Common Values, Unobserved Heterogeneity, and Endogenous Entry in U.S. Offshore Oil Lease Auctions*

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Abstract

We develop and apply an empirical approach for first-price sealed bid auctions with affiliated values, unobserved auction-level heterogeneity, spatial correlation, and endogenous bidder entry. Following Haile, Hong and Shum (2003), we specify a reduced form for bidder entry outcomes and rely on an instrument for entry. However, we avoid their control function requirements and demonstrate that our specification is generated by a model in which a standard entry stage à la Berry (1992) is followed by a standard symmetric affiliated values auction à la Milgrom and Weber (1982). We show that important features of the model are nonparametrically identified and propose a semiparametric estimation approach designed to scale well to the moderate sample sizes typically encountered in practice. Key elements of the model and approach are motivated by our application to U.S. offshore oil and gas lease auctions, where our primary goal is to test for common values. Although an oil lease auction is the classic example cited to motivate a common values model, formal testing has been hindered by the confounding effects of unobserved heterogeneity. Our results show that common values, affiliated private information, and unobserved heterogeneity—three distinct notions with different implications for policy and empirical work—are all present and important in these auctions. Ignoring unobserved heterogeneity obscures the presence of common values. Although common values and affiliation imply theoretical ambiguity regarding the effects of additional competition on bids and revenues, we find that an exogenous increase in competition leads to more aggressive bidding and higher revenue.

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1 Introduction

In many auction settings it is natural to presume that important information commonly known among bidders is unobserved by the econometrician. Ignoring such unobserved heterogeneity can lead to a variety of errors. One may infer too much within-auction correlation in bidders' private information, as well as too much cross-auction variation in this information, leading to incorrect conclusions about such issues as bidder market power, the division of surplus between buyers and sellers, and optimal auction design.[1] In a first-price auction, unobserved heterogeneity presents a particular challenge because standard identification approaches exploit the insight that bidders' equilibrium beliefs about the competition can be inferred from observed distributions of rivals' bids.[2] With unobserved auction-level heterogeneity, bidders' beliefs condition on information unavailable to the econometrician. A further problem is that auction-level unobservables are likely to affect not only bids but also bidder participation. Endogenous bidder entry threatens several identification and testing approaches relying on exogenous variation in the level of competition.[3]

We propose an empirical model of entry and bidding in first-price auctions with affiliated values and unobserved heterogeneity. We show nonparametric identification of key features and propose a semiparametric estimation approach. We apply the approach to auctions of offshore oil and gas leases in the United States Outer Continental Shelf ("OCS") to evaluate the importance of unobserved heterogeneity, test the hypothesis of equilibrium bidding, assess the effect of competition on bidding and

[1]See, e.g., Krasnokutskaya (2011), Krasnokutskaya and Seim (2011), Athey, Levin, and Seira (2011), and Roberts (2013).

[2]See, e.g., Laffont and Vuong (1993), Guerre, Perrigne, and Vuong (2000), Li, Perrigne, and Vuong (2002), Hendricks, Pinkse, and Porter (2003), and Athey and Haile (2006, 2007).

[3]See, e.g., Gilley and Karels (1981), Athey and Haile (2002), Haile, Hong, and Shum (2003) ("HHS"), Guerre, Perrigne, and Vuong (2009), Campo, Guerre, Perrigne, and Vuong (2011), and Gillen (2010).

revenues, and test for the presence of common values. An auction of drilling rights is a classic example cited to motivate a common values environment. However, prior work on these auction has pointed out that formal testing for common values is hindered by the confounding effects of unobserved heterogeneity (Hendricks, Pinkse, and Porter (2003)). Indeed, we reject the private values hypothesis in favor of common values, but also find that ignoring unobserved heterogeneity and endogenous entry obscures the presence of common values. More broadly, we find that affiliated private information, common values, and common knowledge unobservables—three distinct phenomena with different implications for policy and empirical work[4]—are all present and important in OCS auctions.

We are not the first to study first-price auctions with unobserved heterogeneity. Haile and Kitamura (2017) provide an overview of existing approaches. All such methods require compromises. Several (e.g., Krasnokutskaya (2011), Hu, McAdams, and Shum (2013)) require that bidders have independent types, so that all correlation among bids can be attributed to unobserved heterogeneity.[5] Other approaches rely on a control function strategy that requires strict monotonicity between an observed outcome and the unobserved heterogeneity, allowing observables to indirectly control for unobservables (e.g., Campo, Perrigne, and Vuong (2003), HHS, Guerre, Perrigne, and Vuong (2009), Roberts (2013)). Simultaneous work by Kitamura and Laage (2017) proposes a finite mixture approach that allows correlated types (signals), but requires that the unobservable take on only a finite set of values—a limitation shared by, e.g., HHS and Hu, McAdams, and Shum (2013)—and enter the model through a

[4]For example, affiliation leads to the "linkage principle" (Milgrom and Weber (1982), Milgrom (1987)) whereas common values leads to the "winner's curse," each of which have important implications for auction design. Unobserved heterogeneity, which is taken as fixed in auction theory. implies neither affiliation nor common values but creates challenges for identification.

[5]See also the partial identification results in Armstrong (2013). An exception among approaches building on the measurement error literature is Balat (2011). His extension of Hu, McAdams, and Shum (2013) is an exploits observation of potential bidders' entry decisions at two sequential stages.

separable structure similar to that in Krasnokutskaya (2011). Finally, while control function approaches provide strategy for isolating exogenous variation in bidder entry, others do not.

Our approach requires compromises as well. We rely on an index assumption similar to that in Krasnokutskaya (2011) and Kitamura and Laage (2017) and posit a reduced form for bidder entry such that, conditional on observables, participation is weakly increasing in the unobservable. This reduced form rules out selective entry (cf., e.g., Gentry and Li (2014) and Bhattacharya, Roberts, and Sweeting (2014)), and a reduced form entry model cannot be used to evaluate interventions that would alter the map between auction characteristics and the entry outcome.[6] Like HHS, we also require an instrument for entry. But our approach has several advantages as well, and these are particularly attractive for our application. Our model avoids the requirement of independent private values and provides a strategy for exploiting exogenous sources of variation in bidder entry. This combination of features is particularly attractive for our study of OCS auctions, where a primary objective is testing for the presence of common values. Common values settings generally demand that we allow for correlated bidder types, and our test for common values exploits exogenous variation in bidder entry arising through an instrumental variable. We also avoid the strict monotonicity requirement of the control function approach and show that our empirical model, including the instrument, can be derived from a natural two-stage game motivated by our application—an entry stage à la Berry (1992) followed by a competitive bidding stage à la Milgrom and Weber (1982), with auction-specific unobservables that may have arbitrary dimension, may be correlated with auction observables, and may be spatially correlated.

Prior work on testing for common values includes Paarsch (1992), Athey and

[6]An open question is whether, once the bidding model is identified, it may become possible to identify a particular structural model of entry consistent with the reduced form.

Haile (2002), HHS, and Bajari and Hortaçsu (2003). Our approach is most similar to that in HHS, who studied timber auctions. We drop their assumptions of discrete unobserved heterogeneity and entry outcomes that are strictly increasing in the unobservable. This makes the identification problem substantially more challenging and requires a different estimation strategy. Our analysis of OCS auction data is most closely related to that of Hendricks, Pinkse, and Porter (2003) ("HPP"), who focused on testable implications of a pure common values model.[7] Our study is complementary to that of HPP. We allow the pure common values model but do not assume it, and we neither exploit nor rely on estimates of the realized tract values. HPP point out that tests for common values would be difficult to apply due to the fact that bidder entry is correlated with auction-level unobservables.[8] Our study also complements the simultaneous work of Aradillas-Lopez, Haile, Hendricks, and Porter (2017), which evaluates implications of competitive bidding in OCS auctions following the introduction of "area-wide leasing" in 1983. Our analysis here considers only the period 1954–1983, when the evidence in Haile, Hendricks, and Porter (2010) and Aradillas-Lopez, Haile, Hendricks, and Porter (2017) supports the assumption of competitive bidding.

The following section describes our model. In Section 3 we address nonparametric identification. Section 4 describes the estimation method. Sections 5 and 6 are devoted to our study of OCS oil lease auctions, with our baseline specification presented in section 5 and several alternative specifications explored in section 6. We conclude in section 7. Several appendices provide a detailed example, proofs omitted from the text, a small Monte Carlo study, and additional empirical results.

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[7] Earlier work on OCS auctions includes Gilley and Karels (1981), Hendricks, Porter, and Boudreau (1987), Hendricks and Porter (1988), and Hendricks, Porter, and Spady (1989).

[8] HPP partially control for unobserved heterogeneity by conditioning on two categories of tracts–those with high or low numbers of *potential bidders*, measured as the number of large firms ever to bid on a tract within a certain distance of the tract offered.

2 Model

We consider a standard model of first-price sealed bid auctions with symmetric affiliated values, extended to allow for auction-level heterogeneity and endogenous bidder entry. Each auction t is associated with observed characteristics $X_t \in \mathbb{X}$ and a scalar unobservable U_t. Without further loss, we let U_t be uniformly distributed on $[0, 1]$. We also assume independence between X_t and U_t.

Assumption 1. $X_t \perp\!\!\!\perp U_t$.

Neither the restriction to a scalar unobservable nor the independence required by Assumption 1 is without loss in general. However, we show below (see Example 1) that, this representation can be derived from a model, motivated by our application, in which auction-level unobservables have arbitrary dimension and arbitrary dependence with X_t. The same is true of the weak monotonicity conditions imposed below.

For each auction t we postulate a two-stage procedure in which entry is followed by bidding. We will not specify the the form of the entry stage; rather, we will specify a reduced form for the entry outcome and assume Bayes Nash equilibrium in the auction stage. The number of bidders entering auction t is denoted by N_t. Bidders are assumed risk neutral. Bidder i's valuation for the good offered is denoted by V_{it}. Upon entering, i observes a private signal $S_{it} \in [\underline{s}, \overline{s}]$ of his valuation.[9] Let $V_t = (V_{1t}, \ldots, V_{N_t t})$, $S_t = (S_{1t}, \ldots, S_{N_t t})$, and $S_{-it} = S_t \backslash S_{it}$.

In the bidding stage, the realizations of (N_t, X_t, U_t) are common knowledge among bidders, as is the distribution of $(S_t, V_t) \,|\, (N_t, X_t, U_t)$. In addition, each bidder knows his own signal. Given N_t and any set of conditioning variables Ω_t, let $F_{SV}(S_t, V_t | N_t, \Omega_t)$ denote the conditional joint distribution of signals and valuations. We make the following standard assumptions on $F_{SV}(S_t, V_t | N_t, X_t, U_t)$.

[9]Although we write $\underline{s}$ and $\overline{s}$ for expositional clarity, one could let $[\underline{s}, \overline{s}] = [0, 1]$ without loss.

Assumption 2. (i) For all $n \in \text{supp} N_t | (X_t, U_t)$, $F_{SV}(S_t, V_t | n, X_t, U_t)$ has a continuously differentiable joint density that is affiliated, exchangeable in the indices $i = 1, \ldots, n$, and positive on $(\underline{s}, \overline{s})^n \times (\underline{v}, \overline{v})^n$; (ii) $E[V_{it} | S_{it}, S_{-it}, N_t, X_t, U_t]$ exists and is strictly increasing in S_{it}.

Because the bidding stage involves a standard affiliated values model, it nests a variety of special cases. With *private values*, $E[V_{it} | S_{it}, S_{-it}, N_t, X_t, U_t]$ does not depend on S_{-it}. In our setting this is equivalent to bidders' knowing their valuations, i.e., $S_{it} = V_{it}$. When $E[V_{it} | S_{it}, S_{-it}, N_t, X_t, U_t]$ depends on S_{-it}, we have *common values* (or *interdependent values*). A special case of the common values model is that of *pure common values*, where $V_{it} = \bar{V}_t$ for all i.

We impose the following restriction on how the auction characteristics (X_t, U_t) affect bidder valuations.

Assumption 3. (i) $V_{it} = \Gamma(X_t, U_t) V_{it}^0$; (ii) conditional on N_t, $(V_{1t}^0, \ldots, V_{N_t t}^0, S_{1t}, \ldots, S_{N_t t})$ is independent of (X_t, U_t); (iii) Γ is bounded and weakly increasing in U_t.

Assumption 3 is an index restriction, imposing multiplicative separability in (X_t, U_t) and weak monotonicity in U_t.[10] An assumption of multiplicative (or additive) separability has often been relied upon in the auctions literature, including for identification in other settings with unobserved heterogeneity. Our identification result will rely on this assumption as well. Without further loss, we normalize the scale of Γ relative to that of V_{it}^0 by taking an arbitrary point $x^0 \in \mathbb{X}$ and setting

$$\Gamma\left(x^0, 0\right) = 1. \tag{1}$$

We assume initially that the auction is conducted without a binding reserve price,

[10]Without a distributional restriction like that in part (ii) of the assumption, part (i) would have no content. In addition, weak monotonicity in U_t merely specifies that more "desirable" realizations of the unobservable be labeled with larger values.

although below we also consider an extension allowing a random reserve price. Under Assumption 2, the auction stage of our model admits a unique Bayesian Nash equilibrium in weakly increasing strategies; these strategies are in fact strictly increasing in signals.[11] Letting the random variable B_{it} denote the equilibrium bid of bidder i in auction t, we then have

$$B_{it} = \beta\left(S_{it}; X_t, U_t, N_t\right),$$

where $\beta(\cdot; X_t, U_t, N_t) : [\underline{s}, \overline{s}] \to \mathbb{R}$ denotes the symmetric equilibrium bidding strategy for auction t and is strictly increasing in S_{it}. Let B_t denote $(B_{1t}, \ldots, B_{N_t t})$.

