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Mean Ratio Statistic for Measuring predictability *

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Abstract

We propose an alternative Ratio Statistic for measuring predictability of stock prices. Our statistic is based on actual returns rather than logarithmic returns and is therefore better suited to capturing price predictability. It captures not only linear dependence in the same way as the variance ratio statistics of Lo and MacKinlay (1988) but also some nonlinear dependencies. We derive the asymptotic distribution of the statistics under the null hypothesis that simple gross returns are unpredictable after a constant mean adjustment. This represents a test of the weak form of the Efficient Market Hypothesis. We also consider the multivariate extension, in particular, we derive the restrictions implied by the EMH on multiperiod portfolio gross returns. We apply our methodology to test the gross return predictability of various financial series.

JEL classification: C10, C22, G10, G14
Keywords: Variance Ratio Tests, Martingale, Predictability

1 Introduction

Variance ratio tests (Cochrane ,1988; Lo and MacKinlay, 1988; Poterba and Summers, 1988) are widely used to test the (weak form of) Efficient Market Hypothesis (EMH) of no predictability of asset returns. One particular advantage of the variance ratio test over the alternatives, such as the standard Box-Pierce statistic[1], is that the direction of the ratio depends on all the first k autocorrelations and their relative magnitudes, thus providing the direction of the predictability. The original Variance Ratio test, developed by Lo and MacKinlay (1988) and all other modifications thereof focus on the log return predictability, where the log return is defined to be the first difference of the log prices, i.e., $r_t := \log P_t - \log P_{t-1}$. Although very convenient, log returns are just an approximation of the actual return defined

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[1] Recall that the Box-Pierce Q statistic involves just the squared autocorrelations, i.e. $Q = T\sum_{j=1}^{P} \hat{\rho}_j^2$.

by $R_t := \frac{P_t}{P_{t-1}} - 1$, which is much harder to work with. Due to its convenience, most tests of the EMH were developed for the log returns. Here we focus directly on the simple gross return $\mathcal{R}_t := \frac{P_t}{P_{t-1}}$ and derive an alternative ratio statistics to test the hypothesis that risk adjusted gross returns are a martingale difference sequence.

Throughout the paper "$\Longrightarrow$" denotes convergence in distribution.

2 Test Statistic

Suppose that stock prices P_t obey the martingale hypothesis (after a constant risk adjustment which we take to be represented by μ), or more precisely suppose that the gross return series satisfies

$$E\left[\mathcal{R}_{t+1}\,|\mathcal{F}_t\right] = E\left[\frac{P_{t+1}}{P_t}\,|\mathcal{F}_t\right] = (1+\mu) \tag{1}$$

for each t, where $\mathcal{F}_t = \sigma(P_k, k \leq t)$ is a sigma-algebra, containing current and past prices and μ is a constant. The gross return over the most recent j periods can be written as

$$\mathcal{R}_{t+j}(j) = \frac{P_{t+j}}{P_t} = \frac{P_{t+j}}{P_{t+j-1}} \times \frac{P_{t+j-1}}{P_{t+j-2}} \times \cdots \times \frac{P_{t+1}}{P_t} = \mathcal{R}_{t+1} \times \mathcal{R}_{t+2} \times \ldots \times \mathcal{R}_{t+j}, \tag{2}$$

and by the law of iterated expectations it follows that

$$E\left[\mathcal{R}_{t+j}(j)\,|\mathcal{F}_t\right] = (1+\mu)^j \equiv \mu_j \tag{3}$$

for all $j, i \in \mathbb{Z}$ and all t. For motivation we are comparing the mean of K period gross returns with the mean of one period gross returns.

$$\tau_K = \frac{E\left[\mathcal{R}_{t+K}(K)\right]}{E^K\left[\mathcal{R}_{t+1}\right]} = 1 \tag{4}$$

This ratio is the basis of our testing strategy. Alternative statistic $\tau_{K,L,\alpha,\beta}$ can be written as

$$\tau_{K,L,\alpha,\beta} \equiv \frac{\left(E\left[\mathcal{R}_{t+K}(K)\right]\right)^\alpha}{\left(E\left[\mathcal{R}_{t+L}(L)\right]\right)^\beta} = 1, \tag{5}$$

where $\beta/\alpha = K/L$. Unlike the usual variance ratio statistics, this quantity only depends on the first moments of gross returns, but we show below how this quantity captures linear dependence under the alternative hypothesis.

Forming the sample analogue of $\tau_{K,L,\alpha,\beta}$ and approximating it with the first order Taylor expansion we get:

$$\begin{aligned}\widehat{\tau}_{K,L,\alpha,\beta}-1 &\approx \frac{\alpha\mu_K^{\alpha-1}\left(\hat{\mu}_K-\mu_K\right)}{\mu_L^{\beta}}-\frac{\beta\mu_K^{\alpha}\left(\hat{\mu}_L-\mu_L\right)}{\mu_L^{\beta+1}}\\ &= \frac{\alpha\left(\hat{\mu}_K-\mu_K\right)}{\mu_1^K}-\frac{\beta\left(\hat{\mu}_L-\mu_L\right)}{\mu_1^L}.\end{aligned}$$

Suppose that we observe a sample of prices on an unequally spaced grid $\{t_1,\ldots,t_T\}$, P_{t_i}, $i=1,\ldots,T$. Define the spacing of the observations $\delta_i=t_{i+1}-t_i\in\mathbb{Z}_+$, for $i=2,\ldots,T$; regular sampling would have $\delta_i=1$ for all i, but other structures are encountered in practice. Then define for $j=1,2,\ldots$

$$\widehat{\mu}_j=\frac{1}{T_j}\sum_{\{i:\delta_i=j\}}\frac{P_{t_{i+1}}}{P_{t_i}}=\frac{1}{T_j}\sum_{\{i:\delta_i=j\}}\frac{P_{t_i+j}}{P_{t_i}}, \tag{6}$$

where $T_j=\sum_{i=1}^{T-1}1\{\delta_i=j\}$ is the number of observations available to compute the j period return. In the special case that the observations are equally spaced, the spacing is $\delta_i=t_{i+1}-t_i=1$. Then define for $j=1,2,\ldots$

$$\widehat{\mu}_j=\frac{1}{T-j}\sum_{t=1}^{T-j}\frac{P_{t+j}}{P_t}$$

and we might take $L=1$[2] and

$$\widehat{\tau}_K=\frac{\widehat{\mu}_K}{\left(\widehat{\mu}_1\right)^K}. \tag{7}$$

Although we focus here on this particular statistic, alternative statistics and their limiting distributions are discussed in the Appendix 2.

3 Distribution Theory

We now turn to the distribution theory of $\widehat{\tau}_K$ under the null hypothesis. We shall assume that the observations are recorded at equally spaced intervals.

Define the sequence

$$u_{t:t+K}\equiv\frac{P_{t+K}}{P_t}-(1+\mu)^K=\mathcal{R}_{t+K}(K)-(1+\mu)^K, \tag{8}$$

which determines the estimation error in $\widehat{\mu}_K$. We consider two different cases, namely the "M-dependent" case where we take $u_{t:t+K}$ to be the $2(K-1)$- dependent sequence, and the

[2] Note that we could analagously to CLM 2.4.22 calculate $\widehat{\mu}_j$ using non-overlapping observations so that $\widehat{\mu}_j^{no}=\frac{1}{M}\sum_{k=1}^{M}\frac{P_{jk+1}}{P_{jk+1-j}}$, where $Mj+1=T$.

mixing case, where we allow $u_{t:t+K}$ to be an $\alpha-$mixing process.[3]

3.1 $M(T)$-dependent Case

Let $Z_{t,K}$ be the following 2×1 vector

$$Z_{t,K} = \begin{pmatrix} u_{t:t+K} \\ u_{t:t+1} \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{t+K}(K) - (1+\mu)^K \\ \mathcal{R}_{t+1} - (1+\mu) \end{pmatrix} \tag{9}$$

Under the null hypothesis, the autocorrelation function of $Z_{t,K}$ is zero for all lags bigger than $K-1$. Furthermore, if $\mathcal{R}_{t+1}$ are independent then $Z_{t,K}$ is a $2(K-1)$-dependent sequence, i.e., $Z_{t,K}, Z_{s,K}$ are independent when $|t-s| > K-1$. We will not assume that underlying returns are independent over time, but allow them to be "M-dependent" where the order, say $L(T)$,may increase with T. In fact, we will make the high level assumption that $Z_{t,K}$ is $M(T)$ dependent sequence, which is consistent with the underlying return series being $L(T)$ dependent for some $L(T)$. In this case we can apply Berk's (1973) CLT for finitely dependent triangular array of random variables. Sufficient conditions (which we call MD to denote M-dependence) are as follows.

ASSUMPTION MD

MD1. *For some* $\delta > 0$, *for all* t, l $E\left[|Z_{lt,K}|^{2+\delta}\right] \leq C < \infty$ *where* $l = 1, \dots, L$ *with* L *being the row rank of* $Z_{t,K}$ *and* C *is a constant.*
MD2. *For all* i, j, $\text{var}\left(\sum_{t=i+1}^{j} Z_{lt,K}\right) \leq (j-i)C$.
MD3. *The limit below exists and is positive and finite*

$$\lim_{T\to\infty} \frac{1}{T} \text{var}\left(\sum_{t=1}^{T-K} Z_{t,K}\right) \equiv \Omega.$$

MD4. *As* $T \to \infty$, $M(T)^{2+2/\delta}/T \to 0$.

For a stationary process, condition MD2 obviously holds; for nonstationary process maybe a further explanation is required. The moment condition MD1 seems natural. A sufficient condition would be: for some $\delta > 0$, for all t,

$$E\left[\mathcal{R}_{t+K}(K)^{2+\delta}\right] + E\left[\mathcal{R}_t^{2+\delta}\right] \leq C < \infty$$

We can apply this to the case where $\mathcal{R}_{t+1}$ is an independent sequence, in which case

[3] The special case of R_{t+1} being an i.i.d. sequence is considered in the Appendix 1.

$M(T) = K - 1$, where K is fixed so MD4 is automatically satisfied. However, we allow the gross one period returns to be $L(T)$ dependent, where $M(T) = (K-1)L(T)$ satisfies condition MD4.

4 Mixing case

We next allow $Z_{t,K}$, defined by eq. (9), to be the $\alpha-$ mixing sequence. Let $\mathcal{F}_t$ be the natural filtration $\{\mathcal{R}_t, \mathcal{R}_{t-1}, \ldots\}$. For this case the sufficient conditions (which we denote by M to denote mixing) for applying the CLT are as follows

ASSUMPTION M

M1. *For all t, $Z_{t,K}$ satisfies $E[Z_{t,K}|\mathcal{F}_t] = 0$, and the following limit exists*

$$\lim_{T\to\infty} \frac{1}{T}\operatorname{var}\left(\sum_{t=1}^{T-K} Z_{t,K}\right).$$

Denote the above limit by Ω.

