ in the sharpness proof for Theorem 1).

**Proof of Theorem 3**

As above, under Assumption 2 $\mathbb{E}[Y_t^1|\overline{Y}^1_{t-1}=0]=\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=1)$, so that if $\Pr(\overline{Y}_{t-1}=0|\overline{D}_t=1)>0$ then $\mathbb{E}[Y_t^1|\overline{Y}^1_{t-1}=0]$ is point-identified, and if $\Pr(\overline{Y}_{t-1}=0|\overline{D}_t=1)=0$ then $\text{ATETS}_t^{1,0}$ is not defined. If $\Pr(\overline{Y}_{t-1}=0|\overline{D}_t=1)>0$ we have from (A.13)

$$\frac{\left[\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)-1\right]\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_t=0\right)+p_{t-1}(0,0)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_t=1)} \tag{A.15}$$

$$\leq\mathbb{E}\left[Y_t^0|\overline{Y}^1_{t-1}=0\right]\leq$$

$$\frac{\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_t=0\right)-p_{t-1}(0,0)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_t=1)}+1.$$

Because the lower bound is increasing in $p_{t-1}(0,0)$ and the upper bound decreasing in $p_{t-1}(0,0)$ we need the lower bound on this probability. We have

$$p_{t-1}(0,0)=\Pr(Y^1_{t-1}=0,\ldots,Y^1_1=0,Y^0_{t-1}=0,\ldots,Y^0_1=0)=$$

$$\Pr(Y^1_{t-1}=0,Y^0_{t-1}=0|S_{t-2})\Pr(Y^1_{t-2}=0,\ldots,Y^1_1=0,Y^0_{t-2}=0,\ldots,Y^0_1=0).$$

By Assumption 3 either

$$\Pr\left(Y^1_{i,t-1}=0|S_{i,t-2}\right)\leq\Pr\left(Y^0_{i,t-1}=0|S_{i,t-2}\right), \tag{A.16}$$

or

$$\Pr\left(Y^1_{i,t-1}=0|S_{i,t-2}\right)>\Pr\left(Y^0_{i,t-1}=0|S_{i,t-2}\right), \tag{A.17}$$

for all $i$. Assume that (A.16) holds. By Assumption 4 this implies that

$$\Pr(Y^1_{i,t-1}=0,Y^0_{i,t-1}=1|S_{i,t-2})=0,$$

so that

$$\Pr(Y_{i,t-1}^1 = 0|S_{i,t-2}) = \Pr(Y_{i,t-1}^1 = 0, Y_{i,t-1}^0 = 0|S_{i,t-2}) + \Pr(Y_{i,t-1}^1 = 0, Y_{i,t-1}^0 = 1|S_{i,t-2})$$
$$= \Pr(Y_{i,t-1}^1 = 0, Y_{i,t-1}^0 = 0|S_{i,t-2}).$$

Because this holds for all members of the population we omit $i$ in the sequel. Because Assumptions 3 and 4 hold for all $t$, it follows from this equation by recursion that

$$\Pr(Y_{t-1}^1 = 0, \ldots, Y_1^1 = 0, Y_{t-1}^0 = 0, \ldots, Y_1^0 = 0) = \prod_{s=1}^{t-1} \Pr(Y_s^1 = 0|\overline{Y}_{s-1}^1 = 0),$$

so that

$$p_{t-1}(0,0) = \prod_{s=1}^{t-1} \Pr(Y_s^1 = 0|\overline{Y}_{s-1}^1 = 0) = \prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1).$$

If Assumption 3 holds with (A.17), then

$$p_{t-1}(0,0) = \prod_{s=1}^{t-1} \Pr(Y_s^0 = 0|\overline{Y}_{s-1}^0 = 0) = \prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0).$$

We conclude that

$$p_{t-1}(0,0) \geq \min\left\{\prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1), \prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0)\right\} =$$
$$\min\left\{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0)\right\}.$$

As noted below Theorem 3 the bounds simplifies in an obvious way if we have prior knowledge of the direction of the effect of the treatment. The resulting bounds are sharp.

Next, upon substitution of this lower bound on $p_{t-1}(0,0)$ into (A.3) and because the probability $\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right]$ is bounded by zero and one we have

$$\max\left\{0\,, \frac{[\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0) - 1]\Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0\right)}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)} + \frac{\min\left\{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0)\right\}}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)}\right\}.$$

$$\leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq \tag{A.18}$$

$$\min\left\{1,\ 1 + \frac{\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0)\Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0\right)}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)} - \frac{\min\left\{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0)\right\}}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)}\right\},$$

Finally, we combine these bounds with the point-identified $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0]$ to obtain bounds on $\text{ATETS}_t$.