A useful fact is that the separability required by Assumption 3 is is inherited by the equilibrium bidding strategies.[12] Thus, under Assumptions 2 and 3 we may write

$$\beta\left(S_{it}; X_t, U_t, N_t\right) = \Gamma\left(X_t, U_t\right) \beta^0\left(S_{it}; N_t\right), \tag{2}$$

where β^0 denotes the symmetric Bayesian Nash equilibrium bidding strategy for a (possibly hypothetical) auction t at which $\Gamma\left(X_t, U_t\right) = \Gamma\left(x^0, 0\right) = 1$. Following HHS, we refer to $B_{it}^0 = \beta^0\left(S_{it}; N_t\right)$ and V_{it}^0 as "homogenized" bids and valuations.

We link the model of a single auction to the observed sample through Assumption 4. Given Assumption 3, this is the standard (although often implicit) assumption that auctions are i.i.d. conditional on auction characteristics, (N_t, X_t, U_t). However, we do not require U_t to be independent across auctions. This may be important in our application and others, where spatial dependence is likely but typically ignored.

Assumption 4. $(V_t^0, S_t) \perp\!\!\!\perp (V_{t'}^0, S_{t'})$ for $t' \neq t$.

[11]See Theorem 2.1 in Athey and Haile (2007) and the associated references, including Athey (2001), Reny and Zamir (2004), and McAdams (2007). Milgrom and Weber (1982) characterize the equilibrium strategies.

[12]See, e.g., HHS, Athey and Haile (2007), or Krasnokutskaya (2011).

Finally, we specify the outcome of the entry stage by supposing that the number of bidders at auction t satisfies

$$N_t = \eta(X_t, Z_t, U_t) \tag{3}$$

for some function η that is weakly increasing in U_t. Formally, (3) represents the reduced form for the outcome in the entry stage. The weak monotonicity requirement links the interpretation of the unobservable in the entry and bidding stages: unobservables that make the good for sale more valuable also encourage more entry. The new variable Z_t in (3) is an exogenous auction-specific observable that affects bidder entry but is otherwise excludable from the auction model, as formalized below.

Assumption 5. (i) $Z_t \perp\!\!\!\perp U_t | X_t$; (ii) $Z_t \perp\!\!\!\perp (S_t, V_t^0) | N_t$.

The following example, discussed more fully in Appendix A, describes one fully specified two-stage game leading to the structure assumed above.

Example 1. *Consider a model of entry and bidding for an OCS oil and gas lease, where a standard simultaneous move entry stage à la Berry (1992) precedes a competitive bidding stage à la Milgrom and Weber (1982). Players in the game are firms in the industry. The tract offered for lease is associated with observables X_t, which includes (among other relevant covariates) the number of active leases on neighbor tracts and the sets of bidders for those leases.*[13] *The active neighbor leases are owned by Z_t distinct* ***neighbor firms****. Tract-level unobservables are denoted by $E_t \in \mathbb{R}^L$, may be correlated with X_t, and may be spatially correlated. The characteristics X_t and E_t are assumed to scale valuations (multiplicatively) through an unrestricted index function $\lambda(X_t, E_t)$. Firms play a two-stage game. They first choose simultaneously whether to enter, with each entering firm i incurring a signal acquisition cost $c_i(X_t)$. Signal*

[13] Tracts s and t are defined to be neighbors if their borders coincide at some point.

acquisition costs are common knowledge and lower for neighbor firms than other (non-neighbor) firms.[14] *Entrants learn their private signals then participate in a first-price sealed bid auction with symmetric affiliated values. Appendix A shows that all perfect Bayesian equilibria of this game in pure strategies can be represented by the model and assumptions above. This representation is obtained by defining* $U_t = F_\lambda\left(\lambda(x_t, E_t)|x_t\right)$, *where* $F_\lambda(\cdot|x)$ *is the CDF of the random variable* $\lambda(x, E_t)$ *conditional on* $X_t = x$. *Observe that in this case the distribution of* U_t *does not vary with* X_t, *although its interpretation does.*[15]

3 Identification

In this section we develop sufficient conditions for identification of the entry model, the index function Γ, and key features of the bidding model. We address each of these in turn. Throughout we assume that the observables include X_t, Z_t, N_t and B_t. Let $\mathbb{Y}$ denote the support of (X_t, Z_t), and let $\mathbb{Y}(n)$ denote the support of (X_t, Z_t) conditional on $N_t = n$. Let $\underline{n} \geq 0$ denote the minimum value in the support of N_t; let $\overline{n}$ denote the supremum (possibly ∞) of this support. Recalling (1), for convenience we take x^0 such that for some z we have $(x^0, z) \in \mathbb{Y}(\underline{n})$.

3.1 Identification of the Entry Model

We show identification of the entry model under the following regularity condition on the support of $N_t|(X_t, Z_t)$.

Assumption 6. For all $(x, z) \in \mathbb{Y}$, there exist $\underline{n}(x, z)$ and $\overline{n}(x, z)$ such that $\eta(x, z, U_t)$ has support $\left(\underline{n}(x, z), \underline{n}(x, z) + 1, \ldots, \overline{n}(x, z)\right)$.

[14] This structure generalizes that in Hendricks and Porter (1988), where neighbors obtain a signal for free but non-neighbors face an infinite cost of signal acquisition.

[15] In that case, knowledge of the function Γ will not be sufficient to characterize the separate effects of a ceteris paribus change in X_t on bidder valuations. See the additional discussion in Appendix A.

Given Assumption 6, for any $(x, z) \in \mathbb{Y}$, the step function $\eta(x, z, \cdot)$ is characterized by the thresholds

$$\tau_{\underline{n}(x,z)-1}(x, z) \leq \tau_{\underline{n}(x,z)}(x, z) \leq ... \leq \tau_{\overline{n}(x,z)},$$

where

$$\tau_{\underline{n}(x,z)-1}(x, z) = 0, \qquad \tau_{\overline{n}(x,z)}(x, z) = 1, \tag{4}$$

and for $n = \{\underline{n}(x, z), \ldots, \overline{n}(x, z)\}$,

$$\tau_{n-1}(x, z) = \inf\{u \in [0, 1] : \eta(x, z, u) \geq n\}.$$

With this observation, identification of η follows easily.

Theorem 1. *Under Assumptions 1–6, η is identified.*

Proof. Take arbitrary $(x, z) \in \mathbb{Y}$. For each $n \in \{\underline{n}(x, z), \ldots, \overline{n}(x, z)\}$

$$\begin{aligned} \Pr(N_t = n | X_t = x, Z_t = z) &= \Pr(\tau_{n-1}(x, z) \leq U_t \leq \tau_n(x, z) | X_t = x, Z_t = z) \\ &= \tau_n(x, z) - \tau_{n-1}(x, z). \end{aligned} \tag{5}$$

The probabilities on the left are observed. Thus, using (4), equation (5) can be solved iteratively for the unknown thresholds $\tau_n(x, z)$ starting from $n = \underline{n}(x, z)$. □

Identification of η determines the effects of Z_t on participation and provides bounds $\tau_{n_t-1}(x_t, z_t)$ and $\tau_{n_t}(x_t, z_t)$ on the realization of each unobservable U_t. For our purposes, the latter is most important, as we will exploit variation in these bounds with (x_t, z_t) to obtain identification of the index function Γ and the bidding model. A useful implication of Theorem 1 is the following corollary, proved in Appendix B.

Corollary 1. *Under Assumptions 1–6, the distribution of $U_t | X_t, N_t$ is identified.*

3.2 Identification of the Index Function

Define

$$\gamma(x,u) = \ln \Gamma(x,u).$$

We first provide conditions sufficient to identify $\gamma(x,u)$ at each $x \in \mathbb{X}$ and $u \in \mathcal{U}_x$, where

$$\mathcal{U}_x = \bigcup_{\substack{z:(x,z)\in\mathbb{Y} \\ n\in \text{supp } \eta(x,z,U_t)}} \{\tau_{n-1}(x,z), \tau_n(x,z)\}.$$

We then give additional conditions guaranteeing that $\mathcal{U}_x$ includes the entire interval $[0,1]$ for each x.

We begin with the following result, whose proof illustrates a key argument.

Lemma 1. *Under Assumptions 1–6, for all* $n \geq \underline{n}$, *all* $(x,z) \in \mathbb{Y}(n)$, *and all* $(x',z') \in \mathbb{Y}(n)$, *the differences* $\gamma(x', \tau_n(x',z')) - \gamma(x, \tau_n(x,z))$ *and* $\gamma(x', \tau_{n-1}(x',z')) - \gamma(x, \tau_{n-1}(x,z))$ *are identified.*

Proof. By (2) and monotonicity of the equilibrium bid function,

$$\inf\{\ln B_{it} | N_t = n, X_t = x, Z_t = z\} = \gamma(x, \tau_{n-1}(x,z)) + \ln \beta^0(\underline{s}; n).$$

So under Assumptions 1–6, for any n and all (x,z) and (x',z') in $\mathbb{Y}(n)$, the differences

$$\gamma(x', \tau_{n-1}(x',z')) - \gamma(x, \tau_{n-1}(x,z)) \tag{6}$$

are identified.[16] Similarly, since

$$\sup\{\ln B_{it} | N_t = n, X_t = x, Z_t = z\} = \gamma(x, \tau_n(x,z)) + \ln \beta^0(\overline{s}; n),$$

[16]Because $\beta^0(\underline{s};n) = E[V_{it}^0|S_{jt} = \underline{s}, j = 1,\ldots,n]$ and $\beta^0(s;n) \leq E[V_{it}^0|S_{jt} = s, j = 1,\ldots,n]$ for all s (see, e.g., Milgrom and Weber (1982)), both $\beta^0(\underline{s};n)$ and $\beta^0(\overline{s};n)$ are finite under Assumption 2.

we obtain identification of the differences

$$\gamma\left(x', \tau_n\left(x', z'\right)\right) - \gamma\left(x, \tau_n\left(x, z\right)\right) \tag{7}$$

for all n and all (x, z) and (x', z') in $\mathbb{Y}\left(n\right)$. □

Under two additional conditions, the first differences obtained above can be differenced again at well chosen points (allowing cancellation of common terms) to obtain a set of first differences sufficient to pin down the value of the index $\gamma(x, u)$ at all x and $u \in \mathcal{U}_x$.

Assumption 7. For all $n \in \underline{n}+1, \underline{n}+2, \ldots, \overline{n}$, $\mathbb{Y}\left(n-1\right) \cap \mathbb{Y}\left(n\right)$ is nonempty.

Assumption 8. There exists n^* such that
(i) $\forall n \in \{\underline{n}, \ldots, n^*\}$, $\mathbb{Y}$ contains points $\left(x\left(n\right), z\left(n\right)\right)$ and $\left(x\left(n\right), \hat{z}\left(n\right)\right)$ such that $\underline{n}\left(x\left(n\right), z\left(n\right)\right) = n$ and $\underline{n}\left(x\left(n\right), \hat{z}\left(n\right)\right) = n+1$; and
(ii) $\forall n \in \{n^*, \ldots, \overline{n}\}$, $\mathbb{Y}$ contains points $\left(x\left(n\right), z\left(n\right)\right)$ and $\left(x\left(n\right), \hat{z}\left(n\right)\right)$ such that $\overline{n}\left(x\left(n\right), z\left(n\right)\right) = n$ and $\overline{n}\left(x\left(n\right), \hat{z}\left(n\right)\right) = n-1$.

Assumption 7 requires variation in U_t that produces local variation in participation. This rules out trivial cases in which the unobservable has no effect on entry. Assumption 8 requires variation in the instrument Z_t that can induce local variation in the support of the participation outcomes, at least at some values of X_t. This is our most critical assumption on the variation in entry generated by variation in the instrument.

We prove the following results in Appendix B.

Lemma 2. *Under Assumptions 1–8, for all* $n \geq \underline{n}$*, all* $(x, z) \in \mathbb{Y}\left(n\right)$*, and all* $(x', z') \in \mathbb{Y}\left(n\right)$*,* $\gamma\left(x, \tau_n\left(x, z\right)\right) - \gamma\left(x', \tau_{n-1}\left(x', z'\right)\right)$ *is identified.*

Lemma 3. *Under Assumptions 1–8, for all* $n \geq \underline{n}$ *and all* $(x, z) \in \mathbb{Y}\left(n\right)$*, the values of* $\gamma\left(x, \tau_{n-1}\left(x, z\right)\right)$ *and* $\gamma\left(x, \tau_n\left(x, z\right)\right)$ *are identified.*

By Theorem 1, the values of $\tau_{n-1}(x,z)$ and $\tau_n(x,z)$ are known for all n and $(x,z) \in \mathbb{Y}(n)$. Thus, Lemma 3 demonstrates identification of $\gamma(x,u)$ at each $x \in \mathbb{X}$ and $u \in \mathcal{U}_x$. In general, this may deliver only partial identification of the index function γ, so that in practice one may rely on functional form to interpolate between the points $\{x \in \mathbb{X}, u \in \mathcal{U}_x\}$ at which $\gamma(x,u)$ is point identified. However, the following conditions are sufficient to ensure that no such interpolation is necessary.

Assumption 9. For every $x \in \mathbb{X}$ there exists a finite partition $0 = \tau^0(x) < \tau^1(x) < \cdots < \tau^K(x) = 1$ of the unit interval such that for each $k = 1, \ldots, K$ and some $z(k), z'(k) \in \text{supp} Z_t | X_t = x$, $\eta\left(x, z(k), \tau^{k-1}(x)\right) > \eta\left(x, z'(k), \tau^k(x)\right)$.