M2. *$Z_{t,K}$ is α-mixing with coefficient $\alpha(m)$ of size $r/(r-1)$, where $r > 1$, such that for all t and for any $j \geq 0$, there exists some $\delta > 0$ for which $E|Z_{it}Z_{l,t-j}|^{(r+\delta)} < \Delta < \infty$ for all $i, l = 1, 2$.*

Under Assumptions $M1$ and $M2$ the result in eq. (10) holds. Note that one can replace the mixing condition on $Z_{t,K}$ by the same condition on $\mathcal{R}_t$. These conditions do not require stationarity but do require some uniform bound on moments and mixing.

4.1 Central Limit Theorem

Define

$$A_K = \left(\frac{1}{(1+\mu)^K}, \quad \frac{-K}{(1+\mu)}\right)^{\intercal}.$$

THEOREM 1. *Suppose that Assumptions MD1-MD4 or Assumptions M1-M2 are satisfied. Then*

$$\sqrt{T}(\widehat{\tau}_K - 1) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T-K} A_K^{\intercal} Z_{t,K} + o_p(1) \Longrightarrow N(0, W_K), \tag{10}$$

where $W_K \equiv A_K \Omega A_K^{\intercal}$ and Ω is defined by eq. (11).

The proof of Theorem 1 is provided in the Appendix 1. In our case, $Z_{t,K}$ and $Z_{t+j,K}$ are uncorreltated for $|j| \geq K$, and the form of the asymptotic variance is simpler

$$\Omega = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T-K} \left(EZ_{t,K}Z_{t,K}^{\intercal} + \sum_{j=\pm 1}^{\pm(K-1)} \left(1 - \frac{j}{T-K}\right) EZ_{t,K}Z_{(t+j),K}^{\intercal} \right). \tag{11}$$

5 Standard errors and bias correction

This secton is aimed at providing empirical implementation of the Mean Ratio statistics $\widehat{\tau}_K$, including the bias correction, and inference based on the asymptotic result stated in the Theorem 1. First note that there is a simple expression for the asymptotic variance, namely

$$W_K \equiv A_K^{\intercal}\Omega A_K = \frac{1}{(1+\mu)^{2K}}\Upsilon_K + \frac{K^2}{(1+\mu)^2}\Upsilon_1 - \frac{2K}{(1+\mu)^{K+1}}\Upsilon_{K1}, \tag{12}$$

where Υ_1 and Υ_K are given by:

$$\Upsilon_1 \equiv \lim_{T\to\infty} \frac{1}{T} \sum_t E\left[\left(\mathcal{R}_{t+1} - (1+\mu)\right)\left(\mathcal{R}_{t+1} - (1+\mu)\right)\right] = \gamma_1(0)$$

$$\Upsilon_K \equiv \lim_{T\to\infty} \frac{1}{T} \sum_t \sum_s E\left[\left(\mathcal{R}_{t+K}(K) - (1+\mu)^K\right)\left(\mathcal{R}_{s+K}(K) - (1+\mu)^K\right)\right] = \gamma_K(0) + \sum_{j=\pm 1}^{\pm(K-1)} \gamma_K(j)$$

$$\Upsilon_{K,1} \equiv \lim_{T\to\infty} \frac{1}{T} \sum_t \sum_s E\left[\left(\mathcal{R}_{t+K}(K) - (1+\mu)^K\right)\left(\mathcal{R}_{s+1} - (1+\mu)\right)\right] = \gamma_{K,1}(0) + \sum_{j=1}^{K-1} \gamma_{K,1}(j).$$

The detailed derivation of this result is provided in the proof of Theorem 1 in the Appendix 1. Empirically, we should estimate W_K as follows

$$\widehat{W}_K = \frac{1}{(1+\widehat{\mu})^{2K}}\widehat{\Upsilon}_K + \frac{K^2 - 2K}{(1+\widehat{\mu})^2}\widehat{\Upsilon}_1 - \frac{2K}{(1+\mu)^{K+1}}\widehat{\Upsilon}_{K1} \tag{13}$$

$$\widehat{\Upsilon}_1 \equiv \widehat{\gamma}_1(0) \quad \text{and} \quad \widehat{\Upsilon}_K \equiv \widehat{\gamma}_K(0) + \sum_{j=\pm 1}^{\pm(K-1)} \widehat{\gamma}_K(j) \tag{14}$$

with

$$\widehat{\gamma}_K(j) = \frac{1}{T-j-K} \sum_{t=j+1}^{T-j-K} \widehat{u}_{t:t+K}\widehat{u}_{t+j:t+j+K}, \tag{15}$$

where $\widehat{u}_{t:t+K} = \mathcal{R}_{t+K}(K) - (1+\widehat{\mu})^K$ and $\widehat{\mu} = \widehat{\mu}_1 - 1$. Then, $\widehat{W}_K \to W_K$ with probability one. The standard errors can be then easily derived from eq. (12) as the square root of the corresponding variance $\widehat{W}_K$.

Recall, however, that $\widehat{\tau}_K$ is formed as a ratio of two estimated means:

$$\widehat{\tau}_K = \frac{\widehat{\mu}_K}{(\widehat{\mu}_1)^K},$$

which may result in the need of the finite sample bias correction. And indeed, forming the second-order Taylor expansion of $\widehat{\tau}_K$ we have

$$\begin{aligned}\widehat{\tau}_K - 1 &\simeq \frac{\widehat{\mu}_K - \mu_K}{(\mu_1)^K} - K\frac{\mu_K}{(\mu_1)^{K+1}}(\widehat{\mu}_1 - \mu_1) + \frac{K(K+1)\mu_K}{2(\mu_1)^{K+2}}(\widehat{\mu}_1 - \mu_1)^2 \\ &\simeq \frac{\widehat{\mu}_K - \mu_K}{(1+\mu)^K} - \frac{K}{(1+\mu)}(\widehat{\mu}_1 - \mu_1) + \frac{K(K+1)}{2(1+\mu)^2}(\widehat{\mu}_1 - \mu_1)^2 .\end{aligned}$$

Taking expectations of the above expression we can deduce that the bias corrected estimator of $\widehat{\tau}_K^{bc}$ is given by

$$\widehat{\tau}_K^{bc} = \widehat{\tau}_K - \frac{K(K+1)\widehat{V}_1}{2(\widehat{\mu}_1)^2 (T-1)}, \tag{16}$$

where $\widehat{V}_1$ estimates consistently the asymptotic variance of $\sqrt{T}(\widehat{\mu}_1 - \mu_1)$, specifically,

$$\widehat{V}_1 = \frac{1}{T-1}\sum_{t=1}^{T-1}\left(\mathcal{R}_{t+1} - \overline{\mathcal{R}}\right)^2, \quad \overline{\mathcal{R}} = \frac{1}{T-1}\sum_{t=1}^{T-1}\mathcal{R}_{t+1}.$$

Define:

$$se_K \equiv \sqrt{\frac{W_K}{T}} \quad \text{and} \quad \widehat{se}_K \equiv \sqrt{\frac{\widehat{W}_K}{T-K}}. \tag{17}$$

Accounting for the bias correction and estimated standard errors, we obtain the following.

THEOREM 2. *Suppose that Assumptions MD1-MD4 or Assumptions M1-M2 are satisfied. Then*

$$\frac{\widehat{\tau}_K^{bc} - 1}{\widehat{se}_K} \Longrightarrow N(0,1) \tag{18}$$

where $\widehat{\tau}_K^{bc}$ and $\widehat{se}_K$ are defined by eq. (17) and (16).

This version of the CLT for our Mean Ratio statistic is particularly convenient for empirical implementations.

REMARK 1. Note that the expression (12) invokes $2(K-1)$ autocovariances of R_{t+K} and $K-1$ covariances between R_{t+K} and R_{t+1}, thus W_K may or may not be a positive number. This is a well-known problem in long-run variance estimation and different methods exist to ensure that the limiting variance is positive-definite (see Andrews (1991), Newey and West (1987) among others). Majority of the methods are based on the proper scaling of the autocovariances such that the variance terms doniminate, which, however, may come at the price of the distorted empirical size of the test statistic. An existing alternative to deal with this issue is to use the subsampling procedure to directly approximate the limiting distribution, see Politis et al. (1999).

6 Subsampling

With some abuse of notation, the centered and properly scaled test statistic, call it T_K, can be re-written as a function of the data $\{\mathcal{R}_t : t = 1, \ldots, T\}$:

$$T_K = \sqrt{T}\,[\widehat{\tau}_K(\mathcal{R}_1, \ldots, \mathcal{R}_T) - 1]\,.$$

Let

$$G_T(x) = \Pr\left(\sqrt{T}\,[\widehat{\tau}_K(\mathcal{R}_1, \ldots, \mathcal{R}_T) - 1] \leq x\right) \tag{19}$$

denote the distribution function of W_K. Let $\widehat{\tau}_{K,b,t}$ be equal to the statistic $\widehat{\tau}_K$ but evaluated at the subsample $\{\mathcal{R}_t, \ldots, \mathcal{R}_{t+b-1}\}$ of size b, i.e.,

$$\widehat{\tau}_{K,b,t} = \widehat{\tau}_K(\mathcal{R}_t, \mathcal{R}_{t+1}, \ldots, \mathcal{R}_{t+b-1}) \text{ for } t = 1, \ldots, T - b + 1.$$

We note that each subsample of size b (taken *without replacement* from the original data) is indeed a sample of size b from the true sampling distribution of the original data. Hence, it is clear that one can approximate the sampling distribution of T_K using the distribution of the values of $\widehat{\tau}_{K,b,t}$ computed over $T - b + 1$ different subsamples of size b. That is, we approximate the sampling distribution G_T of T_K by

$$\hat{G}_{T,b}(x) = \frac{1}{T - b + 1} \sum_{t=1}^{T-b+1} 1\left(\sqrt{b}\,(\widehat{\tau}_{K,b,t} - \widehat{\tau}_K) \leq x\right).$$

Let $g_{T,b}(1-\alpha)$ denote the $(1-\alpha)$-th sample quantile of $\hat{G}_{T,b}(\cdot)$, i.e.,

$$g_{T,b}(1-\alpha) = \inf\{w : \hat{G}_{T,b}(w) \geq 1 - \alpha\}.$$

We call it the *subsample critical value* of significance level α. Thus, we reject the null hypothesis at the significance level α if $T_K > g_{T,b}(1-\alpha)$. The computation of this critical value is not particularly onerous, although it depends on how big b is. The subsampling method has been proposed in Politis and Romano (1994) and is thoroughly reviewed in Politis, Romano and Wolf (1999). It works in many cases where the standard bootstrap fails: in heavy tailed distributions, in unit root cases, in cases where the parameter is on the boundary of its space, etc.