We now prove that the bounds are sharp, by showing that there is a joint distribution of the latent potential outcomes such that the counterfactual transition probability in (A.2) is equal to the upper bound in (A.18). There are two cases. First, consider the upper bound if

$$1 + \frac{\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0) \Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0\right)}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)} - \tag{A.19}$$

$$\frac{\min\left\{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0)\right\}}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)} \leq 1$$

Combining this upper bound and the equation for the counterfactual transition probability in (A.2) we have to choose the joint distribution of the latent potential outcomes such that

$$p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1) + \tag{A.20}$$

$$\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0) - \min\left\{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0)\right\}$$

and the restrictions

$$p_{t-1}(0,0) + p_{t-1}(0,\neq 0) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_t = 1) \tag{A.21}$$

$$p_{t-1}(0,0) + p_{t-1}(\neq 0,0) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_t = 0)$$

$$p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0) = \Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0),$$

hold. Now set $p_{t-1}(0,0)$ at its lower bound. Initially, consider the case that

$$\min\left\{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0)\right\} = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1),$$

so that the lower bound on $p_{t-1}(0,0)$ is $\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)$, then $p_{t-1}(0,\neq 0) = 0$, $p_{t-1}(\neq 0,0) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0) - \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)$, and the restriction in (A.20) is

$$p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0) = \Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0). \tag{A.22}$$

Therefore to satisfy (A.22) and (A.21) the probabilities $p_t^0(1|0 \neq, 0)$ and $p_t^0(1|\neq 0, 0)$ must satisfy

$$p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0) - p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0) = 0 \tag{A.23}$$

with $p_{t-1}(0,\neq 0) = 0$ and $p_{t-1}(\neq 0,0) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0) - \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)$. Since, $p_{t-1}(0,\neq 0) = 0$ the equation is satisfied if $p_t^0(1|\neq 0,0) = 0$, in which case $p_t^0(1|0,\neq 0)$ is unrestricted and we can set $p_t^0(1|0,\neq 0) = 1$. Therefore if the joint distribution of the potential outcomes is such that $p_{t-1}(0,0) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)$, $p_{t-1}(0,\neq 0) = 0$, $p_{t-1}(\neq 0,0) = \Pr(\overline{Y}_{t-1} = 0|\overline{D}_t = 0) - \Pr(\overline{Y}_{t-1} = 0|\overline{D}_t = 1)$, $p_t^0(1|0,\neq 0) = 1$, $p_t^0(1|\neq 0,0) = 0$ and

$$p_t^0(1|0,0) = \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0)}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)}, \tag{A.24}$$

then the counterfactual transition probability is equal to its upper bound. Note that the right hand side of (A.24) is not greater than 1 if (A.19) holds with

$$\min\left\{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1),\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)\right\}=\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1).$$

Second, consider the upper bound if

$$1+\frac{\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}-. \tag{A.25}$$

$$\frac{\min\left\{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1),\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)\right\}}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}>1$$

Then the $\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1=0\right]$ in (A.2) is at its upper bound 1 if the latent distribution satisfies

$$p_t^0(1|0,0)p_{t-1}(0,0)+p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)=\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1) \tag{A.26}$$

Now set $p_{t-1}(0,0)$ at its lower bound, so that $p_{t-1}(0,0)=\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)$. Then, $p_{t-1}(0,\neq 0)=0$ and $p_{t-1}(\neq 0,0)=\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)-\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)$. If we consider (A.26) for these values of $p_{t-1}(0,0),p_{t-1}(0,\neq 0)$, i.e the equation

$$p_t^0(1|0,0)\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)=\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)$$

then $p_t^0(1|0,0)=1$ and $p_t^0(1|0,\neq 0)$ is unrestricted so that we can set $p_t^0(1|0,\neq 0)=1$. The final restriction in (A.21) is satisfied if

$$p_t^0(1|\neq 0,0)=\frac{\Pr(Y_t=1,\overline{Y}_{t-1}=0|\overline{D}_t=0)-\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}{\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)-\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)}$$

which is non-negative by (A.25) and obviously not greater than 1.

The case that the lower bound on $p_{t-1}(0,0)$ is $\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)$ can be dealt with in an analogous way.

We conclude that in all cases we can find a distribution of the latent potential outcomes so that the ATETS attains the upper bound in Theorem 3. The argument that the lower bound is sharp is analogous.