Assumption 10. (a) For all $x \in \mathbb{X}$, $\text{supp} Z_t | X_t = x$ is connected.
(b) For all $(x,z,u) \in \mathbb{Y} \times (0,1)$ and all $\delta > 0$ such that $(u-\delta, u+\delta) \subset (0,1)$, there exists $\epsilon > 0$ such that if $\|z'-z\| < \epsilon$ then $\eta(x,z',u') = \eta(x,z,u)$ for some $u' \in (u-\delta, u+\delta)$.

Assumption 9 requires that variation in Z_t have sufficient effect on participation to offset some discrete variation in the unobservable U_t. A sufficient condition is that for each x there exist z and z' such that $\eta(x,z,\tau_{n-1}(x,z)) > \eta(x,z',\tau_n(x,z))$ for all $n \in \{\underline{n}(x,z), \ldots, \overline{n}(x,z)\}$; in this case, the set $\left\{\tau_{\underline{n}(x,z)-1}, \ldots, \tau_{\overline{n}(x,z)}\right\}$ would define the partition $\tau^0(x) < \tau^1(x) < \cdots < \tau^K(x)$. More generally, however, there is no need to rely on a single pair z and z' to play the roles of $z(k), z'(k)$ for all k. Assumption 10 rules out discrete instruments and requires a type of continuous substitution between Z_t and U_t in the "production" of bidder entry: it must be possible to offset the effect (on entry) of a small change in Z_t with a small change in U_t.

The following lemma, whose proof is provided in Appendix B, leads us to the point identification of γ (and therefore Γ) demonstrated in Theorem 2.

Lemma 4. *Under Assumptions 1–10, $\tau_{n-1}(X_t, Z_t)$ is continuous in Z_t on the pre-image of $(0,1)$.*

Theorem 2. *Under Assumptions 1–10,* Γ *is identified on* $\mathbb{X} \times [0,1]$.

Proof. We need only show that $\mathcal{U}_x$ includes the entire interval $[0,1]$ for each $x \in \mathbb{X}$. Take arbitrary $x \in \mathbb{X}$ and let $0 = \tau^0(x) < \tau^1(x) < \cdots < \tau^K(x) = 1$ be as in Assumption 9. Take any $k \in \{1, \ldots, K\}$. Since $\eta\left(x, z, \tau^{k-1}\right) > \eta\left(x, z', \tau^k\right)$ for some n, z, z', letting $n = \eta\left(x, z, \tau^{k-1}\right)$ we know $\tau_{n-1}(x, z) \leq \tau^{k-1}(x)$ and $\tau_{n-1}(x, z') \geq \tau^k(x)$. Because the continuous image of a connected set is connected, Lemma 4 and Assumption 10 (part (a)) imply that for every $\tilde{\tau} \in \left[\tau^{k-1}(x), \tau^k(x)\right]$ there exists $z(\tilde{\tau})$ such that $\tau_{n-1}(x, z(\tilde{\tau})) = \tilde{\tau}$. □

3.3 Identification of the Bidding Model

Let

$$w(s; n, x, u) \equiv E\left[V_{it} \middle| S_{it} = \max_{j \neq i} S_{jt} = s, N_t = n, X_t = x, U_t = u\right]$$

denote a bidder's pivotal expectation of the value he would obtain from winning—i.e., his expectation of his valuation conditional on his own signal and on the event that he is tied to win the auction. Our main result in this section demonstrates identification of the joint distribution of the pivotal expected values $(w(S_{1t}; n, x, u), \ldots, w(S_{nt}; n, x, u))$ for all $x \in \mathbb{X}, u \in [0,1]$, and $n = 2, \ldots, \bar{n}$. This will lead to several identification and falsifiability results as corollaries.

Let

$$G_{M|B}(m|b, x, u, n) = \Pr\left(\max_{j \neq i} B_{jt} \leq m \middle| B_{it} = b, X_t = x, U_t = u, N_t = n\right)$$

and let $g_{M|B}(m|b, x, u, n)$ denote the associated conditional density (guaranteed to exist by Assumption 2 and strict monotonicity of the equilibrium bid function). Following Laffont and Vuong (1993), Guerre, Perrigne, and Vuong (2000), and Li, Perrigne, and Vuong (2000, 2002) one can characterize the relationship between each realized $w(s_{it}; n_t, x_t, u_t)$ and the associated equilibrium bid $b_{it} = \beta(s_{it}; x_t, u_t, n_t)$ in terms of the joint distribution of equilibrium bids. In particular, each b_{it} must satisfy the first-order condition[17]

$$w(s_{it}; n_t, x_t, u_t) = b_{it} + \frac{G_{M|B}(b_{it}|b_{it}, x_t, u_t, n_t)}{g_{M|B}(b_{it}|b_{it}, x_t, u_t, n_t)}. \tag{8}$$

Although this equation expresses the expectation $w(s_{it}; n_t, x_t, u_t)$ as a functional of a conditional distribution of bids, the presence of u_t on the right-hand side creates challenges. Because realizations of U_t are not observable or identified, one cannot directly condition on them to identify the functions $G_{M|B}$ and $g_{M|B}$. This precludes obtaining identification directly from (8).[18] With the preceding results, however, we can overcome this problem.

Observe that, like valuations and bids, the pivotal expected values $w(s_{it}; n_t, x_t, u_t)$ have the separable structure

$$w(s_{it}; n_t, x_t, u_t) = w^0(s_{it}; n_t)\Gamma(x_t, u_t), \tag{9}$$

where

$$w^0(s_{it}; n_t) \equiv E\left[V_{it}^0 \middle| S_{it}, \max_{j \neq i} S_{jt} = S_{it}, N_t\right].$$

The first-order condition (8) can therefore be written as

$$w^0(s_{it}; n_t) = b_{it}^0 + \frac{G_{M^0|B^0}(b_{it}^0|b_{it}^0, n_t)}{g_{M^0|B^0}(b_{it}^0|b_{it}^0, n_t)} \tag{10}$$

[17] See, e.g., Athey and Haile (2007) for a derivation in the affiliated values model.

[18] Theorem 1 does demonstrate identification of bounds on each u_t. An open question is whether one could use these bounds to obtain useful partial identification of the bidding model. Because (8) involves the value of $\frac{G_{M|B}(b_{it}|b_{it}, x_t, u_t, n_t)}{g_{M|B}(b_{it}|b_{it}, x_t, u_t, n_t)}$ at a point rather than an expectation, this appears to be a more challenging problem than bounding a regression function with interval-measured covariates (e.g., Manski and Tamer (2002)).

where

$$G_{M^0|B^0}(m|b,n_t) = \Pr\left(\max_{j\neq i} B^0_{jt} \le m \middle| B^0_{it} = b, N_t = n\right)$$

and $g_{M^0|B^0}(m|b,n_t)$ is the associated conditional density.

With the following regularity condition, Theorem 2 and a standard deconvolution argument will yield identification of the joint distribution of $B^0_{1t},\ldots,B^0_{nt}$ for all n.

Assumption 11. For some $x \in \mathbb{X}$ the random variable $\gamma(x,U_t)$ has nonzero characteristic function almost everywhere.

Lemma 5. *Under Assumptions 1–11, conditional on any $N_t = n$, the joint density of $B^0_{1t},\ldots,B^0_{nt}$ is identified.*

Proof. Fix $N_t = n$ and $X_t = x$, where x is as in Assumption 11. Let $\tilde{B}_{it} = \ln(B_{it})$ and $\tilde{B}^0_{it} = \ln(B^0_{it})$. By Assumption 3,

$$\tilde{B}_{it} = \tilde{B}^0_{it} + \gamma(x,U_t).$$

By Assumption 3, $\tilde{B}^0_{it}$ and $\gamma(x,U_t)$ are independent conditional on N_t. Let $\psi_{\tilde{B}}$ denote the characteristic function of the log bids $(\tilde{B}_{1t},\ldots,\tilde{B}_{nt})$ conditional on $X_t = x, N_t = n$. Letting ψ_γ denote the characteristic function of $\gamma(x,U_t)$ conditional on $N_t = n$, for $(r_1,\ldots,r_n) \in \mathbb{R}^n$ we then have

$$\psi_{\tilde{B}}(r_1,\ldots,r_n) = \psi_{\tilde{B}^0}(r_1,\ldots,r_n)\,\psi_\gamma(r_1+\cdots+r_n)$$

where $\psi_{\tilde{B}^0}$ is the characteristic function of the log homogenized bids $\left(\tilde{B}^0_{1t},\ldots,\tilde{B}^0_{nt}\right)$ conditional on $N_t = n$. Since the distribution of $U_t|X_t,N_t$ is known (Corollary 1) and γ is a known function (Theorem 2), ψ_γ is known. So under Assumption 11 we have

$$\psi_{\tilde{B}^0}(r_1,\ldots,r_n) = \frac{\psi_{\tilde{B}}(r_1,\ldots,r_n)}{\psi_\gamma(r_1+\cdots+r_n)}$$

for almost all $(r_1, \ldots, r_n)$. By continuity of characteristic functions this yields identification of $\psi_{\tilde{B}^0}$, implying identification of the joint density of $\left(\tilde{B}^0_{1t}, \ldots, \tilde{B}^0_{nt}\right)$.[19] The result then follows. $\square$

This leads directly to our main identification result.

Theorem 3. *Let Assumptions 1–11 hold. Then for all $x \in \mathbb{X}, u \in [0,1]$, and $n = 2, \ldots, \bar{n}$, the joint distribution of $w(S_{1t}; N_t, X_t, U_t), \ldots, w(S_{N_t t}; N_t, X_t, U_t)$ conditional on $N_t = n$, $X_t = x$, $U_t = u$ is identified.*

Proof. Fix n. From (10), we have

$$w^0(S_{it}; n) = \xi(B^0_{it}; n) \equiv B^0_{it} + \frac{G_{M^0|B^0}(B^0_{it}|B^0_{it}, n)}{g_{M^0|B^0}(B^0_{it}|B^0_{it}, n)}. \tag{11}$$

By Lemma 5, the joint distribution $(\xi(B^0_{1t}; n), \ldots, \xi(B^0_{nt}; n))$ is known. This implies identification of the joint distribution of $(w^0(S_{1t}; n), \ldots, w^0(S_{nt}; n))$. The result then follows immediately from (9) and Theorem 2. $\square$

This yields the following implications.

Corollary 2. *Under Assumptions 1–11,*

(a) the hypothesis of equilibrium bidding in the affiliated values model is testable;

(b) the affiliated private values model is identified;

(c) the hypothesis of private values is testable against the alternative of common values;

(d) if the ex post value of the good to the winner is observed, the hypothesis of equilibrium bidding in the pure common values is testable.

Proof. See Appendix B. $\square$

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[19] Note that since the same argument holds at all $x \in \mathbb{X}$ such that $\gamma(x, U_t)$ is nonvanishing a.s., the argument demonstrating Lemma 5 may often yield overidentification.

4 Estimation

We propose a multi-step semiparametric estimation approach. In the first step we estimate the conditional probabilities $\Pr(N_t = n|X_t = x, Z_t = z)$ that determine the entry thresholds $\tau_\ell(x, z)$. The second step uses a semiparametric sieve quasi-maximum likelihood approach to estimate the index function γ and joint distribution of (log) homogenized bids. Finally, the joint distribution of bidders' pivotal expected values is estimated by plugging draws from the estimated distribution into the equilibrium first-order conditions for the auction.

4.1 Entry Thresholds

We showed in section 3 that each $\tau_\ell(x, z)$ can be expressed as a linear combination of conditional probabilities of the form $\Pr(N_t = \hat{n}|X_t = x, Z_t = z)$. In principle, one could substitute standard nonparametric estimates of these conditional probabilities to estimate the entry thresholds. However, it is useful to exploit the ordered nature of the entry outcomes. Observe that, given any strictly increasing univariate CDF H, we have

$$\begin{aligned}
\Pr(N_t = n|X_t = x, Z_t = z) &= \tau_n(x, z) - \tau_{n-1}(x, z) \\
&= H\left(H^{-1}\left(\tau_n\left(x, z\right)\right)\right) - H\left(H^{-1}\left(\tau_{n-1}\left(x, z\right)\right)\right).
\end{aligned}$$

Letting $\alpha_n(x, z) = H^{-1}\left(\tau_n\left(x, z\right)\right)$, this becomes

$$\Pr(N_t = n|X_t = x, Z_t = z) = H\left(\alpha_n\left(x, z\right)\right) - H\left(\alpha_{n-1}\left(x, z\right)\right).$$

Thus, if ϵ is a random variable with distribution H, entry outcomes are characterized by an ordered response model of the form

$$\{N_t = n | X_t = x, Z_t = z\} \iff \{\epsilon \in (\alpha_{n-1}(x,z), \alpha_n(x,z))\}.$$

We estimate this model using a linear specification

$$\alpha_n(x,z) = \alpha_n - x\lambda_x - z\lambda_z$$

and two alternative specifications of H.[20] The first specifies H as the standard normal CDF, yielding an ordered probit model. The second specifies H nonparametrically using a Hermite polynomial series approximation (see, e.g., Gallant and Nychka (1987)). In the latter case $H(\cdot)$ is a continuous distribution function with density

$$\hat{h}(\epsilon) = \frac{1}{\theta_h}\left(\sum_{k=0}^{\tilde{K}} \sigma_k \epsilon^k\right)^2 \phi(\epsilon), \tag{12}$$

where $\tilde{K}$ is the order of the Hermite polynomial approximation, $\phi(\cdot)$ is the standard normal density, and $\theta_h = \int_{-\infty}^{\infty}\left(\sum_{k=0}^{\tilde{K}} \sigma_k \epsilon^k\right)^2 \phi(\epsilon) d\epsilon$.

We estimate the parameters $(\alpha, \lambda_x, \lambda_z)$ (and $\{\sigma_k\}_{k=1,\ldots,K}$ in the case of the series estimator) by (quasi-) maximum likelihood.[21] From these estimates we calculate the implied values of each conditional entry threshold $\tau_{n-1}(x,z)$ and $\tau_n(x,z)$.