We now show that our subsampling procedure works under a very weak condition on b. In many practical situations, the choice of b will be data-dependent, see Linton, Maasoumi and Whang (2005, Section 5.2) for some methodology for choosing b. To accommodate such possibilities, we assume that $b = \hat{b}_T$ is a data-dependent sequence satisfying

Assumption C: $\Pr[l_T \leq \hat{b}_T \leq u_T] \to 1$ where l_T and u_T are integers satisfying $1 \leq l_T \leq u_T \leq T$, $l_T \to \infty$ and $u_T/T \to 0$ as $T \to \infty$.

The following theorem shows that our test based on the subsample critical value has asymptotically correct size:

Theorem 3: *Suppose Assumptions A, B, and C hold. Then, under the null hypothesis* H_0,

$$\lim_{T\to\infty} \Pr[W_K > g_{T,\hat{b}_T}(1-\alpha)] \leq \alpha.$$

Theorem 3 shows that our test based on the subsampling critical values has asymptotically valid size under the null hypothesis. Under additional regularity conditions, we can extend this pointwise result to establish that our test has asymptotically correct size uniformly over the distributions under the null hypothesis, using the arguments of Andrews and Shi (2013) and Linton, Song and Whang (2010). For brevity, we do not discuss the details of this issue in this paper.

We next establish that the test S_T based on the subsampling critical values is consistent against the fixed alternative H_1.

Theorem 4: *Suppose that Assumptions A, B, and C hold. Then, under the alternative hypothesis* H_1,

$$\lim_{T\to\infty} \Pr[W_K > g_{T,\hat{b}_T}(1-\alpha)] = 1.$$

7 Interpretation under the alternative

In this section we discuss the behaviour of the population statistics under the generic stationary alternative to (1). For illustration consider the special case $K = 2$, when

$$\begin{aligned}
\frac{E\left[\frac{P_{t+2}}{P_t}\right]}{E^2\left[\frac{P_{t+1}}{P_t}\right]} &= \frac{E\left[(1+\mu+\mathcal{R}_1-(1+\mu))\,(1+\mu+\mathcal{R}_2-\mu)\right]}{(1+\mu)^2} \\
&= \frac{(1+\mu)^2+\operatorname{cov}(\mathcal{R}_1,\mathcal{R}_2)}{(1+\mu)^2} \\
&= 1+\frac{\gamma_1(0)}{(1+\mu)^2}.
\end{aligned}$$

The second term captures the linear autocovariances and follows the same direction as the usual variance ratio statistics applied to log returns. This shows that our ratio will be one if and only if $\gamma_1(0) = 0$, where $\gamma_1(0)$ is the first order autocovariance of the net/gross return series. In general, the following formula holds

$$\begin{aligned}
\frac{E\left[\mathcal{R}_{t+K}(K)\right]}{E^K\left[\mathcal{R}_{t+1}\right]} &= 1+\frac{K}{(1+\mu)^2}\sum_{j=1}^{K-1}\left(1-\frac{j}{K}\right)\gamma(j) \\
&+\frac{(K-2)E[(\mathcal{R}_{t+1}-E\mathcal{R}_{t+1})\,(\mathcal{R}_{t+2}-E\mathcal{R}_{t+2})\,(E\mathcal{R}_{t+3}-E\mathcal{R}_{t+3})]}{(1+\mu)^3} \\
&+\frac{(K-3)E[(\mathcal{R}_{t+1}-E\mathcal{R}_{t+1})\,(\mathcal{R}_{t+2}-E\mathcal{R}_{t+2})\,(E\mathcal{R}_{t+4}-E\mathcal{R}_{t+4})]}{(1+\mu)^3} \\
&+\ldots+\frac{E[(\mathcal{R}_{t+1}-E\mathcal{R}_{t+1})\,(\mathcal{R}_{t+2}-E\mathcal{R}_{t+2})\ldots(E\mathcal{R}_{t+K}-E\mathcal{R}_{t+K})]}{(1+\mu)^K}.
\end{aligned}$$

We should associate values of the ratio greater than one with positive dependence/momentum in stock prices and likewise a value of the ratio less than one is associated with negative dependence/contrarian movements in stock prices. In the high frequency situation we might take $\mu \simeq 0$ and all the higher order terms are of smaller order and the ratio is approximately

$$\frac{E\left[\mathcal{R}_{t+K}\right]}{E\left[\mathcal{R}_{t+1}\right]^K} \simeq 1+K\sum_{j=1}^{K-1}\left(1-\frac{j}{K}\right)\gamma_{\mathcal{R}}(j),$$

which is similar to the usual variance ratio and is likewise depending on all the autocovariances (and their relative magnitudes) in a linear fashion. The above ratio shares a similar advantage of providing the direction of the predictability (in comparison to Box-Pierce Q statistic) and provides an additional advantage of dealing directly with gross returns rather than with log returns.

8 Fads

Suppose that the true efficient price obeys

$$\frac{P_{t+1}^*}{P_t^*} = (1+\mu)\, Z_{t+1},$$

where $Z_t > 0$ is iid with mean one. Suppose however that observed price is

$$P_t = P_t^* \eta_t,$$

where $\eta_t > 0$ is an iid (or more generally stationary) misspricing error that has mean one. Then

$$\frac{P_{t+1}}{P_t} = (1+\mu)\, Z_{t+1} \frac{\eta_{t+1}}{\eta_t}.$$

It follows that under the iid assumptions

$$E\left[\frac{P_{t+1}}{P_t} |\mathcal{F}_t\right] = (1+\mu)\, \frac{1}{\eta_t},$$

so that the martingale structure is not present. In terms of the unconditional means we have

$$E\left[\frac{P_{t+1}}{P_t}\right] = (1+\mu)\, E\left[\frac{1}{\eta_t}\right],$$

where by Cauchy-Schwarz inequality

$$(1+\delta_\eta) = E\left[\frac{1}{\eta_t}\right] \geq 1.$$

Furthermore, we have

$$\frac{P_{t+K}}{P_t} = (1+\mu)^K\, Z_{t+1} \times \ldots \times Z_{t+K} \frac{\eta_{t+K}}{\eta_t},$$

since we obtain cancellations of the misspricing errors. It follows that

$$E\left[\frac{P_{t+K}}{P_t}\right] = (1+\mu)^K\, E\left[\frac{1}{\eta_t}\right]$$

and so

$$\frac{E\left[\frac{P_{t+K}}{P_t}\right]}{\left(E\left[\frac{P_{t+1}}{P_t}\right]^K\right)} = (1+\delta_\eta)^{1-K},$$

which tends to zero as $K \to \infty$. Instead we have

$$\frac{\left(E\left[\frac{P_{t+K}}{P_t}\right]\right)^{1/K}}{E\left[\frac{P_{t+1}}{P_t}\right]} = (1+\delta_\eta)^{1/K-1} \to \frac{1}{1+\delta_\eta} < 1.$$

In this case the long run value has an interpretation as representing the magnitude of departure from the martingale hypothesis.

9 Multivariate case

In the multivariate case, we consider directly portfolios. Similarly to the univariate case, for assets $j = 1, \ldots, J$ define

$$E\left[\frac{P_{j,t+K}}{P_{j,t}} \,|\mathcal{F}_t\right] = E\left[\mathcal{R}_{j,t+K}(K)\,|\mathcal{F}_t\right] = (1+\mu_j)^K \equiv \mu_{j,K}.$$

where $\mathcal{F}_t = \sigma(\mathcal{R}_{j,k} : k \leq t, for\ all\ j \in J)$. Let portfolio p be constructed from the assets with weights $\{w_j, j = 1, \ldots J\}$ such that $\sum_{j=1}^J w_j = 1$. It follows that for $K = 1, 2, \ldots$ the expected gross return on the portfolio is

$$\mu_{p,K} = E\left[\sum_{j=1}^J w_j \mathcal{R}_{j,t+K}(K)\right] = \sum_{j=1}^J w_j\,(1+\mu_j)^K = \sum_{j=1}^J w_j \mu_{j,K}.$$

By the binomial theorem, we have

$$\mu_{p,K} = \sum_{j=1}^J w_j \sum_{l=0}^K \binom{K}{l} \mu_j^l = 1 + K\mu_p + \sum_{l=2}^K \binom{K}{l} \sum_{j=1}^J w_j \mu_j^l,$$

where $\mu_p = \sum_{j=1}^J w_j \mu_j$. Define the portfolio ratio statistic

$$\tau_{p,K} = \frac{E\left[\sum_{j=1}^J w_j \mathcal{R}_{j,t+K}(K)\right]}{\left(E\left[\sum_{j=1}^J w_j \mathcal{R}_{j,t+1}\right]\right)^K} = \frac{\mu_{p,K}}{(1+\mu_p)^K} = \frac{1 + K\mu_p + \sum_{l=2}^K \binom{K}{l} \sum_{j=1}^J w_j \mu_j^l}{1 + K\mu_p + \sum_{l=2}^K \binom{K}{l} \mu_p^l}. \tag{20}$$

If $\mu_1 = \cdots = \mu_J = \mu$, then $\tau_{p,K} = 1$. If $\mu_j = \mu + c_j/J^\alpha$ for some c_j with $|c_j| \le c < \infty$, $\alpha > 0$, and J is large, then $\tau_{p,K} \simeq 1$. However, in general this is not the case, and $\tau_{p,K}$ doesn't have a simple limit under the martingale hypothesis. We give a further interpretation of $\tau_{p,K}$. In the case where $w_j \ge 0$, we may think of $\mu_{p,K}$ as an expectation, specifically $E^* X^K$, where X is the random variable with outcome $1 + \mu_j$ with probability w_j. Then $\tau_{p,K} = E^* X^K / (E^* X)^K$. By Liapunov's inequality, $\tau_{p,K} \ge 1$ for all K. For $K = 2$, we have explicitly

$$\tau_{p,K} = \frac{E^* X^2}{(E^* X)^2} = 1 + \frac{\mathrm{var}^*(X)}{(E^* X)^2} \ge 1.$$

Perhaps we might work with assets that have similar means, i.e., sets of assets within which the cross-sectional variability of one period gross expected return is small.