**Proof of Theorem 4**

Using similar reasoning as above, under Assumption 2 $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1=0]=\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=1)$, so that if $\Pr(\overline{Y}_{t-1}^1=0|\overline{D}_{t-1}=1)=\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)>0$ then $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1=0]$ is point-identified, and if $\Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)=0$ then $\text{ATETS}_t^{1,0}$ are not defined.

Next, we have for the counterfactual transition probability

$$\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1=0\right]=\frac{p_t^0(1|0,0)p_{t-1}(0,0)+p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)}{p_{t-1}(0,0)+p_{t-1}(0,\neq 0)}. \tag{A.27}$$

The expression on the right-hand side is increasing in $p_t^0(1|0,\neq 0)$. By Assumption 5 we have the restriction $p_t^0(1|0,\neq 0)\geq p_t^0(1|0,0)$. Then the upper bound is obtained by setting $p_t^0(1|0,\neq 0)=1$ and lower bound by setting $p_t^0(1|0,\neq 0)=p_t^0(1|0,0)$:

$$p_t^0(1|0,0)\leq\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1=0\right]\leq\frac{p_t^0(1|0,0)p_{t-1}(0,0)+p_{t-1}(0,\neq 0)}{p_{t-1}(0,0)+p_{t-1}(0,\neq 0)}.$$

By Assumption 2 and the law of total probability we have using similar reasoning as for Theorem 1:

$$\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | \overline{D}_t = 0) = p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0) \tag{A.28}$$

Solving for $p_t^0(1|0,0)$ gives

$$p_t^0(1|0,0) = \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0) - p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0)}{p_{t-1}(0,0)}$$

and upon substitution

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0) - p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0)}{p_{t-1}(0,0)} \leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq$$

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0) - p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0) + p_{t-1}(0,\neq 0)}{p_{t-1}(0,0) + p_{t-1}(0,\neq 0)}$$

Both the lower and upper bound is decreasing in $p_t^0(1|\neq 0,0)$. By Assumption 5 we have the restriction $p_t^0(1|\neq 0,0) \geq p_t^0(1|0,0)$. Therefore the lower bound is obtained by setting $p_t^0(1|\neq 0,0)$ at 1. The upper bound is obtained by setting $p_t^0(1|\neq 0,0) = p_t^0(1|0,0)$, upon substitution into (A.28) this implies that

$$p_t^0(1|\neq 0,0) = p_t^0(1|0,0) = \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0).$$

Then

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0) - p_{t-1}(\neq 0,0)}{p_{t-1}(0,0)} \leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq$$

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0) - \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0)p_{t-1}(\neq 0,0) + p_{t-1}(0,\neq 0)}{p_{t-1}(0,0) + p_{t-1}(0,\neq 0)}$$

Because

$$\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1) = p_{t-1}(0,0) + p_{t-1}(0,\neq 0)$$
$$\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0) = p_{t-1}(0,0) + p_{t-1}(\neq 0,0)$$

we have

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_t = 0) - \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0) + p_{t-1}(0,0)}{p_{t-1}(0,0)} \tag{A.29}$$

$$\leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq \frac{[\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0) - 1]p_{t-1}(0,0)}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)} + 1.$$

The lower bound is increasing and the upper bound decreasing in $p_{t-1}(0,0)$. Assumption 5 also improves on the Bonferroni inequality for $p_{t-1}(0,0)$. We have

$$p_{t-1}(0,0) = \prod_{s=1}^{t-1} \Pr(Y_s^1 = 0, Y_s^0 = 0|S_{s-1}).$$

By the Bonferroni inequality and the results above

$$\Pr(Y_s^1 = 0, Y_s^0 = 0|S_{s-1}) \geq \max\{1 - \Pr(Y_s^1 = 1|S_{s-1}) - \Pr(Y_s^0 = 1|S_{s-1}), 0\} \geq$$
$$\max\{1 - \Pr(Y_s = 1|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) - \Pr(Y_s = 1|\overline{Y}_{s-1} = 0, \overline{D}_s = 0), 0\} =$$
$$\max\{\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1, 0\},$$

so that

$$p_{t-1}(0,0) \geq \prod_{s=1}^{t-1} \max\{\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1, 0\}. \tag{A.30}$$