[20] The linear specification can be equivalently stated as $\tau_n(x,z) = H(\alpha_n - x\lambda_x - z\lambda_z)$.

[21] Recall that we permit spatial dependence. Consistency follows standard arguments under appropriate weak dependence conditions.

4.2 Index Function and Homogenized Bid Distribution

In the second step we take the estimated entry thresholds as given and estimate the index function γ and joint distribution/density of the log homogenized bids $\left(\tilde{B}^0_{1t}, \ldots, \tilde{B}^0_{nt}\right)$ for each value n. Given our focus on testing for common values, our estimation strategy will prioritize flexibility in how this joint distribution/density can vary with n. We specify γ parametrically as $\gamma(\cdot, \cdot; \theta_\gamma)$; we will use a linear specification below. The joint density of homogenized bids is specified semiparametrically, using a nonparametric (Bernstein polynomial sieve) specification of the common marginal distribution and parametric copula.[22][23] Below we specify a Gaussian copula, with separate covariance parameter ρ_n for each value of n. As we discuss in Appendix , this choice provides a substantial computational advantage when we transform our estimated joint distributions and densities to the conditional distributions and densities that are plugged into bidders' first-order conditions.

We specify the marginal density of a generic bidder's log homogenized bid in an n-bidder auction as

$$\tilde{g}_{B^0_i}(\tilde{b}^0; \theta_b, n) = \sum_{j=0}^{m} \theta^{(j)}_{b,n} q_{j,m}\left(\Phi\left(\tilde{b}^0\right)\right)\phi\left(\tilde{b}^0\right), \tag{13}$$

where $q_{j,m}(\nu) = \binom{m}{j}\nu^j (1-\nu)^{m-j}$ and $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal distribution and density functions, respectively. Here m is a parameter, growing with the sample size, that determines the order of the approximation. Let $\theta_{b,n} = \{\theta^{(j)}_{b,n}\}_{j=0}^{m}$.

[22]Prior work using Bernstein polynomials in estimation of auction models includes Komarova (2017) and Kong (2017a,b). Hubbard, Paarsch, and Li (2012) have previously combined a parametric copula and nonparametric (kernel) specification of marginal densities to estimate auction models.

[23]Our use of a parametric copula reflects in part our choice to let the distribution of bids be fully flexible (i.e., even in finite sample) with respect to the number of bidders n_t, because a primary focus of our application is an evaluation of how variation in n_t affects equilibrium bidding. In other applications one might specify a Bernstein copula and account for the effects of n_t within the sieve approximation.

Note that this parameter vector is different for each n. Thus, we let the marginal density of log homogenized bids vary with n in an unrestricted way, even in finite sample.

Because Bernstein polynomials approximate functions with domain $[0,1]$, in (13) we have applied the Bernstein approximation to marginal density of the transformed variable $\Phi(\tilde{b}^0)$.[24] This type of CDF transformation is useful not only for normalizing the support but also for ensuring that the nonparametric estimator will offer sensible approximations even in modest sample sizes. When $m=0$, for example, the distribution of log-bids will be normal. Thus, our nonparametric specification is based on a sequence of models that starts with a natural (lognormal) parametric specification and adds flexibility as permitted by the sample size.

Let $\tilde{G}_{B_i^0}(\tilde{b}^0;\theta_b,n)$ denote the conditional CDF associated with $\tilde{g}_{B_i^0}(\tilde{b}^0;\theta_b,n)$. Let $\chi(\cdot;\rho_n)$ denote the symmetric Gaussian copula density with covariance parameter ρ_n.[25] For each value of n, the joint density of the log homogenized bids is specified as

$$\tilde{g}_{B^0}\left(\tilde{b}_1^0,\ldots,\tilde{b}_n^0;\theta_{b,n},\rho_n,n\right) = \chi\left(\tilde{G}_{B_i^0}(\tilde{b}_1^0;\theta_{b,n},n),\ldots,\tilde{G}_{B_i^0}(\tilde{b}_n^0;\theta_{b,n},n);\rho_n\right)\tilde{g}_{B_i^0}(\tilde{b}_1^0;\theta_{b,n},n)\ldots\tilde{g}_{B_i^0}(\tilde{b}_n^0;\theta_{b,n},n). \quad (14)$$

[24]For estimation purposes, we add intercepts $\gamma_{n_t}^0$ for each value of n_t in the index function $\gamma(x_t,u_t;\theta_\gamma)$, implying that in this step we actually estimate joint densities of *centered* log homogenized bids $\tilde{b}_t^0-\gamma_{n_t}^0$. We then adjust each estimated density with the appropriate intercept estimate to obtain the density of (uncentered) log homogenized bids that is relevant to bidders' first-order conditions. When homogenization is performed via OLS, this type of centering procedure is required for consistency (see HHS and Athey and Haile (2007)). In our case this is not essential but offers several practical advantages by ensuring that the log homogenized bids are centered at zero prior to transformation by the normal CDF, ensuring that the location of the estimated bid distribution can move freely with n_t (again, prioritizing flexibility with respect to n_t), and freeing the Bernstein coefficients to capture features of the marginal density other than its location.

[25]The symmetric Gaussian copula density is given by:

$$\chi\left(h_1,\ldots,h_n;\rho_n\right)=\frac{1}{|R|^{1/2}}\exp\left(\frac{1}{2}\varphi(h)'\left(R^{-1}-I\right)\varphi(h)\right),$$

where $\varphi(h)=\left\{\Phi^{-1}\left(h_1\right),\ldots,\Phi^{-1}\left(h_n\right)\right\}$ and the matrix R has ones on the diagonal and covariance ρ_n in all off-diagonal entries.

A further advantage of the Bernstein polynomial approximation used here is the ease of imposing otherwise complex functional restrictions through linear restrictions on the vector of Bernstein coefficients. For instance, a necessary and sufficient condition for the function to integrate to one is that $\sum_{j=0}^{m} \theta_{b,j} = m+1$. Additionally, the Bernstein polynomials allow easy transformation between the density and associated cumulative distribution. In particular,

$$\tilde{G}_{B_i^0}(\tilde{b}^0; \theta_m, n) = \sum_{j=0}^{m+1} \tilde{\theta}_{j,m+1} q_{j,m+1}\left(\Phi\left(\tilde{b}^0\right)\right), \tag{15}$$

where $\tilde{\theta}_{m+1} = (\tilde{\theta}_{0,m+1}, \ldots, \tilde{\theta}_{m+1,m+1})' = M\theta_m$ for a known matrix M. This is useful here because we require the CDF of the bid marginal when applying the copula $\chi(\cdot; \rho_n)$ to compute the joint density (14). The transformation from density to CDF with Bernstein polynomials involves only a linear transformation of parameters and is exact. This is much faster and precise than a transformation obtained by numerical integration.

We estimate using a quasi-maximum-likelihood approach that can permit spatial correlation in the auction-level unobservables U_t. Letting $\theta_b = \{\theta_{b,n}\}_{n=\underline{n},\ldots,\overline{n}}$ and $\rho = \{\rho_n\}_{n=\underline{n},\ldots,\overline{n}}$, the likelihood of the observed log bids $\tilde{b}_{it}$ in a single auction t (taking the entry thresholds as known) can be written

$$\int_{\tau_{n_t-1}(x_t,z_t)}^{\tau_{n_t}(x_t,z_t)} \frac{\tilde{g}_{b^0}\left(\tilde{b}_{1t} - \gamma(x_t,u;\theta_\gamma), \ldots, \tilde{b}_{n_t t} - \gamma(x_t,u;\theta_\gamma); \theta_{b,n_t}, \rho_{n_t}, n_t\right)}{\tau_{n_t}(x_t,z_t) - \tau_{n_t-1}(x_t,z_t)} du.$$

We estimate the parameters $(\theta_\gamma, \theta_b, \rho)$ by maximizing the quasi-log-likelihood

$$\mathcal{L}(\theta_\gamma, \theta_b, \rho) = \sum_t \frac{1}{\tau_{n_t}(x_t,z_t) - \tau_{n_t-1}(x_t,z_t)} \times$$
$$\log\left(\int_{\tau_{n_t-1}(x_t,z_t)}^{\tau_{n_t}(x_t,z_t)} \tilde{g}_{b^0}\left(\tilde{b}_{1t} - \gamma(x_t,u;\theta_\gamma), \ldots, \tilde{b}_{n_t t} - \gamma(x_t,u;\theta_\gamma); \theta_{b,n_t}, \rho_{n_t}, n_t\right) du\right). \tag{16}$$

Given the participation thresholds $\{\tau_{n_t-1}(x_t, z_t)\, \tau_{n_t}(x_t, z_t)\}_{t=1}^{T}$, consistency can be confirmed by adapting the results of White and Wooldridge (1991) for weakly dependent time-series data to the case of weak spatial dependence.[26]

In practice we approximate the integral in (16) by simulation. We use 100 simulation draws in the Monte Carlo simulations and application below.

4.3 Inverting Equilibrium First-Order Conditions

With estimates of the index function γ and joint distribution of homogenized bids, estimation of the relevant auction primitives is straightforward. The equilibrium first-order condition (10) can be written in terms of the distribution of log homogenized bids as

$$w^0(s_{it}; n_t) = \exp(\tilde{b}^0_{it}) \left(1 + \frac{\tilde{G}_{M|B}(\tilde{b}^0_{it}|\tilde{b}^0_{it}, n_t)}{\tilde{g}_{M|B}(\tilde{b}^0_{it}|\tilde{b}^0_{it}, n_t)} \right) \tag{17}$$

where $\tilde{G}_{M|B}(\tilde{b}^0_{it}|\tilde{b}^0_{it}, n_t)$ and $\tilde{g}_{M|B}(\tilde{b}^0_{it}|\tilde{b}^0_{it}, n_t)$ are the CDF and pdf of $\max_{j\neq i} \tilde{B}^0_{jt}$ conditional on the event $\tilde{B}^0_{it} = \tilde{b}^0_{it}$.

We begin by transforming the estimated joint distributions and densities obtained from the sieve-QMLE procedure to the conditional distributions and densities appearing in these first-order conditions (see Appendix for details). Then, by drawing log homogenized bids from their estimated marginal distributions and plugging these into (17), we obtain pseudo-samples of the vectors

$$(w^0(s_{it}; n_t), \ldots, w^0(s_{n_t t}; n_t))$$

for many simulated auctions t and each value of n_t. These vectors can be scaled by the estimated value of $\Gamma(x, u)$ to yield pseudo-samples of the pivotal expected values

[26]In particular, under a standard "expanding domain" asymptotics, White and Wooldridge's uniform consistency result for stationary α-mixing time series data (Corollary 2.6) can be extended using a Bernstein-type inequality for α-mixing random fields on $\mathbb{Z}^2$ (e.g., Yao (2003)).

$(w(s_{it}; n_t, x, u), \ldots, w(s_{n_t t}; n_t, x, u))$ at any x, u, although for some purposes this will not be necessary. The pseudo-samples can then be used to construct estimates of the primitives or counterfactual quantities of interest. For example, in the case of an affiliated private values model, the simulated draws of $(w(S_{1t}; n_t, x, u), \ldots, w(S_{n_t t}; n_t, x, u))$ will correspond to draws of the valuations $(V_{1t}, \ldots, V_{nt})$, and standard estimators of the joint distribution or joint density can then be applied, e.g., following Li, Perrigne, and Vuong (2002). For our test for common values below, we estimate the marginal distributions of the homogenized pivotal expected values $w^0(S_{it}; n)$ for each n by constructing empirical distribution functions from each pseudo-sample.

In Appendix C, we present the results of a Monte Carlo study, where we find that the estimation procedure performs well, even in moderate sample sizes.

5 OCS Auctions

We apply our method to study auctions of oil and gas leases in the U.S. Outer Continental Shelf (OCS) from 1954 to 1983. The auctioneer was Mineral Management Service, an agency of the Interior Department. Our goal is to examine these data through the lens of our model in order to assess several fundamental features of the market such as the market power of bidders, whether the data support the hypothesis of equilibrium bidding, the effects of competition on bids and revenues, the significance of unobserved heterogeneity and of correlation among bidders' private information, and whether there is empirical support for the presence of common values. The last of these is of particular interest from both positive and normative perspectives. Institutional features of these auctions may suggest a common values model, and these auctions are often cited as examples of a common values setting. However, Li, Perrigne, and Vuong (2000) have pointed out the need for formal testing, while HPP have explained that such testing is difficult due to the likely presence of unobserved heterogeneity that affects both expected lease value and the level of bidder participation.

Extensive discussion of the OCS auctions can be found in, e.g., Hendricks and Porter (1988), Hendricks, Porter, and Spady (1989), and HPP, and we refer readers to that prior work for a more complete description. Briefly, however, auctions were held for the right to lease a specified tract for exploration and production of oil, gas, and other minerals. Production on a tract was subject to royalty payments from the leaseholder at a pre-specified rate (usually 1/6). Bids at an auction were offers of an additional up-front "bonus" payment for the right to become the leaseholder. At a given "sale," many tracts were offered for lease simultaneously through separate first-price sealed bid auctions. No exploratory drilling was permitted prior to the auction, although in some cases exploration and production would have already occurred on adjacent "neighbor" tracts and would be publicly observable. Bids would also reflect information obtained through evaluation of seismic data. Although the collection of seismic data was often funded jointly, each firm relied on its own experts for modeling and analysis of the data. Differences in expert assessments may be one source of heterogeneity in bidder beliefs about the value of a given tract. These features lead us to treat bidder entry as a decision to acquire a costly signal about the value of the tract (recall Example 1 and Appendix A).