We now turn to estimation of $\tau_{p,K}$. We can form the sample analogue of $\mu_{j,K}, \ \forall\, j \in J$ and $\forall\, K = 1, 2, \ldots < T$

$$\widehat{\mu}_{j,K} = \frac{1}{T-K} \sum_{t=1}^{T-K} \mathcal{R}_{j,t+K}(K).$$

Then the sample analogue of $\tau_{p,K}$ is given by:

$$\widehat{\tau}_{p,K} = \frac{\sum_{j=1}^{J} w_j \widehat{\mu}_{j,K}}{\left(\sum_{j=1}^{J} w_j \widehat{\mu}_{j,1}\right)^K}. \tag{21}$$

Define $u_{j,t:t+K} \equiv \mathcal{R}_{j,t+K}(K) - \mu_{j,K}$. Making use of the first order Taylor expansion we get:

$$\begin{aligned} \widehat{\tau}_{p,K} - \tau_{p,K} &\approx \frac{\sum_{j=1}^{J} w_j \left(\widehat{\mu}_{j,K} - \mu_{j,K}\right)}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^K} - K \frac{\left(\sum_{j=1}^{J} w_j \mu_{j,K} (\widehat{\mu}_{j,1} - \mu_{j,1})\right)}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^{K+1}} = \\ &= \frac{1}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^K} \left[\sum_{j=1}^{J} w_j \frac{1}{T-K} \sum_{t=1}^{T-K} \left(u_{j,t:t+K} - K \frac{\mu_{j,K} u_{j,t:t+1}}{\sum_{j=1}^{J} w_j \mu_{j,1}} \right) \right] \end{aligned} \tag{22}$$

Since eq.(21) is just a linear combination of martingales adapted to the same filtration $\mathcal{F}_t$, we still have asymptotic normality by the CLT for stationary ergodic martingale difference sequences. Define:

$$\Gamma_K(j,i) \equiv \gamma_K(j,i)(0) + \sum_{l=\pm 1}^{\pm(K-1)} \gamma_K(j,i)(l) \tag{23}$$

$$\Gamma_1(j,i) \equiv \gamma_1(j,i)(0) \tag{24}$$

to be the longish-run variances of $u_{j,t:t+K}$ and $u_{j,t:t+1}$ respectively, where:

$$\gamma_K(j,i)(l) = E\left[\mathcal{R}_{j,t+K}(K)\mathcal{R}_{i,t-l+K}(K)\right] - \mu_{j,K}\mu_{i,K} \quad \text{for} \quad l = 0,1,2,\dots \tag{25}$$

$$\gamma_1(j,i)(l) = E\left[\mathcal{R}_{j,t+1}\mathcal{R}_{i,t-l+1}\right] - \mu_{j,1}\mu_{i,1} \quad \text{for} \quad l = 0,1,2,\dots \tag{26}$$

Define also

$$\Gamma_{K,1}(j,i) \equiv \gamma_{K,1}(j,i)(0) + \sum_{l=1}^{(K-1)} \gamma_{K,1}(j,i)(l), \tag{27}$$

where

$$\gamma_{K,1}(j,i)(l) = E\left[\mathcal{R}_{j,t+K}(K)\mathcal{R}_{i,t-l+1}\right] - \mu_{j,K}\mu_{i,1} \quad \text{for} \quad l = 0,1,2,\dots \tag{28}$$

THEOREM 3. *Suppose that the gross return process is stationary, ergodic and square-integrable* $\forall\ j \in J$, *and* $E\left[\mathcal{R}_{j,t+1}\,|\mathcal{F}_t\right] = (1+\mu_j) = \mu_{j,1}\ \forall\ j \in J$, *where* $\mathcal{F}_t = \sigma(\mathcal{R}_{j,k} : k \leq t, for\ all\ j \in J)$. *Then*

$$\sqrt{T}\left(\widehat{\tau}_{p,K} - \tau_{p,K}\right) \Longrightarrow N(0, MV_K)$$

with variance MV_K *given by*

$$MV_K = \sum_{j=1}^{J}\sum_{i=1}^{J} w_j w_i \left(\Gamma_K(j,i) - \frac{2K\mu_{j,K}}{\mu_p}\Gamma_{K,1}(j,i) + \frac{K^2\mu_{j,K}\mu_{i,K}}{\mu_p^2}\Gamma_1(j,i)\right),$$

where $\Gamma_K(j,i)$,$\Gamma_1(j,i)$ *and* $\Gamma_{K,1}(j,i)$ *are defined by eq. (23)-(27) and* $\mu_p \equiv \sum_{j=1}^{J} w_j(1+\mu_j)$.

The proof of Theorem 3 can be found in the Appendix 1.

10 Numerical Results

10.1 Applications

We next employ our methodology on different datasets: stock market index, high and low-cap stocks, and exchange rate data. We first present the graphs for the shape of the test statistics calculated for daily S&P500 and Dow Jones stocks prices.

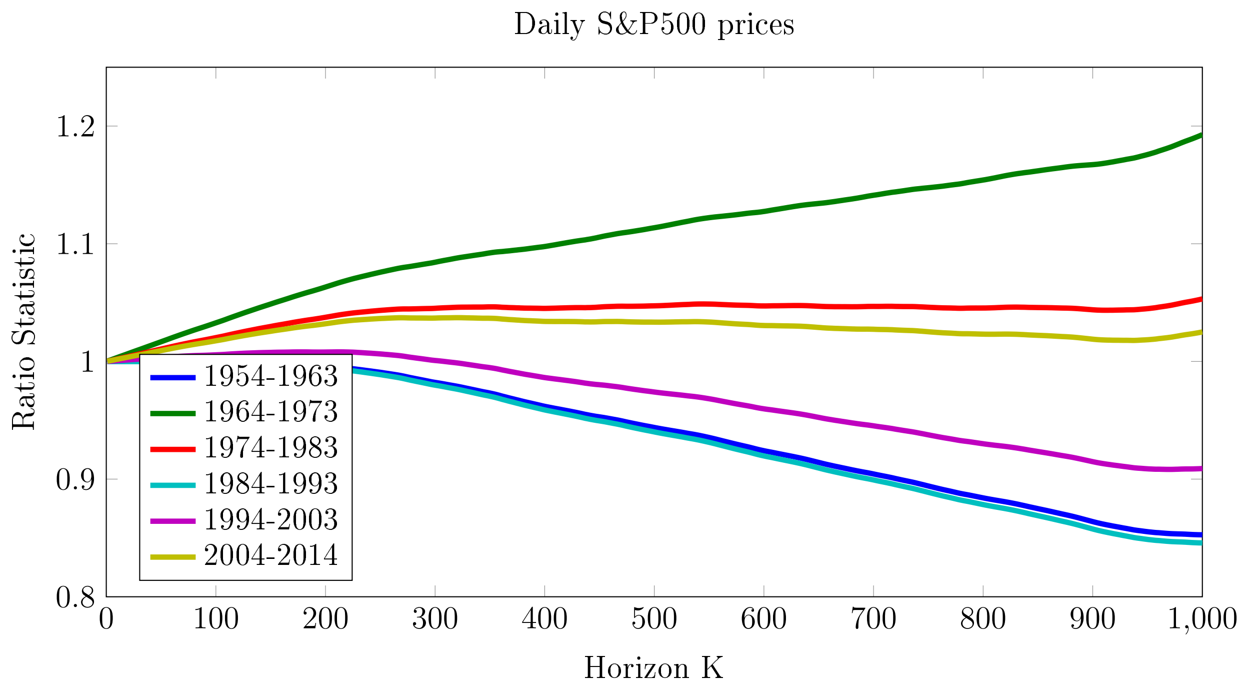

Figure 1: The above figure shows the shape of the ratio statistics $\hat{\tau}_K$ for S&P500 daily prices separately for each decade starting from 1954.

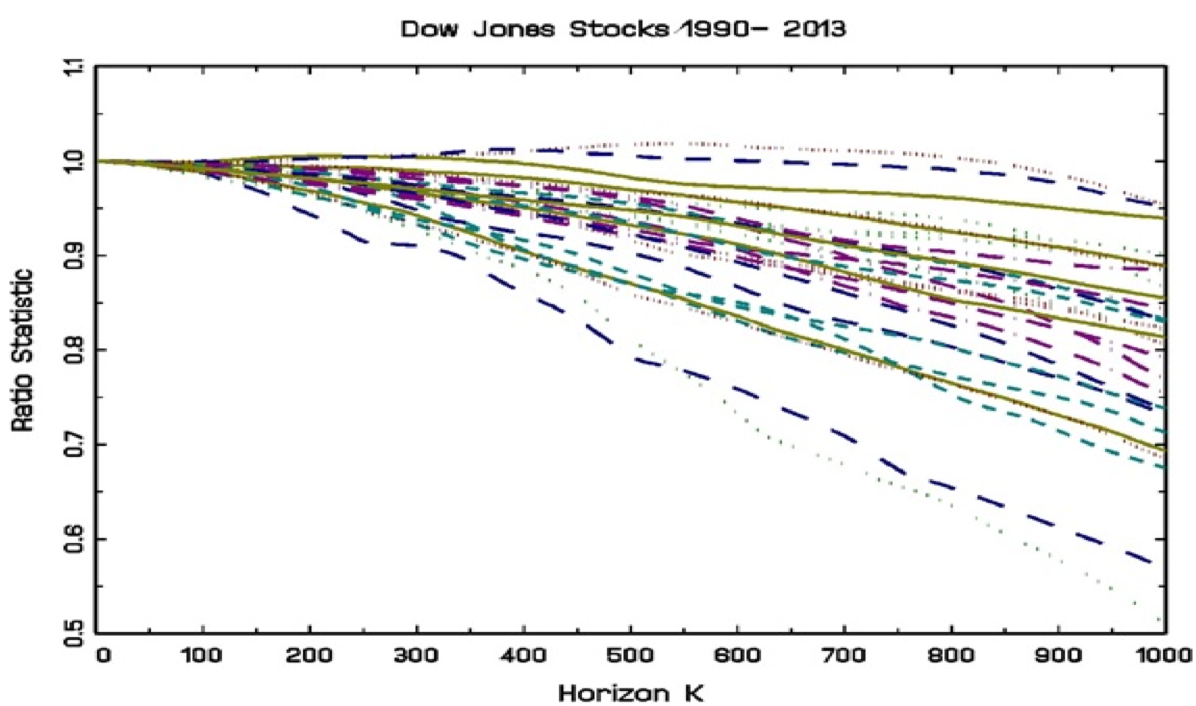

Figure 2: The above figure shows the shape of the ratio statistics $\hat{\tau}_K$ for several Dow Jones stocks.