We compare this to the lower bound

$$\max\left\{\prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1, 0\right\}$$

that we obtained in the proof of Theorem 1. First, if there is an $1 \leq s' \leq t-1$ so that

$$\Pr(Y_{s'} = 0|\overline{Y}_{s'-1} = 0, \overline{D}_{s'} = 1) + \Pr(Y_{s'} = 0|\overline{Y}_{s'-1} = 0, \overline{D}_{s'} = 0) - 1 < 0,$$

then

$$\prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1 =$$
$$\Pr(Y_{s'} = 0|\overline{Y}_{s'-1} = 0, \overline{D}_{s'} = 1) \prod_{s=1, s\neq s'}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) +$$
$$\Pr(Y_{s'} = 0|\overline{Y}_{s'-1} = 0, \overline{D}_{s'} = 1) \prod_{s=1, s\neq s'}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1 < 0$$

so that if the new lower bound is 0, so is the previous one. Finally, if for all $s = 1, \ldots, t-1$

$$\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1 > 0,$$

then

$$\prod_{s=1}^{t-1} \left[\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1\right] \geq$$
$$\prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \prod_{s=1}^{t-1} \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1.$$

If $\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1 \leq 0$ for some $s \leq t$ the lower bound on $p_{t-1}(0,0)$ is 0. In that case the lower bound in (A.29) is non-positive and the upper bound is greater than or equal to 1 so that $0 \leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq 1$.

If $\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1 > 0$ for all $s = 1, \ldots, t-1$ we have upon substitution of the lower bound on $p_{t-1}(0,0)$ in (A.30) into (A.29) and because the probability $\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right]$ is bounded by zero,

$$\max\left\{0, \frac{(\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0) - 1)\Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0\right)}{\prod_{s=1}^{t-1}[\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1]} + 1\right\}$$

$$\leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0\right] \leq 1 - \frac{1 - \Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0)}{\Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1)} \cdot \tag{A.31}$$

$$\cdot \prod_{s=1}^{t-1}[\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1].$$

Finally, we combine these bounds with the point-identified $\mathbb{E}[Y_t^1|\overline{Y}_{t-1}^1 = 0]$ to obtain bounds on $\text{ATETS}_t$.

We now prove that the bounds are sharp. We will show that there is a joint distribution of the latent potential outcomes such that the counterfactual transition probability in (A.2) is equal to the lower bound in (A.31). There are two cases. First, consider the lower bound if

$$\frac{(\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0) - 1)\Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0\right)}{\prod_{s=1}^{t-1}[\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1]} + 1 > 0 \tag{A.32}$$

Combining the counterfactual transition probability in (A.2) and this lower bound we have to choose the joint distribution of the latent potential outcomes such that

$$p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0) = \Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1\right) + \tag{A.33}$$
$$\frac{(\Pr(Y_t = 1|\overline{Y}_{t-1} = 0, \overline{D}_t = 0) - 1)\Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0\right)\Pr\left(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1\right)}{\prod_{s=1}^{t-1}[\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1]}$$

and the restrictions

$$\begin{aligned} p_{t-1}(0,0) + p_{t-1}(0,\neq 0) &= \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 1) \qquad \text{(A.34)} \\ p_{t-1}(0,0) + p_{t-1}(\neq 0,0) &= \Pr(\overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0) \\ p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0) &= \Pr(Y_t = 1, \overline{Y}_{t-1} = 0|\overline{D}_{t-1} = 0), \end{aligned}$$

as well as the restrictions given by Assumption 5

$$p_t^0(1|0,\neq 0) \geq p_t^0(1|0,0), \qquad p_t^0(1|\neq 0,0) \geq p_t^0(1|0,0) \tag{A.35}$$

hold. Now set $p_{t-1}(0,0)$ at its lower bound:

$$p_{t-1}(0,0) = \prod_{s=1}^{t-1}\max\{\Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0|\overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1, 0\}.$$

If that bound is 0, then the counterfactual transition probability is equal to $p_t^0(1|0,\neq 0)$ that is unrestricted so that the $[0,1]$ interval gives sharp bounds. If the lower bound is larger than

0, then the lower bound in (A.32) is strictly positive. So we have the find a joint distribution of the potential outcomes such that $p_t^0(1|0,\neq 0)$ is equal to the positive lower bound in (A.32) and not equal to 0. Thus,

$$p_{t-1}(0,\neq 0) = \Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=1)- \tag{A.36}$$
$$\prod_{s=1}^{t-1}[\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1]$$

and

$$p_{t-1}(\neq 0,0) = \Pr(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0)- \tag{A.37}$$
$$\prod_{s=1}^{t-1}[\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1].$$