The MMS typically announced a small minimum acceptable bid—usually either \$15 or \$25 per acre—but also retained the option to reject all bids when it deemed bids to be noncompetitive. Such rejections typically involved tracts attracting only one bid. We initially treat both the announced minimum bid and the MMS bid rejection policy as nonbinding, as in Li, Perrigne, and Vuong (2000).[27] However, we

[27] The empirical distribution of bids has a very thin left tail, with more than 99% of all bids above twice the announced minimum bid. This suggests that if the announced minimum bid sometimes binds—i.e., is such that any bidder's signal indicates an expected value of winning the lease below the minimum bid—it does so only very rarely.

also consider a variation of the model in which, following Hendricks, Porter, and Spady (1989), the bid rejection policy is modeled with a stochastic secret reserve price that responds to tract characteristics and the number of bids received. Limited forms of joint bidding were permitted in these auctions. Following the prior literature, we treat each bid as coming from a generic "bidder," which might be solo firm or a bidding consortium. Hendricks and Porter (1992) and Hendricks, Porter, and Tan (2008) examine empirical and theoretical aspects of joint bidding in these auctions. Our measure of the number of neighbor firms accounts for the presence of joint bidding by linking together firms that have bid together previously, following the criteria developed by Aradillas-Lopez, Haile, Hendricks, and Porter (2017).

We have data on all auctions attracting at least one bidder.[28] We analyze this full sample, although we will evaluate equilibrium bids themselves only among auctions attracting at least two bids.[29] We do not separate wildcat, development, and drainage tracts; instead, we account for the presence/absence of active neighbor leases/production and allow for asymmetries in neighbor/non-neighbor costs of signal acquisition in a way that generalizes the structure considered in Hendricks and Porter (1988) (see Example 1 and Appendix A).

Typically a given tract will have eight neighbors, only some (or none) of which will be "active" (under lease). Recall that we use the term "neighbor firm" to refer to the owner of an active neighbor lease. We focus on five auction covariates to summarize observable heterogeneity across tracts: the number of active neighbor leases, the number of firms that bid for neighbor leases, whether the tract was offered previously (attracting no bidders or being relinquished by a prior leaseholder), the number of neighbor tracts already drilled, and the number of neighbor tracts with

[28]Appendix A demonstrates that with the model of entry and bidding sketched in Example 1, selection on attraction of at least one bid does not introduce a problem for our approach.

[29]In section 6.4 we incorporate one-bidder auctions as well.

proven reserves ("hits") at the time of the sale. We present a summary of the auction characteristics by number of bidders in Table 1.

Table 1: Summary Statistics

		$\bar{x}$					$\bar{z}$
n	$\boldsymbol{T_n}$	**# neighbor leases**	**# firms bid neighbors**	**Re-offered tract**	**# neigh. drilled**	**# neigh. hits**	**# neigh. firms**
1	1380	1.8355	2.7167	0.4123	2.0051	0.7783	0.992
2	709	1.8505	2.8336	0.3173	2.0649	0.8124	1.0804
3	402	1.7139	2.602	0.291	1.9552	0.7836	0.9801
4	306	1.4216	2.1373	0.2092	1.6503	0.7614	0.8431
5	216	1.5231	2.4444	0.2037	1.7593	0.7222	0.8611
6	153	1.2549	1.8954	0.1438	1.4706	0.6013	0.8039
7	124	0.8145	1.3306	0.0645	0.9677	0.3952	0.5323
8	92	0.9783	1.6957	0.1304	0.9891	0.4674	0.6413
9	66	1.2424	2.2879	0.0909	1.4697	0.6515	0.9242
10	61	0.7541	1.1475	0.0984	0.8033	0.2623	0.5082
11	47	0.9787	1.8936	0.1064	1.2128	0.3191	0.617
12	35	0.8	1.1429	0.0857	0.8571	0.1714	0.3714
13	27	0.2963	0.6667	0.0741	0.4815	0.1852	0.2963
14	17	0.2941	1.1765	0	0.3529	0.1765	0.2353
15	9	0.2222	1.2222	0	0.4444	0.2222	0.3333
16	6	0	0	0	0	0	0
17	1	1	2	0	1	0	1
18	1	0	0	0	0	0	0

Notes: Average values of covariates and instruments by number of bidders. Column T_n reports the number of auctions.

Following Example 1 (see Appendix A), we consider one instrument for participation: the number of neighbor firms. This variable is likely to affect bidder entry because ownership of a neighbor tract is likely to reduce the cost of assessing the value of the current tract. As discussed in Appendix A, when we condition on the number of neighbor tracts and the set of firms that previously bid on for those tracts, variation in the number of neighbor firms is determined entirely by the realizations of bidder signals at prior auctions, and therefore independent of U_t under our maintained assumption (Assumption 4) that latent correlation across tracts arises only through the auction-level unobservables.

5.1 Entry Model Estimates: Baseline Specification

Table 2 shows our estimates of the entry model parameters. As discussed in section 2 (see also section 5.2.3 and Appendix A), the function η characterizes the effect of Z_t on entry but may not reveal the causal effects of X_t on entry. Thus we must interpret the coefficients here with caution.[30] We see, however, that the coefficient on Z_t is positive and significant, supporting its value in providing exogenous variation in bidder entry.

Table 2: Entry Model Estimates

		Est.	**Std. err.**
# active neighbor leases	X	-0.226	0.040
# firms that bid for neighbors		-0.071	0.032
re-offered tract		-0.266	0.021
# neighbors drilled		-0.020	0.043
# neighbor hits		0.136	0.031
Time controls		Sale year dummies	
# neighbor firms	Z	0.122	0.039

Notes: Estimates for the entry model by ordered probit. Dependent variable is the number of bidders.

We can exploit our identification analysis to provide a clearer examination of the

[30] If we give the estimated coefficients on X_t their usual interpretation, only the coefficient on the number of firms bidding previously on neighbor tracts has a statistically significant estimate with counterintuitive sign. Note that, conditional on the number of neighbor firms, a larger number of neighbor leases indicates lower participation in auctions of neighbor tracts, so the negative coefficient may be expected in that case.

"relevance" of our instrument. Assumption 8 was the most important condition we placed on the variation in entry induced by the changes in the instrument.[31] Figures 1a and 1b provide some evidence on the satisfaction of this condition. There we let y denote the sample values of the estimated index $\hat{\lambda}_x' x_t$ and plot the lower boundaries of the level sets of N_t in y–z space. In the notation of Assumption 8, we could let $n^* = 3$ (other values would also work). We see in Figure 1a that for every $n \leq n^*$, we can find y, z and $\hat{z}$ such that $\underline{\hat{n}}(y, z) = n$ and $\underline{\hat{n}}(y, \hat{z}) = n + 1$. Put more simply, for each $n = 2, 3, 4$ one can find a vertical line in the figure that would cross the boundary associated with $\underline{n}(y, z) = n$. Similarly, in Figure 1b we see that for $n \geq n^*$, there exists a vertical line crossing the boundary for $\overline{n}(y, z) = n$.[32]

5.2 Bidding Model Estimates: Baseline Specification

From the entry model estimates, we obtain estimates of the bounds $\tau_{n_t-1}(x_t, z_t)$ and $\tau_{n_t}(x_t, z_t)$ on the unobserved heterogeneity u_t for each auction t. We use these bounds in the estimation of the bidding model. We specify the index function as

$$\gamma(X_t, U_t) = X_t \gamma_x + \gamma_u U_t.$$

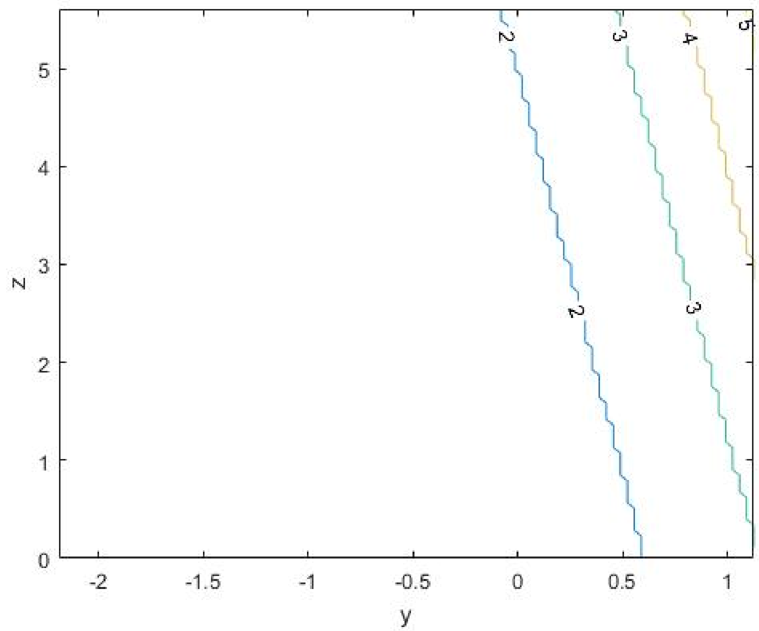
In general we estimate separate distributions of homogenized bids for each value of n_t. However, for large values of n_t we observe relatively few auctions (recall Table 1), leading us to use a specification in which all auctions with $n_t \geq 11$ share the same marginal distribution of homogenized bids and the same copula correlation parameter.

[31] Assumption 9 is a second such requirement. It is more difficult to check, and our identification argument makes clear that the primary role of that condition (like the continuous variation in Z_t required by Assumption 10a) is to fill in gaps between points at which the index function is identified without this condition. Because our instrument is discrete (and more generally, because our sample is finite), we are likely to rely to some degree on functional form for this interpolation.

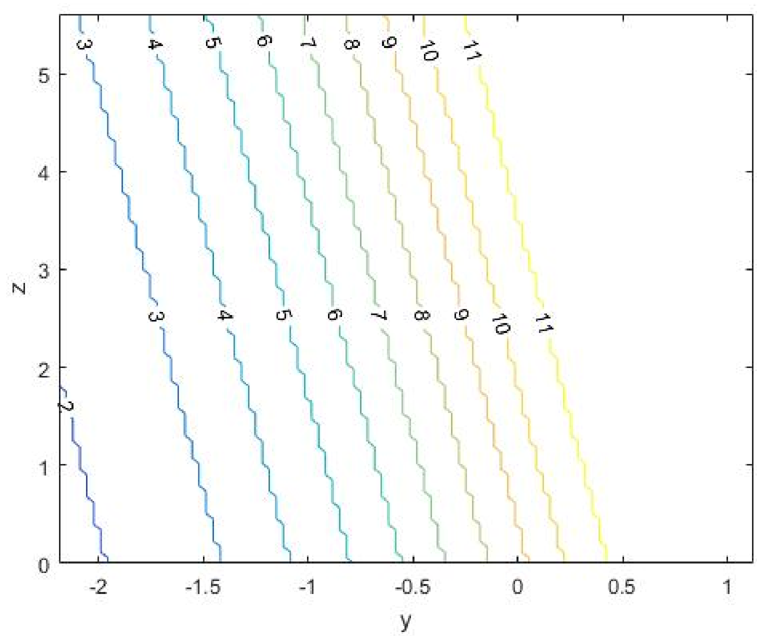
[32] Note that when the entry model is estimated by ordered probit, $\hat{\eta}(y, z, 0) = 1$ and $\hat{\eta}(y, z, 1) = 11$ for all y, z. Because this is only an artifact of the parametric approximation, we approximate $\underline{n}(y, z)$ and $\bar{n}(y, z)$ with $\hat{\eta}(y, z, 0.1)$ and $\hat{\eta}(y, z, 0.9)$, respectively.

Figure 1: Variation Induced by the Instrument

(a) $\underline{n}(y,z) = \hat{\eta}(y,z,0.1)$



(b) $\bar{n}(y,z) = \hat{\eta}(y,z,0.9)$



Notes: Contour curves for predicted participation $\hat{\eta}(y,z,u)$ using the estimates in Table 2, with y representing sample values of the estimated index $\hat{\lambda}_x' x$.

As described in Section 4, estimation of the distribution of homogenized bids follows a semiparametric quasi-maximum likelihood approach. Dependence among the homogenized bids is governed by the Gaussian copula, while marginals are flexibly estimated using a Bernstein polynomial sieve.

5.2.1 The Index Function

In Table 3, we show the estimates of γ_x and γ_u along with bootstrap standard errors.[33] The estimates also suggest a strong effect of the unobserved heterogeneity on bids. Because U_t is normalized to have a uniform distribution, we can see that shifting the unobserved heterogeneity from its 25th percentile its 75th percentile is estimated to drive up bids by roughly 37%. We again caution that estimated coefficients on X_t do not have the usual interpretation unless we add an assumption that the underlying unobserved heterogeneity is independent of X_t. With that caveat, the statistically significant coefficients on X_t have the expected signs: re-offered tracts appear to be worth less while tracts appear to be more valuable when neighbor tracts have attracted more bidders and when those tracts have been drilled.[34]

5.2.2 Copula Correlation

We will not report estimates of the Bernstein polynomial parameters. However, Table 4 shows our estimates of the Gaussian copula correlation parameter ρ_n for each estimation bin. These imply significant positive correlation among the homogenized bids. Because the correlation among homogenized bids reflects indicates correlation

[33] We use a standard nonparametric bootstrap procedure in which we resample auctions and re-estimate the entire entry and bidding model on each bootstrap sample.

[34] Note that here, because we are not controlling for the number of neighbor firms, a larger number of neighbor tracts is naturally associated with higher value.

Table 3: Index Function Estimates

		Est.	Std. err.
# active neighbor leases	X	0.0339	0.0165
# firms that bid for neighbors		0.0186	0.0054
re-offered tract		-0.4022	0.0531
# neighbors drilled		0.0498	0.0157
# neighbor hits		-0.0168	0.0187
Time controls		Sale year dummies	
Unobserved heterogeneity	U	0.7395	0.2732

among bidders' signals,[35] this finding is of some importance on its own. Common knowledge unobservables and correlated private information are two distinct phenomena with different implications for behavior and policy. Typically only one of these two sources of correlation between bids has been permitted in applications. The evidence in Tables 2–4 indicates that both features are present and significant in our data.