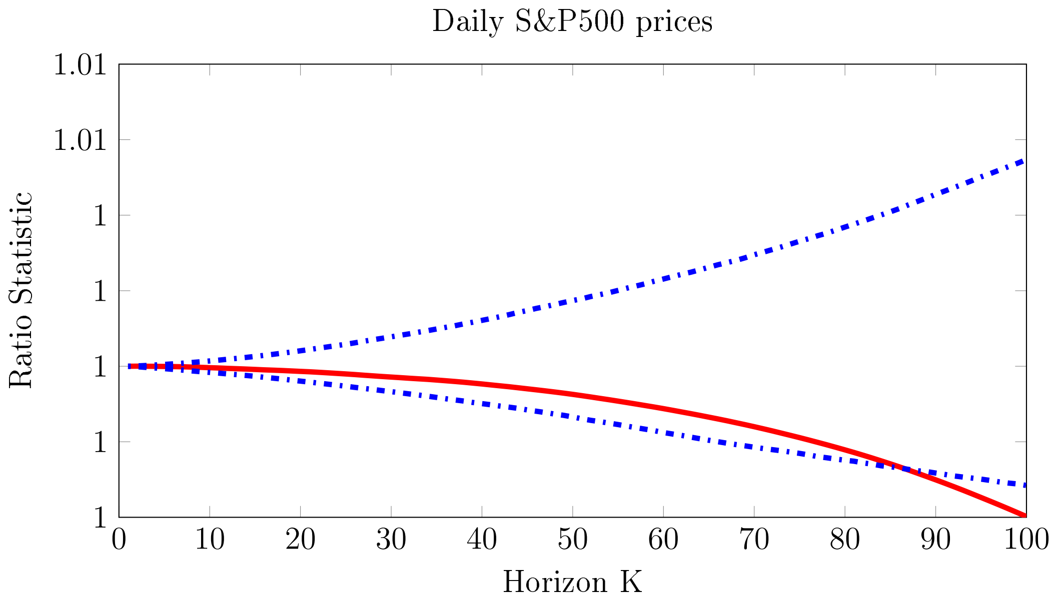

Figure 3: The figure plots ratio $\hat{\tau}_K$ for S&P500 daily prices. The dotted lines represent 95% confidence bands obtained with subsampling.

In fact, empirically we find that it is not possible to reject the null (of martingale hypothesis) for S&P500 prices for small K. The rejection occurs only at $K = 90$ days, i.e. approximately 3 months horizon. Similar picture can be seen for the high-cap stock, which we choose to be IBM. The prices are spanned from January, 1962 till August, 2014.

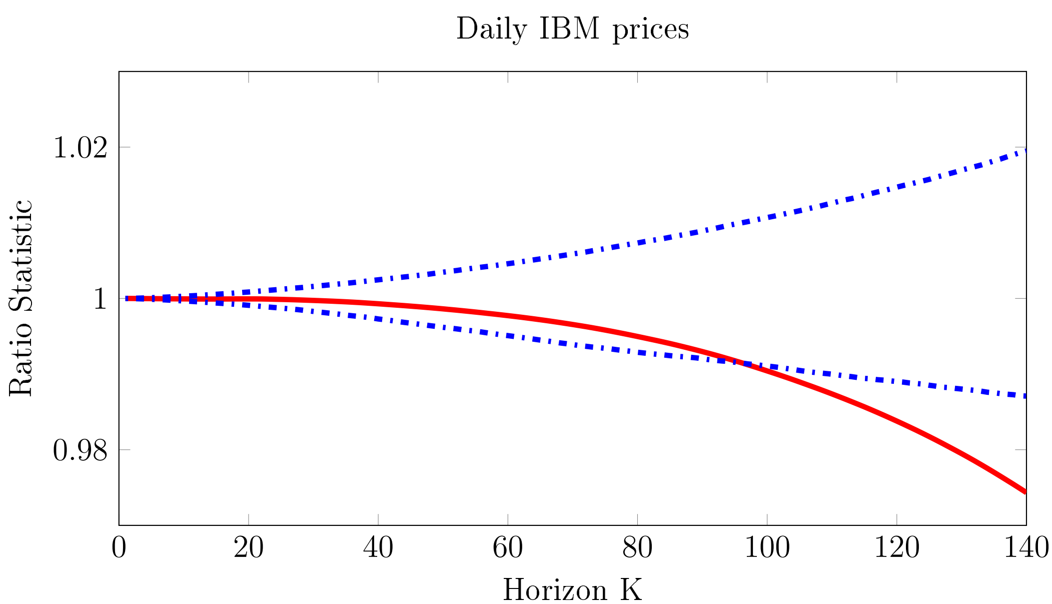

Figure 4: The figure plots ratio $\hat{\tau}_K$ for IBM daily prices. The dotted lines represent 95% confidence bands obtained with subsampling.

Similar to S&P500 prices, the null of mean predictability is not rejected at short horizons, the rejection occurs only at $K \approx 100$. We next employ out test statistics for the exchange rate data, namely GBP/US daily prices, spanned from April, 1971 till August, 2014.

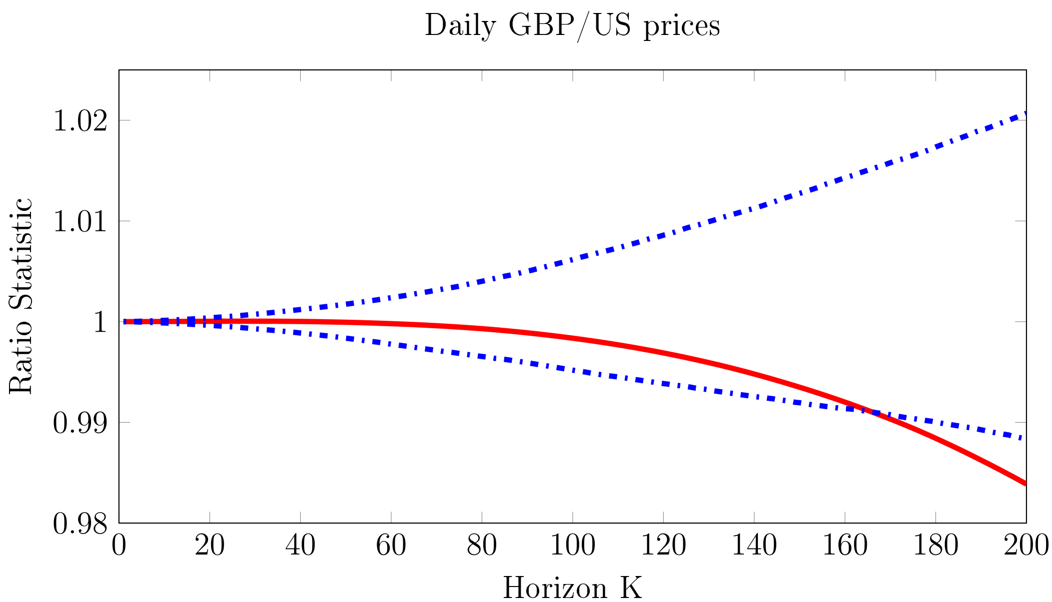

Figure 5: The figure plots ratio $\hat{\tau}_K$ for GBP/US exchange rate daily prices. The dotted lines represent 95% confidence bands obtained with subsampling.

For daily GBP/US prices we do not reject the null for short horizons, but do so for $K \approx 175$. In comparison to S&P500 it takes almost double the time (i.e. horizon) for the null to be rejected, providing an evidence that there is even less predictability for exchange rate data even at long horizons such as half a year.

And finally we apply the methodology on the low cap stocks, which sometimes exhibit different (from high-cap stocks) behavior. We choose INFN (Informational Technology) to represent the low-cap stock. It turns out, that although, τ_K decreases more rapidly compared to the high-cap stocks, the overall picture is very similar.

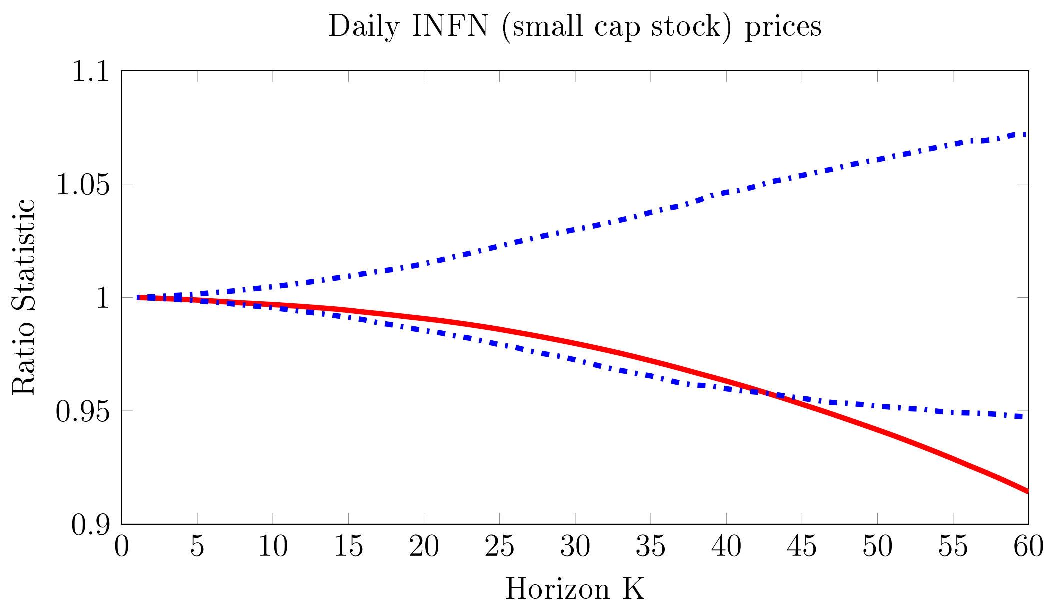

Figure 6: The figure plots ratio $\hat{\tau}_K$ for INFN daily prices. The dotted lines represent 95% confidence bands obtained with subsampling.

10.2 Simulation Study

In this section we present Monte Carlo simulations to investigate a power and a size of the univariate and multivariate versions of the Mean Ratio statistic. The calculations below show the results for the theoretical ratio in Theorem 1, when asymptotic variance is calculated according to eq. (12), however in practical applications whenever subsampling is used to calculate the standard errors the correct size is guaranteed by construction.

10.2.1 Size

To investigate the size of the test statistics we simulate the data under the H_0 as follows

$$H_0: \qquad P_{t+1} = (1+\mu)\, P_t u_{t+1}$$

where $u_t \sim \mathcal{U}(0,2)$ such that $E[u_t] = 1$, $\mu = 0.3$ and $P_1 = 1$. Then under H_0 it holds that

$$E\left[\frac{P_{t+K}}{P_t}\right] = E[\mathcal{R}_{t+K}] = (1+\mu)^K$$

Multivariate version uses the same H_0 for each of $j = 1, \ldots, J$ assets and portfolio is formed with equal weights $w_j = 1/J \ \ \forall\, j \in J$.

Table 1: Empirical size of the nominal **5%** Univariate Mean Ratio statistics.

	$T = 5000$		$T = 10000$	
# of lags	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 10\%$
K=2	0.0522	0.1047	0.0520	0.1051
K=4	0.0569	0.1054	0.0495	0.1018
K=8	0.0605	0.1117	0.0591	0.1099
K=10	0.0749	0.1262	0.0679	0.1191
K=12	0.0938	0.1584	0.0869	0.1321
K=16	0.1603	0.2184	0.1337	0.1855

Note: Number of replications $N = 10000$.