Next, (A.35) is satisfied if we set $p_t^0(1|0,\neq 0)=p_t^0(1|0,0)$ and $p_t^0(1|\neq 0,0)=1$. Then (A.33) is also satisfied if

$$p_t^0(1|0,0)=p_t^0(1|0,\neq 0)= \tag{A.38}$$
$$\frac{(\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)-1)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)}{\prod_{s=1}^{t-1}[\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1]}+1.$$

Note that the right hand side of (A.38) is not greater than 1 if (A.32) holds. This together with $p_t^0(1|\neq 0,0)=1$ satisfies (A.34), since by $p_t^0(1|\neq 0,0)=1$ and (A.38) we have

$$p_t^0(1|0,0)p_{t-1}(0,0)+p_t^0(1|\neq 0,0)p_{t-1}(\neq 0,0)=\Pr(Y_t=1,\overline{Y}_{t-1}=0|D_{t-1}=0).$$

Therefore if the joint distribution of the potential outcomes is given by (A.36), (A.37),(A.38) and $p_t^0(1|\neq 0,0)=1$ then the counterfactual transition probability is equal to its lower bound.

Second, consider the upper bound if

$$\frac{(\Pr(Y_t=1|\overline{Y}_{t-1}=0,\overline{D}_t=0)-1)\Pr\left(\overline{Y}_{t-1}=0|\overline{D}_{t-1}=0\right)}{\prod_{s=1}^{t-1}[\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1]}+1<0. \tag{A.39}$$

Then the $\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1=0\right]$ in (A.2) is at its lower bound 0 if the latent distribution satisfies

$$p_t^0(1|0,0)p_{t-1}(0,0)+p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)=0 \tag{A.40}$$

Now set $p_{t-1}(0,0)$ at its lower bound. As above, if that bound is 0, then the counterfactual transition probability is equal to $p_t^0(1|0,\neq 0)$ that is unrestricted so that the $[0,1]$ interval gives sharp bounds. If the lower bound on $p_{t-1}(0,0)$ is $\prod_{s=1}^{t-1}[\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=1)+\Pr(Y_s=0|\overline{Y}_{s-1}=0,\overline{D}_s=0)-1]$, then the lower bound in (A.32) is strictly positive. So we have the find a joint distribution of the potential outcomes such that $p_t^0(1|0,\neq 0)$ is equal to the positive lower bound in (A.32) and not equal to 0. Thus, $p_{t-1}(0,\neq 0)$ is given by (A.36) and $p_{t-1}(\neq 0,0)$ is given by (A.37). If we consider (A.40) for these values of $p_{t-1}(0,0), p_{t-1}(0,\neq 0)$ then this equations is satisfied if $p_t^0(1|0,0)=p_t^0(1|0,\neq 0)=0$. The final restriction in (A.34) is satisfied if

$$p_t^0(1|\neq 0,0)=$$

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0)}{\Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0) - \prod_{s=1}^{t-1} [\Pr(Y_s = 0 | \overline{Y}_{s-1} = 0, \overline{D}_s = 1) + \Pr(Y_s = 0 | \overline{Y}_{s-1} = 0, \overline{D}_s = 0) - 1]}$$

which is less than 1 by (A.10) and obviously non-negative.

We conclude that in all cases we can find a distribution of the latent potential outcomes so that the ATETS attains the lower bound in (A.31). The argument that the upper bound is sharp is analogous.

**Proof of Theorem 5**

Using similar reasoning as for the proof of Theorem 4 we have under Assumptions 2 and 5:

$$\mathbb{E}[Y_t^1 | \overline{Y}_{t-1}^1 = 0] = \Pr(Y_t = 1 | \overline{Y}_{t-1} = 0, \overline{D}_t = 1)$$

and

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0 | \overline{D}_t = 0) - \Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0) + p_{t-1}(0,0)}{p_{t-1}(0,0)}$$
$$\leq \mathbb{E}\left[Y_t^0 | \overline{Y}_{t-1}^1 = 0\right] \leq \frac{[\Pr(Y_t = 1 | \overline{Y}_{t-1} = 0, \overline{D}_t = 0) - 1] p_{t-1}(0,0)}{\Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1)} + 1.$$