Table 4: Estimated Gaussian Copula Correlation

n	Est.	Std. err.
2	0.1173	0.0387
3	0.1433	0.0308
4	0.1639	0.0289
5	0.1677	0.0276
6	0.1274	0.0275
7	0.1393	0.0316
8	0.1455	0.0350
9	0.2667	0.0474
10	0.1646	0.0464
11 - 18	0.1073	0.0161

[35] Recall that the homogenized bids are strictly increasing (but typically nonlinear) functions of signals.

5.2.3 Decomposition of Correlation and Variance

The correlation in private information documented in Table 4 is just one reason bids (or valuations) are correlated within an auction; auction observables and unobservables also play a role. These auction characteristics also contribute to the overall variance of the observed bids. We can use our estimates to describe the contributions of each factor.

Table 5 shows a decomposition of within-auction pairwise correlation and overall variance of the log bids.[36] The figures presented here reflect calculations using simulated draws from the estimated model. Figures in the first column, labeled "$\log B^0_{it}$," are for the homogenized log bids. Here the pairwise correlation reflects the correlation among signals, the nonlinearity of the bidding strategy (and log transformation), and the fact that, all else equal, bid levels vary with the number of competitors in the auction. Similarly, the variance in this column reflects the variability in bidders' assessments of tract values, as well as variation in bidding strategies across auctions with different numbers of entrants.

A natural way to characterize the contributions of unobservables is to examine the correlation and variance arising from variation in $\gamma(x, U_t)$ at a representative value of x. With our linear specification of $\gamma(X_t, U_t)$, this variation is identical for all x. Thus, in the second column, labeled "$\log B^0_{it} + \gamma_u U_t$," we add the contribution of auction-level unobservables.

Quantifying the contribution of the observables X_t can be more problematic, at least when we allow our model to represent an environment in which the underlying unobservables may be correlated with X_t (see Example 1 and Appendix A). In that case, variation with X_t in the conditional mean $E_{U_t}[\gamma(X_t, U_t)|X_t]$ or in the uth quantile $\gamma(X_t, u)$ would correctly describe variation in bids associated with variation in

[36]By "within-auction pairwise correlation" we refer to the Pearson correlation coefficient for pairs within the same auction.

Table 5: Decomposition of Log Bid Correlation and Variance

$\log B^0_{it}$	$\log B^0_{it} + \gamma_u U_t$	$\log B^0_{it} + \gamma_u U_t + X_t'\gamma_x$	$\log B_{it}$
Within-Auction Pairwise Correlation			
0.188	0.233	0.251	0.431
Variance			
1.725	1.778	1.821	2.398

X_t, but need not represent the variation *caused* by variation in X_t alone—i.e., that arising when the actual unobservables (E_t in Example 1) are held fixed.[37] However, once we know the contributions of the log homogenized bids and the unobservables, we know that all remaining correlation/variance in the log bids arises from the auction observables. Again exploiting our linear model, we split the contributions of observables into those of the year fixed effects and those of the auction-level covariates. The third column of the table, labeled "$\log B^0_{it} + \gamma_u U_t + X_t'\gamma_x$," adds the variation due to auction-level covariates, while the final column, labeled "$\log B_{it}$," adds the contribution of the year fixed effects to yield the total correlation and variance of the observed bids.

Given the wide time span of our data set and the substantial variation across time in underlying market conditions, it is not surprising that the fixed effects capture a substantial portion of the variation.[38] Perhaps more interesting is a comparison of the contributions of the auction-level covariates and the auction-level unobservables. The latter appears to be at least as important as the former. In the case of the within-

[37] Of course, under an additional (and usual) independence assumption, the validity of this calculation would be unambiguous.

[38] In section 6 we will examine a specification that omits these fixed effects.

auction correlation, the contribution of the unobservables is nearly three times as large as that of the observed covariates. This is particularly noteworthy because we selected covariates X_t from an unusually rich set of observables based on explanatory power. This may suggest that unobserved heterogeneity is likely to be even more important in other applications with a more limited set of covariates.

Together, the evidence here indicates that observed heterogeneity, unobserved heterogeneity, and correlated private information all contribute significantly to the variation and dependence observed in the equilibrium bids.

5.3 Test For Common Values: Baseline Specification

Following HHS, we test the null hypothesis of private values against the alternative of common values by examining how the marginal distribution of homogenized pivotal expected values $w^0(S_{it}; N_t)$ varies with N_t. Under the null, the distribution of these pivotal expected values is the same for all N_t. Under the common values alternative, $w^0(S_{it}; N_t)$ is stochastically decreasing (in terms of FOSD) in N_t. This is due to the presence of the winner's curse, which is present in (and only in) common values auctions, and which becomes more severe as N_t increases. A comparison of these distribution functions also permits a specification test of the model: under the maintained assumptions of the model, $w^0(S_{it}; N_t)$ must be weakly stochastically decreasing in N_t. Violations of this requirement indicate a rejection of at least one of our maintained hypotheses.[39]

As discussed in section 4, we perform the test by first drawing homogenized bids from their estimated distributions and plugging these into the first-order conditions to produce pseudo-samples of pivotal expected values for different values of N_t. We then

[39] Another testable implication of the maintained hypotheses is that the inverse homogenized bid functions—i.e., the right-hand-side of the first order conditions (17)—be strictly increasing as functions of the log homogenized bid (see Guerre, Perrigne, and Vuong (2000)). We find no violation of this requirement (even before allowing for sampling error) among any of the specifications reported.

use the pseudo-samples to construct empirical distribution functions. We construct these distributions for pooled ranges ("low," "medium," and "high") for N_t. In Figure 2a, we show the result for our baseline specification discussed above. The plots suggest the stochastic ordering implied by the common values model. The largest shift in the distributions occurs when moving from low to medium levels of entry. As discussed by HHS, this is natural, as the change in the severity of the winner's curse implied by adding a bidder is typically largest when the number of bidders is low.

Figure 2b, shows the results when we used the estimates obtained ignoring unobserved heterogeneity. Here, there is no clear stochastic ordering. This finding is not a necessary implication of ignoring unobserved heterogeneity, since bidders' first-order conditions are mis-specified when unobserved heterogeneity is present but ignored. However, because ignoring unobserved heterogeneity implies ignoring endogenous bidder entry, this finding is very intuitive. In particular, if more valuable tracts attract more bidder entry, the implied correlation between entry and tract value works against the causal effects of entry on the winner's curse. Thus, it may be natural to expect that the stochastic dominance expected under common values would diminish (or even reverse) when one ignores the problem of unobserved heterogeneity and endogenous entry. Here we see that ignoring unobserved heterogeneity obscures the presence of common values.

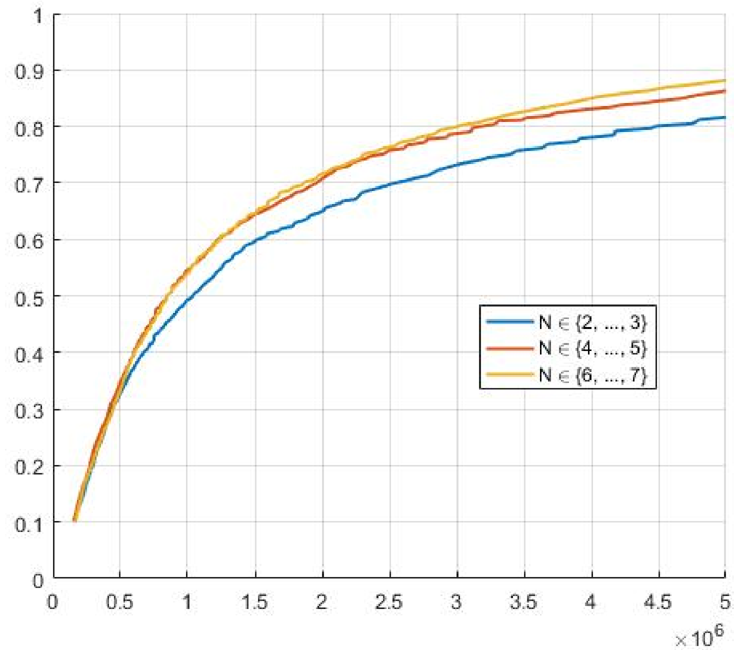
In Table 6 we present the results of the formal tests. We compare pairs of distributions using the one-sided Cramer-Von Mises type statistic

$$CVM = \int_{-\infty}^{\infty} \left[\hat{F}_w(w; \mathbb{N}_2) - \hat{F}_w(w; \mathbb{N}_1) \right]_+^2 dw, \tag{18}$$

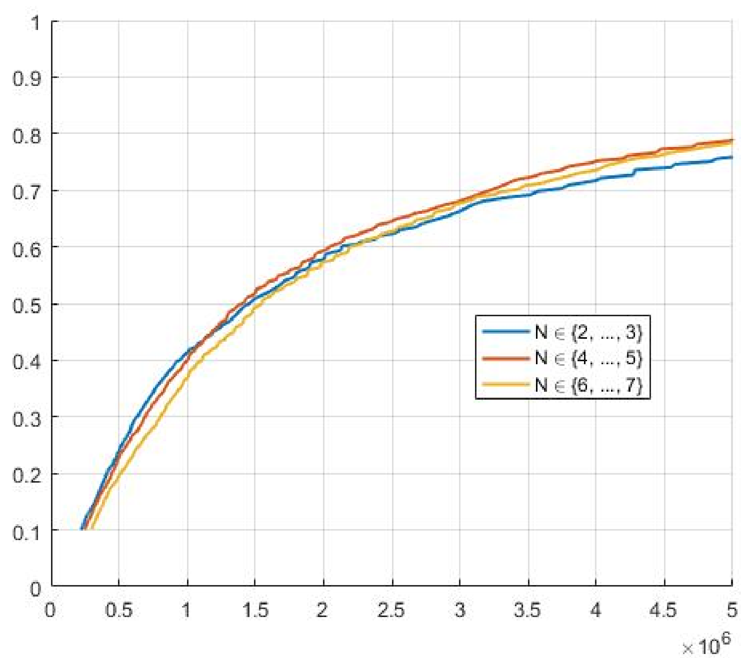
where $\hat{F}_w(w; \mathbb{N})$ is the estimated distribution of $w^0(S_{it}, S_{it}; N_t)$ conditional on N_t lying in a range of values defined by $\mathbb{N}$. In addition to the pairwise tests, we construct a single test statistic for the full sample based on the maximum across all adjacent pairs

Figure 2: Test for Common Values, Baseline Specification

(a) With Unobserved Heterogeneity



(b) No Unobserved Heterogeneity



of sets:

$$CVM^{max} = \max_j \int_{-\infty}^{\infty} \left[\hat{F}_w(w; \mathbb{N}_{j+1}) - \hat{F}_w(w; \mathbb{N}_j)\right]_+^2 dw \tag{19}$$

We construct critical values (or p-values) for the test statistics using the (re-centered) nonparametric bootstrap.

The formal results align with what we saw in Figure 2. First consider the results when we account for the presence of unobserved heterogeneity and endogenous bidder entry. We reject the private values hypothesis in favor of common values at standard confidence levels when comparing low and medium values of N_t, or when considering all three distribution functions jointly. When comparing medium and high values of N_t the differences between the two distributions are small and not statistically significant. When we ignore unobserved heterogeneity (and therefore eliminate the possibility of bidder entry that is correlated with tract value), however, we find no evidence of the stochastic ordering implied by common values.

Table 6: Test for Common Values

Pairwise Tests

		Unobs het		No unobs het	
$\mathbb{N}_1$	$\mathbb{N}_2$	Test Stat	p-value	Test Stat	p-value
$\{2,3\}$	$\{4,5\}$	0.0053	0.0224	0.0007	0.2281
$\{4,5\}$	$\{6,7\}$	0.0005	0.2220	0.0001	0.4785

Maximum Test

Unobs het		No unobs het	
Test Stat	p-value	Test Stat	p-value
0.0053	0.0351	0.0007	0.4283

6 Alternative Specifications

In this section we consider several alternative specifications, focusing on the test for common values.

6.1 Alternatives to Year Fixed Effects

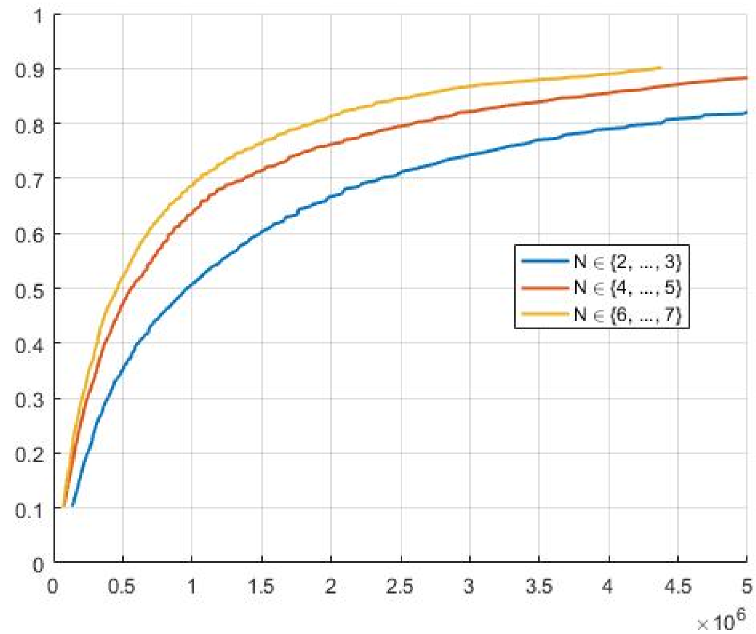
Our baseline specification included year fixed effects. This provides flexible control for a number of common-knowledge time-varying factors such as macro shocks, variation in oil and gas prices, changes in industry structure, regulatory changes, etc., which vary substantially over the three decades of our sample. We saw the importance of this temporal variation in Table 5. However, there may be other reasonable treatments of time-varying factors. On one hand, because year fixed effects will absorb some of the variation in our instrument, we might consider a model that drops the fixed effects. Such an exercise may also suggest the importance of accounting for unobserved heterogeneity in settings where sample sizes do not permit estimation with year fixed effects. On the other hand, we might want to allow not only the intercept but also the slopes of the index function $\gamma(X_t, U_t)$ to vary over time.