For large number of lags, i.e., $K \leq 10$ the Univariate test has a proper size, however simulations show that for $K > 10$ the test statistic has severe size distortions that do not recover even when the sample size T becomes relatively large. In fact, this will hold not only for the univariate version but for the multivariate test statistics as well.

Table 2: Empirical size of the nominal **5%** Multivariate Mean Ratio statistics.

# of lags/assets	J=2	J=4	J=8	J=16	J=24
K=2	0.0521	0.0527	0.0489	0.0528	0.0510
K=4	0.0525	0.0467	0.0530	0.0590	0.0610
K=8	0.0609	0.0544	0.0538	0.0470	0.0480
K=10	0.0741	0.648	0.0593	0.0554	0.0550
K=16	0.1272	0.1023	0.0921	0.0828	0.0760

Note: Simulations are based on $N = 10000$ replications, $T = 5000$.

As in Table 1 for the univariate test statistics the Multivariate analogue has a proper empirical size for $K < 10$. This result does not depend on the number of assets, rather the number of lags is what matters. For large values of K the test, similar to the univariate case, has severe size distortions. This problem can not be resolved by simply increasing the sample size T as shown in the Table 3 below, unless the number of assets is very small, e.g. $J = 2$.

Table 3: Empirical size of the nominal **5%** Multivariate Mean Ratio statistics.

# of lags/assets	J=2	J=4	J=8	J=16	J=24
K=16	0.0590	0.0870	0.0610	0.0610	0.0600
K=20	0.0490	0.1030	0.0900	0.0860	0.0790

Note: The table presents the empirical size of 5% nominal Multivariate statistics for large (i.e. $K > 10$) lags. Simulations are based on $N = 1000$ replications, $T = 15000$. The reduced number of replications is due to dimensionality problem.

10.2.2 Power

In order to investigate how powerful is our Mean Ratio Statistic against different alternatives, we consider two alternatives: $H_1^{(1)}$, representing slowly varying mean and thus being close to H_0; and $H_1^{(2)}$ under which prices follow stationary AR(2) process - an alternative quite different to H_0. Consider first the fist alternative:

$$H_1^{(1)}: \qquad P_{t+1} = \Big(1 + \mu_t + \epsilon_t\Big) P_t$$

where $\epsilon_t \sim \mathcal{N}(0, 0.5)$[4]. Once again we have $E[\epsilon_t] = 0$ and

$$E\left[\frac{P_{t+K}}{P_t}\right] = E[\mathcal{R}_{t+K}] = (1 + \mu_t)^K$$

In order to simulate μ_t we simulate returns r_t according to the GARCH(1,1) model and define

$$\mu_t = \mathrm{e}^{r_t} - 1,$$

where

$$r_t = \sigma_t z_t$$

$$\sigma_t^2 = \alpha + \beta\sigma_{t-1}^2 + \gamma r_{t-1}^2$$

with $z_t \sim \mathcal{N}(0, 1)$ and $[\alpha, \beta, \gamma] = [0.01, 0.95, 0.03]$ and $\sigma_1^2 = 0$.

The other alternative we consider is that prices follow a stationary AR(2) process.

$$H_1^{(2)}: \qquad P_{t+1} = \alpha_0 + \alpha_1 P_t + \alpha_2 P_{t-1} + \eta_t,$$

[4]In this case we chose $\sigma_\epsilon^2 = 0.5$ such that prices stay positive. This approach is easier than imposing an indicator function which will depend on μ_t

where $\eta_t \sim \mathcal{N}(0,1)$ and $\alpha_0 = 0.1$; $\alpha_1 = 0.9$; and $\alpha_2 = 0.8$.
As before for the multivariate version we simulate prices according to the same $H_1^{(1)}$ and $H_1^{(2)}$ for each of $j = 1, \ldots, J$ assets and portfolio is formed with equal weights $w_j = 1/J \;\; \forall\, j \in J$.

Table 4: Power of the Univariate Test against H_1^1 and H_1^2.

K	H_1^1	H_1^2
2	0.0890	1
4	0.0770	0.9998
8	0.0630	0.5196
16	0.0611	1

Note: Simulations are based on $N = 5000$ replications, $T = 5000$. Nominal test size is 5%.

For the Multivariate Mean Ratio test the power against H_1^1, similarly to the univariate test, is quite low and doesnt exceed 9% no matter how big J is. This result is quite expected since H_1^1 was specifically designed to be as close to H_0 as possible. More interesting question is how the Multivariate test statistic peforms when the alternative is quite different from the null. The table below answers this question.

Table 5: Power of the Multivariate Test Statistics against H_1^2

# of lags/assets	J=2	J=4	J=8	J=16	J=24
K=2	1.0000	1.0000	1.0000	1.0000	1.0000
K=4	1.0000	1.0000	1.0000	1.0000	1.0000
K=6	0.2656	0.8522	0.9998	1.0000	1.0000
K=10	0	0.9990	1.0000	1.0000	1.0000
K=12	1.0000	1.0000	1.0000	1.0000	1.0000
K=16	1.0000	1.0000	1.0000	1.0000	1.0000
K=20	1.0000	1.0000	1.0000	1.0000	1.0000

Note: Simulations are based on $N = 5000$ replications, $T = 5000$. Nominal test size is 5%.

11 Conclusion

We propose an alternative Ratio Statistic for measuring mean predictability, which represents the test of the weak form of the EMH. We propose different versions the statistics can be stated and derive their limiting distributions. Applying our methodology to different financial series we conclude that there is no mean predictability at short horizons, however the null of the mean predictability is rejected at longer ($K > 80$ days) horizons.

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Appendix 1

Proof of Theorem 1.

First observe that

$$V_{T,K} \equiv \mathrm{Var}\left\{\sqrt{T}\left(\hat{\mu}_K - (1+\mu)^K\right)\right\} = \mathrm{Var}\left\{(T-K)^{0.5}\left(\frac{1}{T-K}\sum_{t=1}^{T-K}\mathcal{R}_{t+K}(K) - (1+\mu)^K\right)\right\}$$

$$\mathrm{Var}\left\{(T-K)^{-0.5}\sum_{t=1}^{T-K}\left[\mathcal{R}_{t+K}(K) - (1+\mu)^K\right]\right\} = \mathrm{Var}\left((T-K)^{-0.5}\sum_{t=1}^{T-K}u_{t:t+K}\right) =$$

$$= \sum_{t=1}^{T-K}\sum_{s=1}^{T-K}\left(\frac{1}{T-K}\right)\mathrm{cov}\left(u_{t:t+K},\, u_{s:s+K}\right) = \sum_{t=1}^{T-K}\sum_{s=1}^{T-K}\left(\frac{1}{T-K}\right)E\left(u_{t:t+K}\cdot u_{s:s+K}\right) =$$

$$\frac{1}{T-K}\sum_{t=1}^{T-K}\sum_{s=1}^{T-K}\gamma_{t-s} = \gamma_0 + 2\sum_{j=1}^{T-K-1}\left(\frac{T-K-j}{T-K}\right)\gamma_j = \gamma_0 + 2\sum_{j=1}^{T-K-1}\left(1-\frac{j}{T-K}\right)\gamma_j$$

$$V_K \equiv \lim_{T\to\infty} V_{T,K} = \gamma_0 + \sum_{j=-\infty}^{\infty}\gamma_j = \gamma_0 + \sum_{j=\pm 1}^{\pm(K-1)}\gamma_j$$

We have for $K = 1, 2, \ldots$ and $j = 0, 1, 2, \ldots, K-1$

$$\begin{aligned}\gamma_K(j) &= \mathrm{cov}\left(\mathcal{R}_{t+K}(K), \mathcal{R}_{t+j+K}(K)\right)\\ &= E\left[\mathcal{R}_{t+K}(K)\mathcal{R}_{t+j+K}(K)\right] - (1+\mu)^{2K}\\ &= \begin{cases}(1+\mu)^j\, E\left[\mathcal{R}^2_{t+K}\mathcal{R}^2_{t+K-1}\cdots\mathcal{R}^2_{t+j}\mathcal{R}_{t+j-1}\ldots\mathcal{R}_{t+1}\right] - (1+\mu)^{2K} & \text{if } j>0\\ (1+\mu)^{-j}\, E\left[\mathcal{R}^2_{t+j+K}\mathcal{R}^2_{t+j+K-1}\cdots\mathcal{R}^2_{t}\mathcal{R}_{t-1}\ldots\mathcal{R}_{t+j}\right] - (1+\mu)^{2K} & \text{if } j<0\end{cases}\end{aligned}$$

Furthermore

$$\gamma_1(0) = \mathrm{var}\left(\frac{1}{T}\sum_t \mathcal{R}_{t+1}\right) = \frac{1}{T^2}\sum_t \mathrm{var}\left(\mathcal{R}_{t+1}\right)$$

because $\mathcal{R}_{t+1} - E[\mathcal{R}_{t+1}]$ is a martingale difference sequence.

We start by performing the first-order Taylor expansion of $\widehat{\tau}_K - 1$ around the point (μ_K, μ_1)

is given by:

$$\begin{aligned}
\widehat{\tau}_K - 1 &\simeq \frac{\widehat{\mu}_K - \mu_K}{(\mu_1)^K} - K\frac{\mu_K}{(\mu_1)^{K+1}}(\widehat{\mu}_1 - \mu_1) \\
&= \frac{\widehat{\mu}_K - (1+\mu)^K}{(1+\mu)^K} - \frac{K}{(1+\mu)}(\widehat{\mu}_1 - (1+\mu)) \\
&= \frac{1}{(1+\mu)^K}\frac{1}{T-K}\sum_{t=1}^{T-K}\left\{\left[\mathcal{R}_{t+K}(K) - (1+\mu)^K\right] - K(1+\mu)^{K-1}\left[\mathcal{R}_{t+1} - (1+\mu)\right]\right\} \\
&= \frac{1}{(1+\mu)^K}\frac{1}{T-K}\sum_{t=1}^{T-K}\left\{u_{t:t+K} - K(1+\mu)^{K-1}u_{t:t+1}\right\}
\end{aligned}$$

Define

$$\Upsilon_1 \equiv \lim_{T\to\infty}\frac{1}{T}\sum_t\sum_s E\left[(\mathcal{R}_{t+1} - (1+\mu))(\mathcal{R}_{s+1} - (1+\mu))\right] = \gamma_1(0)$$

and

$$\Upsilon_K \equiv \lim_{T\to\infty}\frac{1}{T}\sum_t\sum_s E\left[\left(\mathcal{R}_{t+K}(K) - (1+\mu)^K\right)\left(\mathcal{R}_{s+K}(K) - (1+\mu)^K\right)\right] = \gamma_K(0) + \sum_{j=\pm 1}^{\pm(K-1)}\gamma_K(j)$$

to be the "long-runish" variances of $u_{t:t+1}$ and $u_{t:t+K}$ respectively, where $\gamma_K(j)$ are defined above