The lower bound on $\mathbb{E}\left[Y_t^0 | \overline{Y}_{t-1}^1 = 0\right]$ is increasing and the upper bound on $\mathbb{E}\left[Y_t^0 | \overline{Y}_{t-1}^1 = 0\right]$ is decreasing in $p_{t-1}(0,0)$. By the proof of Theorem 3 we have under Assumptions 3 and 4

$$p_{t-1}(0,0) \geq \min\left\{\Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0)\right\},$$

so that

$$\max\left\{0, \frac{(\Pr(Y_t = 1 | \overline{Y}_{t-1} = 0, \overline{D}_t = 0) - 1) \Pr\left(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0\right)}{\min\left\{\Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0)\right\}} + 1\right\} \leq \mathbb{E}\left[Y_t^0 | \overline{Y}_{t-1}^1 = 0\right] \leq$$

$$\frac{1 - \Pr(Y_t = 1 | \overline{Y}_{t-1} = 0, \overline{D}_t = 0)}{\Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1)} \times \min\left\{\Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 1), \Pr(\overline{Y}_{t-1} = 0 | \overline{D}_{t-1} = 0)\right\} + 1.$$

Together with the results for $\mathbb{E}[Y_t^1 | \overline{Y}_{t-1}^1 = 0]$ this gives the bounds. Sharpness follows using the same reasoning as for the proofs of Theorems 3 and 4.

## Appendix B: Heterogenous effects (for online publication only)

Table 3: Bounds on $ATETS^{1,0}$ for the Illinois job bonus experiment. Heterogenous effects for blacks and non-blacks

| | Panel A: Blacks | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | No assumption bounds | | | | MTR+CS+PCO | | | |
| | Lower-CI | LB | UB | Upper-CI | Lower-CI | LB | UB | Upper-CI |
| | (1) | (2) | (3) | (4) | (1) | (2) | (3) | (4) |
| Week | | | | | | | | |
| 1-2 | -0.006 | 0.021 | 0.021 | 0.049 | 0.000 | 0.021 | 0.021 | 0.043 |
| 3-4 | -0.124 | -0.111 | 0.059 | 0.071 | -0.012 | 0.005 | 0.028 | 0.044 |
| 5-6 | -0.180 | -0.167 | 0.058 | 0.070 | -0.004 | 0.015 | 0.043 | 0.062 |
| 7-8 | -0.243 | -0.230 | 0.044 | 0.057 | -0.007 | 0.010 | 0.044 | 0.061 |
| 9-10 | -0.290 | -0.277 | 0.048 | 0.060 | -0.005 | 0.012 | 0.048 | 0.064 |
| 11-12 | -0.352 | -0.342 | 0.030 | 0.040 | -0.013 | 0.001 | 0.030 | 0.044 |
| 13-14 | -0.395 | -0.384 | 0.032 | 0.043 | -0.012 | 0.002 | 0.032 | 0.045 |
| 15-16 | -0.449 | -0.439 | 0.025 | 0.035 | -0.020 | -0.007 | 0.025 | 0.037 |
| 17-18 | -0.496 | -0.485 | 0.028 | 0.039 | -0.021 | -0.007 | 0.028 | 0.042 |
| 19-20 | -0.532 | -0.520 | 0.037 | 0.049 | -0.007 | 0.010 | 0.037 | 0.053 |
| 21-22 | -0.605 | -0.596 | 0.019 | 0.029 | -0.028 | -0.016 | 0.019 | 0.031 |
| 23-24 | -0.635 | -0.623 | 0.039 | 0.051 | -0.011 | 0.006 | 0.039 | 0.055 |
| | PCO [C] | | | | MTR+CS+PCO [D] | | | |
| | Panel B: Non-blacks | | | | | | | |
| | No assumption bounds | | | | MTR+CS+PCO | | | |
| | (1) | (2) | (3) | (4) | (1) | (2) | (3) | (4) |
| Week | | | | | | | | |
| 1-2 | 0.005 | 0.022 | 0.022 | 0.040 | 0.009 | 0.022 | 0.022 | 0.035 |
| 3-4 | -0.158 | -0.148 | 0.106 | 0.116 | 0.002 | 0.016 | 0.040 | 0.053 |
| 5-6 | -0.293 | -0.284 | 0.080 | 0.090 | -0.010 | 0.003 | 0.044 | 0.058 |
| 7-8 | -0.392 | -0.382 | 0.090 | 0.100 | 0.003 | 0.017 | 0.062 | 0.076 |
| 9-10 | -0.523 | -0.513 | 0.077 | 0.087 | 0.001 | 0.011 | 0.074 | 0.086 |
| 11-12 | -0.639 | -0.629 | 0.075 | 0.085 | 0.006 | 0.015 | 0.075 | 0.087 |
| 13-14 | -0.773 | -0.763 | 0.066 | 0.076 | -0.006 | 0.004 | 0.066 | 0.077 |
| 15-16 | -0.889 | -0.879 | 0.062 | 0.071 | 0.003 | 0.013 | 0.062 | 0.072 |
| 17-18 | -0.991 | -0.942 | 0.058 | 0.068 | 0.000 | 0.010 | 0.058 | 0.069 |
| 19-20 | -1.002 | -0.946 | 0.054 | 0.064 | -0.013 | -0.003 | 0.054 | 0.064 |
| 21-22 | -1.002 | -0.940 | 0.060 | 0.071 | -0.024 | -0.013 | 0.060 | 0.073 |
| 23-24 | -1.008 | -0.936 | 0.064 | 0.076 | -0.011 | 0.001 | 0.064 | 0.079 |