We first consider a specification that drops the year fixed effects. Tables in Appendix D shows the parameter estimates for the entry and bidding models with and without unobserved heterogeneity. We show the estimated distributions of pivotal expected values used for testing in Figure 3. When we account for unobserved heterogeneity and endogenous entry we see a somewhat stronger stochastic ordering than in our baseline model. When we ignore unobserved heterogeneity the plots reveal no evidence of common values but suggest the possibility of mis-speification.

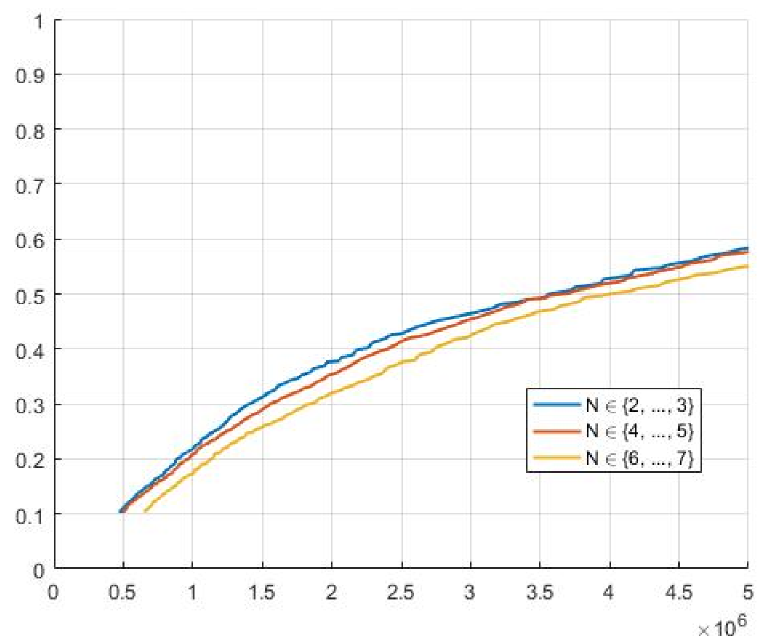
Next we consider a specification that permits time-varying coefficients. We estimate separate coefficients on each covariate for each decade, both in the participation model and in the bidding model. We also replace year fixed effects with separate lin-

Figure 3: Dropping Time Fixed Effects

(a) With Unobserved Heterogeneity



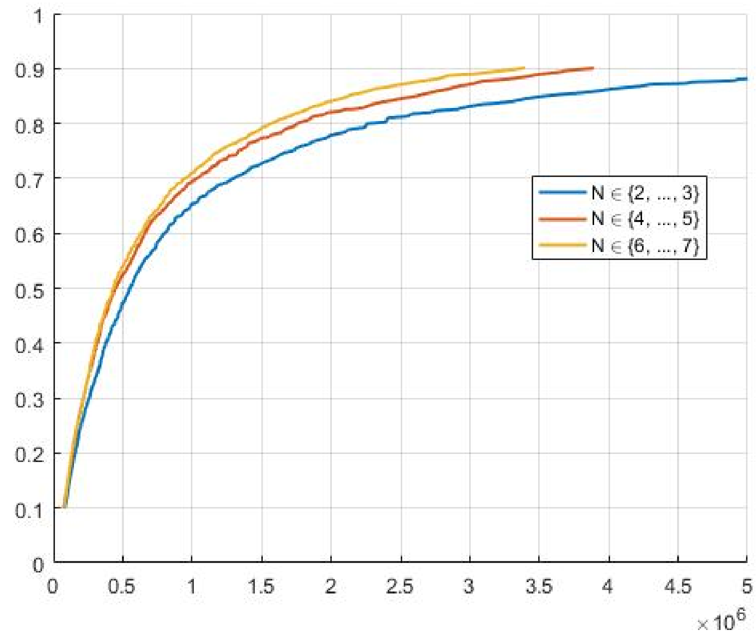
(b) No Unobserved Heterogeneity



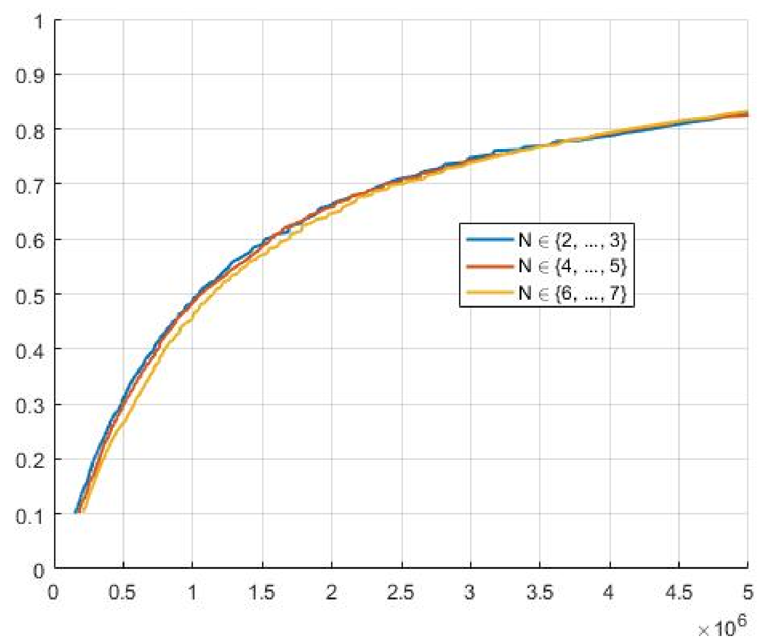
ear time trends for each decade. Parameter estimates are reported in Tables 13 and 14 of Appendix D. Figure 4 shows the distributions of pivotal expected values. Here the results are very similar to those in our baseline specification. We see a clear stochastic ordering consistent with common values when we account for unobserved heterogeneity and endogenous participation, but not such ordering when we ignore unobserved heterogeneity

Figure 4: Time-Varying Coefficients

(a) With unobserved heterogeneity



(b) No unobserved heterogeneity



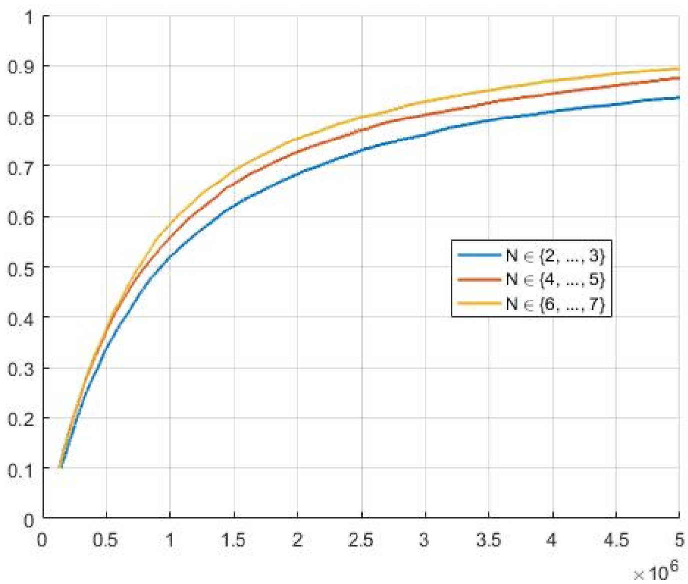
6.2 Semi-nonparametric Entry Model

In all results presented so far, the entry model was estimated as an ordered probit. In the notation of Section 4, $H(\cdot)$ was the standard normal distribution. Recall from section 4 that the distribution function H could be chosen arbitrarily if we had specified each $\alpha_n(x, z)$ nonparametrically. But because we use a linear specification of each $\alpha_n(x, z)$, the choice of H can matter. Here we examine the robustness of our results to choices other than the standard normal. Following the discussion in Section 4, we consider a version of the entry model in which $H(\cdot)$ is estimated flexibly using Hermite polynomials.[40]

In Figure 5 we plot the CDFs of pivotal expected values obtained under the semi-nonparametric entry model. These distributions exhibit patterns very similar to those from the original model, with clear stochastic ordering consistent with the common values hypothesis. We conclude from this that our results are robust to the alternative specification of the entry model.

[40] We set $K = 3$ as Hermite flexibility (see equation 12).

Figure 5: SNP Entry Model



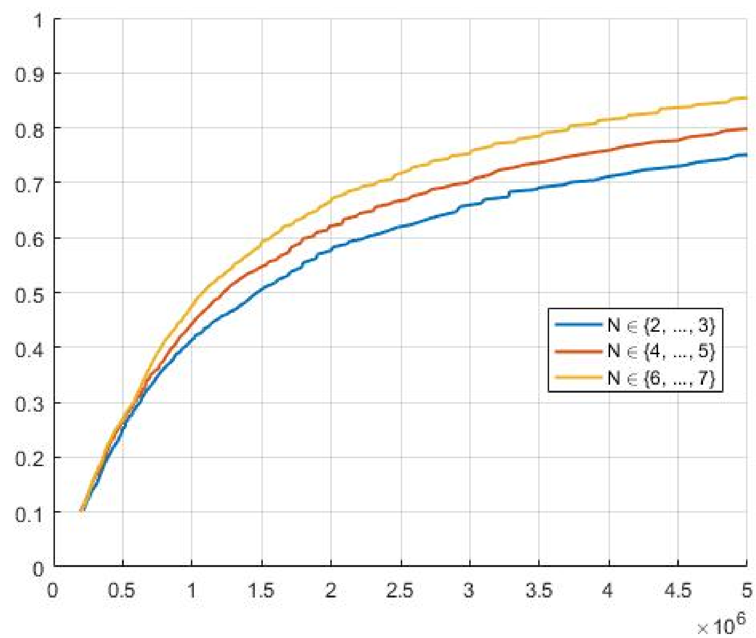
6.3 Non-Isolated Tracts

We have also estimated the baseline model restricting the sample to "non-isolated" tracts—those with at least one active neighbor. We do this for two reasons. First, isolated tracts might differ from tracts with active neighbors in ways not fully captured by our covariates. Second, our instrument has no variation among isolated tracts. The latter does not alone imply that we should restrict attention to non-isolated tracts; if the model is correctly specified, including all tracts in the estimation will only aid efficiency. But together these factors suggest examining our test for common values when we restrict attention to non-isolated tracts.

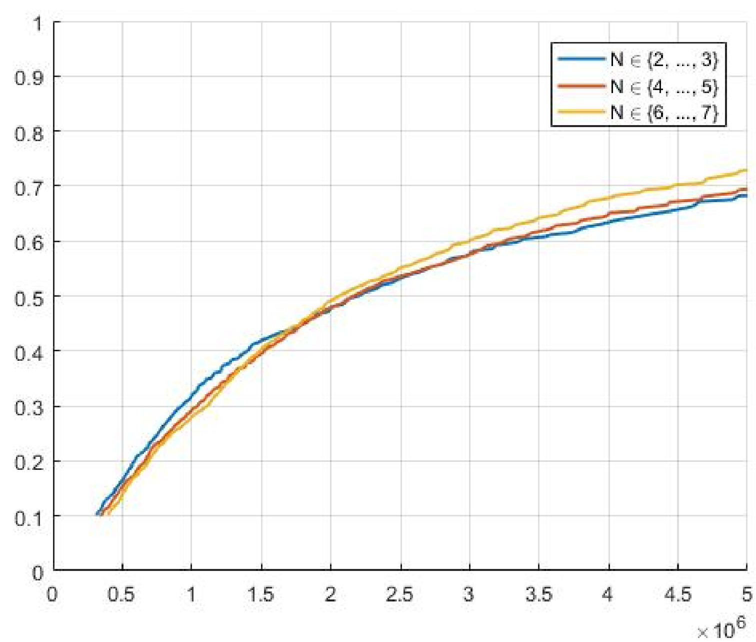
Figure 6 shows the estimated distribution functions used for the test. The results are quite similar to those in the baseline specification, although the stochastic ordering observed when we permit unobserved heterogeneity is somewhat stronger.

Figure 6: Non-Isolated Tracts

(a) With unobserved heterogeneity



(b) No unobserved heterogeneity



6.4 Random Reserve Price

As discussed in section 5, the MMS occasionally exercised its right to reject all bids, typically when the number of bidders or the level of the bids was viewed as low. The first two columns of Table 8 below show the fraction of tracts that were sold to the highest bidder. Most rejections involved one-bidder auctions, where the winning bid was rejected more than 20 percent of the time. For two- and three-bidder auctions, rejections occurred less than 10 percent of the time. For auctions with more than four bidders, virtually all tracts were sold to the highest bidder. But the potential for rejection of the high bid could have significantly affected bidding behavior, particularly in auctions with low levels of entry.