Define also

$$\Upsilon_{K,1} \equiv \lim_{T\to\infty}\frac{1}{T}\sum_t\sum_s E\left[\left(\mathcal{R}_{t+K}(K) - (1+\mu)^K\right)(\mathcal{R}_{s+1} - (1+\mu))\right] = \gamma_{K,1}(0) + \sum_{j=1}^{K-1}\gamma_{K,1}(j)$$

Since $E\left[\sqrt{T}(\widehat{\tau}_K - 1)\right] = 0$, then the asymptotic variance of $\sqrt{T}(\widehat{\tau}_K - 1)$ is given by

$$W_{T,K} \equiv \operatorname{Var}\left\{\sqrt{T}\left(\hat{\tau}_K - 1\right)\right\} = \frac{1}{(1+\mu)^{2K}}\frac{1}{(T-K)}\operatorname{Var}\left\{\sum_{t=1}^{T-K}\left[u_{t:t+K} - K(1+\mu)^{K-1}u_{t:t+1}\right]\right\} =$$

$$\frac{1}{(1+\mu)^{2K}}\frac{1}{(T-K)}\left\{\operatorname{Var}\left[\sum_{t=1}^{T-K}u_{t:t+K}\right]\right\} + \frac{1}{T-K}\frac{K^2(1+\mu)^{2(K-1)}}{(1+\mu)^{2K}}\left\{\operatorname{Var}\left[\sum_{t=1}^{T-K}u_{t:t+1}\right]\right\} -$$

$$-\frac{1}{T-K}\frac{2K(1+\mu)^{K-1}}{(1+\mu)^{2K}}\left\{\operatorname{Cov}\left(\sum_{t=1}^{T-K}u_{t:t+K}, \sum_{t=1}^{T-K}u_{t:t+1}\right)\right\} =$$

$$\frac{1}{(1+\mu)^{2K}(T-K)}\sum_{t=1}^{T-K}\sum_{s=1}^{T-K}E\left(u_{t:t+K}\cdot u_{s:s+K}\right) + \frac{K^2}{(1+\mu)^2(T-K)}\sum_{t=1}^{T-K}\sum_{s=1}^{T-K}E\left(u_{t:t+1}\cdot u_{s:s+1}\right) -$$

$$-\frac{2K}{(1+\mu)^{K+1}(T-K)}E\left[\sum_{t=1}^{T-K}u_{t:t+K}\cdot\sum_{t=1}^{T-K}u_{t:t+1}\right] =$$

$$\frac{1}{(1+\mu)^{2K}}\sum_{t=1}^{T-K}\sum_{s=1}^{T-K}\frac{1}{T-K}E\left(u_{t:t+K}\cdot u_{s:s+K}\right) + \frac{K^2}{(1+\mu)^2}\sum_{t=1}^{T-K}\sum_{s=1}^{T-K}\frac{1}{T-K}E\left(u_{t:t+1}\cdot u_{s:s+1}\right) -$$

$$-\frac{2K}{(1+\mu)^{K+1}}\frac{1}{T-K}E\left[\sum_{t=1}^{T-K}u_{t:t+K}\cdot\sum_{t=1}^{T-K}u_{t:t+1}\right] =$$

$$\frac{1}{(1+\mu)^{2K}}\left[\gamma_K(0) + 2\sum_{j=1}^{T-K-1}\left(1-\frac{j}{T-K}\right)\gamma_K(j)\right] + \frac{K^2}{(1+\mu)^2}\left[\gamma_1(0) + 2\sum_{j=1}^{T-K-1}\left(1-\frac{j}{T-K}\right)\gamma_1(j)\right]$$

$$-\frac{2K}{(1+\mu)^{K+1}}\left[\gamma_{K,1}(0) + 2\sum_{j=1}^{T-K-1}\left(1-\frac{j}{T-K}\right)\gamma_{K,1}(j)\right]$$

Taking the limit, we get the asymptotic variance:

$$W_K \equiv \lim_{T\to\infty} W_{T,K} = \frac{1}{(1+\mu)^{2K}}\left[\gamma_K(0) + \sum_{j=\pm 1}^{\pm(K-1)}\gamma_K(j)\right] + \frac{K^2}{(1+\mu)^2}\left[\gamma_1(0) + \sum_{j=\pm 1}^{\pm(K-1)}\gamma_1(j)\right] -$$

$$-\frac{2K}{(1+\mu)^{K+1}}\left[\gamma_K(K-1) + \sum_{j=\pm 1}^{\pm(K-1)}\gamma_K(K-1+j)\right]$$

Making use of Υ_1, Υ_K and $\Upsilon_{K,1}$, the asymptotic variance W_K can be expressed as

$$W_K = \frac{1}{(1+\mu)^{2K}}\Upsilon_K + \frac{K^2}{(1+\mu)^2}\Upsilon_1 - \frac{2K}{(1+\mu)^{K+1}}\Upsilon_{K,1}$$

This completes the proof.

REMARK 2. If we assume that R_{t+1} is i.i.d. sequence, then the asymptotic variance of the statistics $\widehat{\tau}_K$ is given by the following simplified formula

$$\mathrm{var}(\widehat{\tau}_K) \quad = \quad \frac{1}{(1+\mu)^{2K}}\mathrm{var}(\widehat{\mu}_K) + \frac{K^2}{(1+\mu)^2}\mathrm{var}(\widehat{\mu}_1) - \frac{2K}{(1+\mu)^{K+1}}\mathrm{cov}(\widehat{\mu}_K, \widehat{\mu}_1),$$

where

$$\mathrm{var}(\widehat{\mu}_1) = E\left[\left(\frac{1}{T}\sum_{t=1}^{T-1}\mathcal{R}_{t+1} - (1+\mu)\right)^2\right] = \frac{1}{T}\mathrm{var}(\mathcal{R}_{t+1}).$$

and

$$\mathrm{var}(\widehat{\mu}_K) = \frac{1}{T}\Bigg(\left[\left(\mathrm{var}(\mathcal{R}_{t+1}) + (1+\mu)^2\right)^K - (1+\mu)^{2K}\right] + \\ + 2\sum_{j=1}^{K-1}\left[(1+\mu)^{2(K-j)}\left(\mathrm{var}(\mathcal{R}_{t+1}) + (1+\mu)^2\right)^j - (1+\mu)^{2K}\right]\Bigg)$$

Finally,

$$\mathrm{cov}(\widehat{\mu}_K, \widehat{\mu}_1) = \frac{1}{T}K\left((1+\mu)^{K-1}\left[\mathrm{var}(\mathcal{R}_{t+1}) + (1+\mu)^2\right] - (1+\mu)^{K+1}\right)$$

PROOF OF THEOREM 2.

The first-order Taylor expansion of $\widehat{\tau}_{p,K}$ is given by:

$$\widehat{\tau}_{p,K} - \frac{\sum_{j=1}^J w_j\mu_{j,K}}{\left(\sum_{j=1}^J w_j\mu_{j,1}\right)^K} \approx \frac{\sum_{j=1}^J w_j\left(\hat{\mu}_{j,K} - \mu_{j,K}\right)}{\left(\sum_{j=1}^J w_j\mu_{j,1}\right)^K} - K\frac{\sum_{j=1}^J w_j\mu_{j,K}\left(\widehat{\mu}_{j,1} - \mu_{j,1}\right)}{\left(\sum_{j=1}^J w_j\mu_{j,1}\right)^{K+1}} =$$

$$= \frac{1}{\left(\sum_{j=1}^J w_j\mu_{j,1}\right)^K}\left[\sum_{j=1}^J w_j\frac{1}{T-K}\sum_{t=1}^{T-K}u_{j,t:t+K} - K\frac{\sum_{j=1}^J w_j\mu_{j,K}\frac{1}{T-K}\sum_{t=1}^{T-K}u_{j,t:t+1}}{\sum_{j=1}^J w_j\mu_{j,1}}\right]$$

$$MV_{T,K} = Var\left[\sqrt{T}\left(\widehat{\tau}_{p,K} - \frac{\sum_{j=1}^{J} w_j \mu_{j,K}}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^K}\right)\right] =$$

$$= Var\left[\frac{\sqrt{T}}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^K}\left(\sum_{j=1}^{J} w_j \frac{1}{T-K}\sum_{t=1}^{T-K} u_{j,t:t+K} - K\frac{\sum_{j=1}^{J} w_j \mu_{j,K} \frac{1}{T-K}\sum_{t=1}^{T-K} u_{j,t:t+1}}{\sum_{j=1}^{J} w_j \mu_{j,1}}\right)\right] =$$

$$\frac{T}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^{2K}} \text{Var}\left\{\left(\sum_{j=1}^{J} \frac{w_j}{T-K}\sum_{t=1}^{T-K}\left[u_{j,t:t+K} - K\frac{\mu_{j,K} u_{j,t:t+1}}{\sum_{l=1}^{J} w_l \mu_{l,1}}\right]\right)\right\} =$$

$$= \frac{T}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^{2K}} \text{Var}\left\{\frac{1}{T-K}\left[\sum_{j=1}^{J} w_j \sum_{t=1}^{T-K}\left(u_{j,t:t+K} - K\frac{\mu_{j,K} u_{j,t:t+1}}{\sum_{l=1}^{J} w_l \mu_{l,1}}\right)\right]\right\} =$$

$$= \frac{1}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^{2K}}\left(\frac{1}{T-K}\right)\left[\sum_{j=1}^{J}\sum_{i=1}^{J} w_j w_i \sum_{t=1}^{T-K}\sum_{s=1}^{T-K} E\left(u_{j,t:t+K} - K\frac{\mu_{j,K} u_{j,t:t+1}}{\sum_{l=1}^{J} w_l \mu_{l,1}}\right) \times\right.$$

$$\left. \times \left(u_{i,s:s+K} - K\frac{\mu_{i,K} u_{i,s:s+1}}{\sum_{l=1}^{J} w_l \mu_{l,1}}\right)\right] =$$

$$= \frac{1}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^{2K}}\left(\frac{1}{T-K}\right)\left[\sum_{j=1}^{J}\sum_{i=1}^{J} w_j w_i \sum_{t=1}^{T-K}\sum_{s=1}^{T-K} E\left(u_{j,t:t+K} u_{i,s:s+K} - K\frac{\mu_{j,K} u_{j,t:t+K}}{\sum_{l=1}^{J} w_l \mu_{l,1}} u_{i,s:s+1} -\right.\right.$$

$$\left.\left. - K\frac{\mu_{i,K} u_{i,s:s+1}}{\sum_{l=1}^{J} w_l \mu_{l,1}} u_{j,t:t+K} + K^2 \frac{\mu_{j,K}\mu_{i,K} u_{j,t:t+1} u_{i,s:s+1}}{\left(\sum_{l=1}^{J} w_l \mu_{l,1}\right)^2}\right)\right] =$$

$$= \frac{1}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^{2K}}\left\{\sum_{j=1}^{J}\sum_{i=1}^{J} w_j w_i \left(\left[\gamma_K(j,i)(0) + \sum_{l=\pm 1}^{\pm(T-K-1)}\left(\frac{T-K-l}{T-K}\right)\gamma_K(j,i)(l)\right] -\right.\right.$$

$$- 2K\frac{\mu_{j,K}}{\sum_{r=1}^{J} w_r \mu_{r,1}}\left[\gamma_{K,1}(j,i)(0) + \sum_{l=\pm 1}^{\pm(T-K-l)}\left(\frac{T-K-l}{T-K}\right)\gamma_{K,1}(j,i)(l)\right] +$$

$$\left.\left. + K^2 \frac{\mu_{j,K}\mu_{i,K}}{\sum_{r=1}^{J} w_r \mu_{r,1}} \gamma_1(j,i)(0)\right)\right\}$$