Notes: CI is 95% confidence intervals. Variances and covariances used to obtain the CI are estimated using bootstrap (399 replications).

Table 4: Bounds on $ATETS^{1,0}$ for the Illinois job bonus experiment. Heterogenous effects for low and high income workers

| | **Panel A: Below median income** | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | No assumption bounds | | | | MTR+CS+PCO | | | |
| | Lower-CI | LB | UB | Upper-CI | Lower-CI | LB | UB | Upper-CI |
| | (1) | (2) | (3) | (4) | (1) | (2) | (3) | (4) |
| Week | | | | | | | | |
| 1-2 | -0.006 | 0.016 | 0.016 | 0.038 | -0.001 | 0.016 | 0.016 | 0.032 |
| 3-4 | -0.172 | -0.161 | 0.089 | 0.100 | -0.002 | 0.013 | 0.030 | 0.044 |
| 5-6 | -0.285 | -0.274 | 0.067 | 0.078 | -0.012 | 0.002 | 0.033 | 0.047 |
| 7-8 | -0.355 | -0.344 | 0.080 | 0.091 | 0.009 | 0.024 | 0.057 | 0.072 |
| 9-10 | -0.461 | -0.450 | 0.067 | 0.078 | 0.003 | 0.014 | 0.067 | 0.080 |
| 11-12 | -0.568 | -0.558 | 0.050 | 0.060 | -0.010 | 0.000 | 0.050 | 0.062 |
| 13-14 | -0.644 | -0.634 | 0.047 | 0.056 | -0.001 | 0.008 | 0.047 | 0.058 |
| 15-16 | -0.727 | -0.717 | 0.041 | 0.051 | -0.009 | 0.001 | 0.041 | 0.052 |
| 17-18 | -0.802 | -0.792 | 0.040 | 0.050 | -0.008 | 0.002 | 0.040 | 0.051 |
| 19-20 | -0.883 | -0.869 | 0.049 | 0.063 | -0.008 | 0.003 | 0.049 | 0.062 |
| 21-22 | -1.021 | -0.959 | 0.041 | 0.053 | -0.024 | -0.014 | 0.041 | 0.053 |
| 23-24 | -1.015 | -0.958 | 0.042 | 0.053 | -0.021 | -0.010 | 0.042 | 0.055 |
| | PCO [C] | | | | MTR+CS+PCO [D] | | | |
| | **Panel B: Above median income** | | | | | | | |
| | No assumption bounds | | | | MTR+CS+PCO | | | |
| | (1) | (2) | (3) | (4) | (1) | (2) | (3) | (4) |
| Week | | | | | | | | |
| 1-2 | 0.010 | 0.028 | 0.028 | 0.047 | 0.014 | 0.028 | 0.028 | 0.043 |
| 3-4 | -0.125 | -0.113 | 0.099 | 0.111 | 0.001 | 0.015 | 0.045 | 0.059 |
| 5-6 | -0.238 | -0.227 | 0.081 | 0.092 | -0.002 | 0.012 | 0.059 | 0.073 |
| 7-8 | -0.342 | -0.331 | 0.076 | 0.087 | -0.003 | 0.008 | 0.068 | 0.000 |
| 9-10 | -0.447 | -0.435 | 0.070 | 0.082 | -0.002 | 0.010 | 0.070 | 0.084 |
| 11-12 | -0.535 | -0.524 | 0.074 | 0.086 | 0.012 | 0.024 | 0.074 | 0.089 |
| 13-14 | -0.666 | -0.653 | 0.066 | 0.078 | -0.012 | 0.000 | 0.066 | 0.080 |
| 15-16 | -0.772 | -0.760 | 0.060 | 0.072 | 0.003 | 0.015 | 0.060 | 0.072 |
| 17-18 | -0.881 | -0.870 | 0.058 | 0.070 | -0.003 | 0.009 | 0.058 | 0.070 |
| 19-20 | -1.008 | -0.952 | 0.048 | 0.059 | -0.010 | 0.000 | 0.048 | 0.058 |
| 21-22 | -1.010 | -0.947 | 0.053 | 0.065 | -0.024 | -0.013 | 0.053 | 0.065 |
| 23-24 | -0.999 | -0.929 | 0.071 | 0.086 | 0.004 | 0.018 | 0.071 | 0.086 |

Notes: CI is 95% confidence intervals. Variances and covariances used to obtain the CI are estimated using bootstrap (399 replications).