Taking the limit as $T \to \infty$ the multivariate asymptotic variance becomes:

$$MV_K \equiv \lim_{T\to\infty} MV_{T,K} = \frac{1}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^{2K}} \left\{ \sum_{j=1}^{J}\sum_{i=1}^{J} w_j w_i \left(\left[\gamma_K(j,i)(0) + \sum_{l=\pm 1}^{\pm(K-1)} \gamma_K(j,i)(l) \right] - \right.\right.$$

$$\left.\left. - 2K \frac{\mu_{j,K}}{\sum_{r=1}^{J} w_r \mu_{r,1}} \left[\gamma_{K,1}(j,i)(0) + \sum_{l=\pm 1}^{\pm(K-1)} \gamma_{K,1}(j,i)(l) \right] + K^2 \frac{\mu_{j,K}\mu_{i,K}}{\sum_{r=1}^{J} w_r \mu_{r,1}} \gamma_1(j,i)(0) \right) \right\}$$

Define

$$\Gamma_K(i,j) \equiv \gamma_K\left(j,i\right)\left(0\right) + \sum_{l=\pm 1}^{\pm(K-1)} \gamma_K\left(j,i\right)\left(l\right)$$

and

$$\Gamma_1(i,j) \equiv \gamma_1\left(j,i\right)\left(0\right)$$

to be the longish-run variances of $u_{j,t:t+K}$ and $u_{j,t:t+1}$ respectively, where

$$\gamma_K(j,i)(l) \equiv \lim_{T\to\infty} \frac{1}{T} \sum_t \sum_s E\left[\mathcal{R}_{j,t+K}(K)\mathcal{R}_{i,s-l+K}(K)\right] - \mu_{j,K}\mu_{i,K} \quad \text{for} \quad l = 0,1,2,\ldots$$

and

$$\gamma_1(j,i)(l) \equiv \lim_{T\to\infty} \frac{1}{T} \sum_t \sum_s E\left[\mathcal{R}_{j,t+1}\mathcal{R}_{i,s-l+1}\right] - \mu_{j,1}\mu_{i,1} \quad \text{for} \quad l = 0,1,2,\ldots$$

Define also

$$\Gamma_{K,1}(i,j) \equiv \gamma_{K,1}(j,i)(0) + \sum_{l=1}^{(K-1)} \gamma_{K,1}(j,i)(l),$$

where

$$\gamma_{K,1}(j,i)(l) \equiv \lim_{T\to\infty} \frac{1}{T} \sum_t \sum_s E\left[\mathcal{R}_{j,t+K}(K)\mathcal{R}_{i,s-l+1}\right] - \mu_{j,K}\mu_{i,1} \quad \text{for} \quad l = 0,1,2,\ldots$$

Making use of the notation above, MV_K can be expressed as

$$MV_K = \frac{1}{\left(\sum_{j=1}^{J} w_j \mu_{j,1}\right)^{2K}} \sum_{j=1}^{J}\sum_{i=1}^{J} w_j w_i \left(\Gamma_K(i,j) - \frac{2K\left(\mu_{j,K}\right)}{\sum_{l=1}^{J} w_l \mu_{l,1}} \Gamma_{K,1}(i,j) + \frac{K^2 \mu_{j,K}\mu_{i,K}}{\left(\sum_{l=1}^{J} w_l \mu_{l,1}\right)^2} \Gamma_1(i,j) \right)$$

This completes the proof.

Appendix 2

Alternatively statistic τ_K can be written as

$$\tau_K' \equiv \frac{E^{1/K}\left[\mathcal{R}_{t+K}(K)\right]}{E\left[\mathcal{R}_{t+1}\right]} = 1 \tag{29}$$

or taking logs of the above equation we get

$$\tau_K'' \equiv \ln E\left[\mathcal{R}_{t+K}(K)\right] - K \ln E\left[\mathcal{R}_{t+1}\right] = 0. \tag{30}$$

First, consider statistics τ_K':

$$\tau_K' = \frac{E^{1/K}\left[\mathcal{R}_{t+K}(K)\right]}{E\left[\mathcal{R}_{t+1}\right]} = 1$$

Forming the sample analogue of τ_K' and approximating it with the first order Taylor expansion we get:

$$\begin{aligned}
\widehat{\tau'}_K - 1 &\approx \frac{1}{K}\frac{\mu_K^{(1-K)/K}\left(\hat{\mu}_K - \mu_K\right)}{\mu_1} - \frac{\mu_K^{1/K}\left(\hat{\mu}_1 - \mu_1\right)}{\mu_1^2} + o_P\left(T^{-1/2}\right) = \\
&= \frac{1}{K}\frac{\hat{\mu}_K - (1+\mu)^K}{(1+\mu)^K} - \frac{(1+\mu)\left[\hat{\mu}_1 - (1+\mu)\right]}{(1+\mu)^2} + o_P\left(T^{-1/2}\right) = \\
&= \frac{1}{K}\frac{\hat{\mu}_K - (1+\mu)^K}{(1+\mu)^K} - \frac{\left[\hat{\mu}_1 - (1+\mu)\right]}{1+\mu} + o_P\left(T^{-1/2}\right) = \frac{1}{K}\left(\widehat{\tau}_K - 1\right) + o_P\left(T^{-1/2}\right)
\end{aligned}$$

Thus, by the CLT we have:

$$\sqrt{T}\left(\widehat{\tau'}_K - 1\right) \Longrightarrow N(0, \frac{1}{K^2}W_K)$$

where W_K is given in the Theorem 1. The standard errors for $\widehat{\tau'}_K$ will be different from those of $\widehat{\tau}_K$ by the factor of 1/K. However, due to the slightly different Taylor expansion, the bias correction term will be different. More precisely, the second-order Taylor expansion is given by

$$\widehat{\tau'}_K - 1 \approx \frac{1}{K}\frac{\mu_K^{(1/K-1)}\left(\widehat{\mu}_K - \mu_K\right)}{\mu_1} - \frac{\mu_K^{1/K}\left(\widehat{\mu}_1 - \mu_1\right)}{\mu_1^2} + \frac{1}{2}\frac{1}{K}\left(\frac{1}{K}-1\right)\frac{\left(\mu_K\right)^{1/K-2}}{\mu_1}\left(\widehat{\mu}_K - \mu_K\right)^2 +$$

$$+ \frac{\mu_K^{1/K}}{\mu_1^3}\left(\widehat{\mu}_1 - \mu_1\right)^2 + o_P\left(T^{-1/2}\right) =$$

$$+ = \frac{1}{K}\frac{\widehat{\mu}_K - (1+\mu)^K}{(1+\mu)^K} - \frac{(1+\mu)\left[\widehat{\mu}_1 - (1+\mu)\right]}{(1+\mu)^2} + \frac{(1-K)}{2K^2(1+\mu)^{2K}}\left(\widehat{\mu}_K - \mu_K\right)^2 + \frac{\left(\widehat{\mu}_1 - \mu_1\right)^2}{(1+\mu)^2} + o_P\left(T^{-1/2}\right)$$

Taking expectations of the above expression we can deduce that the bias corrected estimator of $\widehat{\tau'}_K^{bc}$ is given by

$$\widehat{\tau'}_K^{bc} = \widehat{\tau'}_K - \frac{(1-K)\widehat{V}_K}{2K^2\left(\widehat{\mu}_1\right)^{2K}(T-K)} + \frac{\widehat{V}_1}{\left(\widehat{\mu}_1\right)^2(T-1)}, \tag{31}$$

where $\widehat{V}_K$ estimates consistently the asymptotic variance of $\sqrt{T}\left(\widehat{\mu}_K - \mu_K\right)$, specifically,

$$\widehat{V}_K = \frac{1}{T-K}\sum_{t=1}^{T-K}\left(\mathcal{R}_{t+K}(K) - \overline{\mathcal{R}}_K\right)^2, \quad \overline{\mathcal{R}}_K = \frac{1}{T-K}\sum_{t=1}^{T-K}\mathcal{R}_{t+K}(K).$$

and

$$\widehat{V}_1 = \frac{1}{T-1}\sum_{t=1}^{T-1}\left(\mathcal{R}_{t+1} - \overline{\mathcal{R}}_1\right)^2, \quad \overline{\mathcal{R}}_1 = \frac{1}{T-1}\sum_{t=1}^{T-1}\mathcal{R}_{t+1}.$$

For the third alternative ratio statistic, which we denote τ''_K we have:

$$\tau''_K = \ln E\left[\mathcal{R}_{t+K}(K)\right] - K\ln E\left[\mathcal{R}_{t+1}\right] = 0.$$

Forming the sample analogue and making use of the first order Taylor expansion we have:

$$\widehat{\tau''}_K \approx \frac{\hat{\mu}_K - \mu_K}{\mu_K} - K\frac{\hat{\mu}_1 - \mu_1}{\mu_1} = \frac{\hat{\mu}_K - (1+\mu)^K}{(1+\mu)^K} - K\frac{\left[\hat{\mu}_1 - (1+\mu)\right]}{1+\mu} = \widehat{\tau}_K - 1$$

This means that $\widehat{\tau''}_K$ has the same Taylor expansion as $\widehat{\tau}_K$, and thus resulting in the same limiting distribution:

$$\sqrt{T}\widehat{\tau''}_K \Longrightarrow N(0, W_K)$$

where W_K is given in the Theorem 1. Since $\widehat{\tau}''_K$ has the same asymptotic distribution, the standard errors and the bias correction coincide with those of $\widehat{\tau}_K$.