## Appendix C: Average treatment effect on survivors (for online publication only)

In this appendix we consider the average effect when averaging over the subpopulation of individuals who would have survived until $t$ under both treatment and no-treatment. We call this average effect the Average Treatment Effect on Survivors, $\text{ATES}_t$:

**Definition 2** *Average Treatment Effect on Survivors (ATES)*

$$\text{ATES}_t = \mathbb{E}\left(Y_t^1|\overline{Y}_{t-1}^1 = 0, Y_{t-1}^0 = 0\right) - \mathbb{E}\left(Y_t^0|\overline{Y}_{t-1}^1 = 0, Y_{t-1}^0 = 0\right)$$

The bounds for $\text{ATES}_t$ are given in Theorem 6.

**Theorem 6 (Bounds on ATES)** *Suppose that Assumption 1 holds. If* $\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1 \leq 0$*, then* $\text{ATES}_t$ *is not defined.*

*If* $\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1 > 0$*, then we have the following sharp bounds*

$$\max\left\{0, \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}\right\} -$$

$$\min\left\{1, \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0)}{\Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) - 1}\right\} \leq \text{ATES}_t \leq$$

$$\min\left\{1, \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}\right\} -$$

$$\max\left\{0, \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0) + \Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) - 1}{\Pr\overline{Y}_{t-1} = 0|D = 1) + \Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) - 1}\right\}.$$

**Proof:** First, consider bounds on $\mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] = p_t^1(1|0,0)$. By Assumption 2

$$\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) = \Pr(Y_t^1 = 1, \overline{Y}_{t-1}^1 = 0).$$

By the law of total probability

$$\Pr(Y_t^1 = 1, \overline{Y}_{t-1}^1 = 0) = p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)$$

Therefore,

$$\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) = p_t^0(1|0,0)p_{t-1}(0,0) + p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)$$

Solving for $p_t^1(1|0,0) = \mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right]$ gives

$$\mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] = \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) - p_t^0(1|0,\neq 0)p_{t-1}(0,\neq 0)}{p_{t-1}(0,0)}$$

The expression on the right-hand side is decreasing in $p_t^0(1|0, \neq 0)$. The lower bound is obtained by setting $p_t^0(1|0, \neq 0)$ at 1 and the upper bound by setting $p_t^0(1|0, \neq 0)$ at 0.

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) - p_{t-1}(0, \neq 0)}{p_{t-1}(0, 0)}$$

$$\leq \mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{p_{t-1}(0, 0)}.$$

Because

$$\Pr(\overline{Y}_{t-1} = 0|D = 1) = p_{t-1}(0, 0) + p_{t-1}(0, \neq 0)$$

we have

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) - \Pr(\overline{Y}_{t-1} = 0|D = 1) + p_{t-1}(0, 0)}{p_{t-1}(0, 0)}$$

$$\leq \mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{p_{t-1}(0, 0)}.$$

The upper bound is decreasing and the lower bound is increasing in $p_{t-1}(0, 0)$. From the proof of theorem 1 we have

$$p_{t-1}(0, 0) \geq \max\left\{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1, 0\right\}.$$

If $\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1 > 0$ then we are sure that there are survivors in both treatment arms. Upon substitution of this lower bound

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}$$

$$\leq \mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}.$$

By an analogous argument we have

$$\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0) + \Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) - 1}{\Pr \overline{Y}_{t-1} = 0|D = 1) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}$$

$$\leq \mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0)}{\Pr\left(\overline{Y}_{t-1} = 0|D = 1\right) + \Pr\left(\overline{Y}_{t-1} = 0|D = 0\right) - 1}.$$

Substitution of these results for $\mathbb{E}\left[Y_t^1|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right]$ and $\mathbb{E}\left[Y_t^0|\overline{Y}_{t-1}^1 = 0, \overline{Y}_{t-1}^0 = 0\right]$ and because both probabilites are bounded by zero and one gives the bounds on $\text{ATES}_t$.