A natural way to model the MMS bid rejection policy is to assume MMS uses a secret random (from the bidders' perspective) reserve price for each auction (see, e.g., Hendricks, Porter, and Spady (1989) and Hendricks, Porter, and Wilson (1994)). Here we revisit the data allowing for a random reserve price R_t at each auction t. We let this reserve price vary with the number of bidders and with the auction characteristics (X_t, U_t). The dependence of the reserve price on (X_t, U_t) is assumed to mirror the effect these have on tract values. In particular, we assume

$$R_t = R_t^0 \times \Gamma(X_t, U_t)$$

where R_t^0 is a random variable whose distribution may vary with N_t but which is independent of (X_t, U_t, S_t, V_t^0) conditional on N_t. In this formulation, a homogenized winning bid M_t^0 is accepted if, and only if, $M_t^0 \geq R_t^0$.

Only the final step of the estimation procedure—the inversion of bidder first-order conditions, following the sieve QMLE estimation of the homogenized bid distributions—is affected by introducing a random reserve price. Let $H_R(\cdot|N_t;\theta_R)$ be the distribution function of $R_t^0|N_t$, which we specify parametrically. Taking the estimates of the participation thresholds and bid homogenization parameters θ_γ as given, we can use the observed bid acceptance decisions

$$Y_{it} = 1\{M_{it} \geq R_t\}$$

to estimate θ_R using the quasi-likelihood function

$$\mathcal{L}(\theta_R; \hat{\gamma}) = \prod_{t=1}^{T} E\Big[H_R\left(m_t - \gamma(x_t, u_t; \hat{\theta}_\gamma)|n_t; \theta_R\right)^{y_t} \times \\ \left(1 - H_R\left(m_t - \gamma(x_t, u_t; \hat{\theta}_\gamma) \mid n_t; \theta_R\right)\right)^{1-y_t} \Big| \, x_t, z_t, n_t\Big],$$

or, equivalently,

$$\mathcal{L}(\theta_R; \hat{\gamma}) = \prod_{t=1}^{T} \frac{1}{\tau_{n_t}(x_t, z_t) - \tau_{n_t-1}(x_t, z_t)} \int_{\tau_{n_t-1}(x_t,z_t)}^{\tau_{n_t}(x_t,z_t)} H_R\left(m_t - \gamma(x_t, u; \hat{\theta}_\gamma)|n_t; \theta_R\right)^{y_t} \times \\ \left(1 - H_R\left(m_t - \gamma(x_t, u; \hat{\theta}_\gamma)|n_t; \theta_R\right)\right)^{1-y_t} du. \quad (20)$$

Table 7 shows the parameter estimates we obtain, assuming that the homogenized random reserve price is drawn from a log-normal distribution. Motivated by the first two columns of Table 8, we estimate separate distributions of R_t^0 for $N_t = 1$, $N_t \in \{2, 3\}$ and $N_t \in \{4, \ldots, 18\}$. The third column of Table 8 shows the fitted bid acceptance rates for each value of N_t for comparison to the actual rates.

Under the null hypothesis of private values, bidder's pivotal expected values can still be identified using a revised first-condition

$$w^0(s_{it}; n_t) = b_{it}^0 + \frac{G_{M^0|B^0}(b_{it}^0; b_{it}^0, n_t) H_R(b_{it}^0; n_t)}{G_{M^0|B^0}(b_{it}^0; b_{it}^0, n_t) h_R(b_{it}^0; n_t) + g_{M^0|B^0}(b_{it}^0; b_{it}^0, n_t) H_R(b_{it}^0; n_t)}, \quad (21)$$

or its analog in terms of log homogenized bids (cf. (17).

Accounting for the bid rejection possibility offers several advantages beyond mere

Table 7: Random Reserve Price: Model Estimates

$$\log(R_0)|N = n \sim N(\mu_n, \sigma_n)$$

n	$\hat{\mu}_n$	$\hat{\sigma}_n$
1	10.10	4.08
2 - 3	9.95	3.04
4 - 18	10.44	1.88

Table 8: Actual and Fitted Sale Frequencies

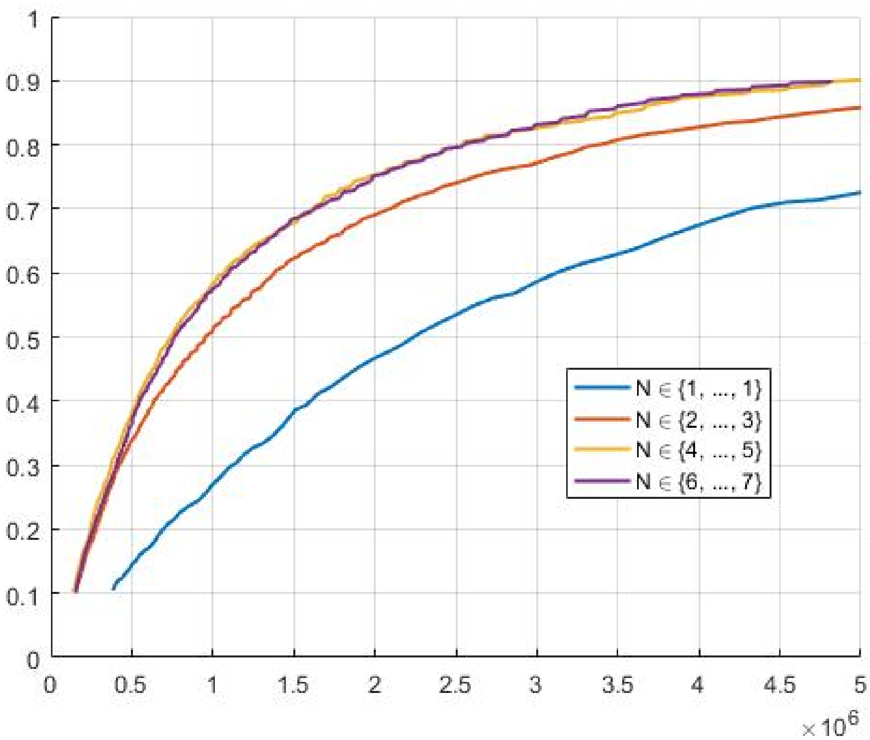
n	Sold (actual)	Sold (predicted)
1	0.770	0.771
2	0.904	0.908
3	0.943	0.935
4	0.974	0.984
5	0.995	0.989
6	1.000	0.993
7	1.000	0.993
8	0.989	0.996
9	1.000	0.997
10	1.000	0.998
11	1.000	0.998
12	1.000	0.999
13	1.000	0.999
14	1.000	0.999
15	1.000	0.999
16	1.000	1.000
17	1.000	1.000
18	1.000	1.000

realism. Absent such a possibility, the theory predicts that bids in one-bidder auctions would equal the publicly announced reserve price. Thus in one-bidder auctions under our baseline specification we can neither rationalize bids above the publicly announced minimum nor invert bidder first-order conditions. Incorporating the random reserve price enables both. And incorporating one-bidder auctions is particularly advantageous in a test for common values because the change in severity of the winner's curse associated with adding a bidder is likely to be largest when starting from a one-bidder auction (where there is no winner's curse). Thus, by including one-bidder auctions—of which there are also many in the sample—the presence of common values may become especially easy to detect.

Using our estimates of H_R, h_R, G, and g, Figure 7 shows the implied marginal distributions of $w^0(S_{it}, ; n_t)$. As expected, under the common values alternative, we see a particularly large stochastic shift when moving from one-bidder auctions to two- and three-bidder auctions. The remaining distribution functions follow the pattern seen in the prior specifications.

This evidence is consistent with the presence of common values. However, there is another more subtle reason that the one-bidder auctions are particularly helpful here. When we permit a random reserve price, the model makes a less sharp prediction about the distribution of pivotal expected values under the common values alternative. In particular, when $n_t \geq 2$, an increase in the number of bidders need not imply a first-order stochastic shift. One can easily confirm that, under the alternative hypothesis of common values, what is recovered from the first-order condition (21) is

Figure 7: Random Reserve Price



not $w^0(s_{it}, s_{it}; n_t)$ but a weighted average[41]

$$\varphi(s_{it}, n_t)\underline{w}^0(s_{it}; n_t) + [1 - \varphi(s_{it}, n_t)]\, w^0(s_{it}; n_t) \tag{22}$$

where

$$\underline{w}^0(s_{it}; n_t) = E\left[V_t^0 \middle| S_{it} = s_{it}, \max_{j \neq i} S_{jt} \leq s_{it}, N_t = n_t\right]. \tag{23}$$

Although both $\underline{w}^0(s_{it}; n_t)$ and $w^0(s_{it}; n_t)$ are decreasing in n_t in a common values model, the weights $\varphi(s_{it}, n_t)$ also vary with n_t, leaving an ambiguous prediction regarding how the distribution of the weighted average varies with n_t.

However, a comparison of one-bidder auctions to others is free from this problem. When $n_t = 1$ there is no winner's curse, $\underline{w}^0(s_{it}; 1)$ and $w^0(s_{it}; 1)$ are identical, and they exceed both $\underline{w}^0(s_{it}; n)$ and $w^0(s_{it}; n)$ for any $n > 1$. Thus, under the common values alternative, we again obtain a clean prediction of first-order stochastic dominance. Of course, within the context of our model, a rejection of the private values hypothesis can only imply the presence of common values. But the lack of ambiguity about the implications of the alternative provides additional assurance that the evidence we see is that predicted by common values.[42] Thus, we take the evidence in this section as providing further support for the robustness of our conclusions regarding the presence of common values.

[41] The weight $\varphi(s_{it}, n_t)$ is equal to

$$h(b_{it}^0, n_t)G_{M|B}(b_{it}^0; b_{it}^0, n_t) / \left[h(b_{it}^0, n_t)G_{M|B}(b_{it}^0; b_{it}^0, n_t) + H(b_{it}^0, n_t)g_{M|B}(b_{it}^0; b_{it}^0, n_t)\right].$$

[42] More broadly, this implies that—even in the model with a random reserve price—not only is the private values hypothesis falsifiable, but the presence of common values is identified, as it is in the baseline model.

7 Conclusion

We have proposed an empirical approach to first-price sealed bid auctions that allows affiliated values, unobserved auction-level heterogeneity, spatial correlation, and endogenous bidder entry. We provided conditions sufficient for nonparametric identification and proposed a semiparametric estimation strategy. All approaches to unobserved heterogeneity in first-price auctions require compromises in one or more dimensions, and the best approach will surely vary with the application and questions at hand. However our approach offers several conceptual and practical advantages for our empirical study of OCS auctions. Our analysis of the data leads to rejection of the private values model in favor common values, a conclusion that is robust across a variety of specifications. More broadly, our analysis indicates that, common values, affiliated private information, and unobserved heterogeneity are all present and important in our sample of OCS auctions. We also found that ignoring unobserved heterogeneity can hide the presence of common values, likely due to the fact that positive correlation between bidder entry and tract values works directly against the effects that exogenous have on the severity of the winner's curse.

Appendices

A Equilibrium Entry in a Model of OCS Auctions

Here we consider a particular extensive form game of entry and bidding that is motivated by our application and yields the reduced form (3) for the entry outcome presented in the text.[43] The discussion here also demonstrates how our model can accommodate auction-specific unobservables that are of arbitrary dimension and correlated with auction-specific observables, despite the apparent contradiction to our assumption that U_t is a scalar and independent of X_t. Accommodation of such correlation requires that we allow the interpretation of U_t to vary with the vector X_t. This precludes identification of causal effects of covariates on the auction, but poses no barrier to answering many questions that motivate estimation of an auction model (see below). Finally, we also discuss here the selection on U_t that could be implied by considering only auctions attracting at least one bid, as necessitated by our data. We demonstrate that such selection introduces only an additional way in which the interpretation of U_t varies with X_t.

A.1 Model

Consider a game of entry and bidding for the lease of a tract t. Let $\mathcal{I}$ denote the set of all potential bidders ("firms"), and let $I = |\mathcal{I}|$. The set $\mathcal{I}$ can be partitioned into the set of "neighbor firms" $\mathcal{Z}_t$—holders of active leases on adjacent ("neighbor") tracts—and all other firms, $\mathcal{I}\backslash\mathcal{Z}_t$. Denote the number of neighbor firms by $Z_t = |\mathcal{Z}_t|$. Let V_{it} denote the value of the lease to firm i (i's "valuation"). Let X_t and E_t denote, respectively, observed and unobserved (to us) characteristics of lease t that

[43] Example 1 in the text provided a sketch.

affect bidders' valuations. Let X_t include (among other relevant characteristics) the number of active leases on neighboring tracts and the set of bidders for each of those leases.[44] We make no restriction on the dimension of E_t or the joint distribution of (X_t, E_t).

The game consists of two stages. In the second stage, lease t is offered by first-price auction to the N_t bidders who enter in the first stage. We assume there is no binding reserve price in the auction.[45] In the first stage, bidders simultaneously choose whether to incur an entry cost in order to acquire a signal and participate in the auction.[46] Let $c_i(x_t)$ denote the entry cost for firm i. Neighbors have lower entry costs. In particular, $c_i(x_t) = c(x_t)$ for a neighbor firm, whereas non-neighbor firms have entry costs $c_i(x_t) = c(x_t) + \delta_i(x_t)$, with $\delta_i(x_t) > 0$ for all i.

Firms acquiring signals become "bidders." Let N_t denote the number of bidders. Let S_{it} denote the signal received by bidder i. Given $N_t = n$, let $S_t = (S_{1t}, \ldots, S_{nt})$ and $V_t = (V_{1t}, \ldots, V_{nt},)$, where without loss we relabel bidders as firms $i = 1, \ldots n$. For any conditioning set $\Omega \subseteq (X_t, Z_t, E_t)$, let $F_{SV}(S_t, V_t|N_t, \Omega)$ denote the conditional distribution of bidders' signals and valuations. We assume $F_{SV}(S_t, V_t|N_t, X_t, Z_t, E_t)$ satisfies standard smoothness, symmetry, affiliation, and nondegeneracy conditions, as described in Assumption 2 in the text. We assume that Z_t alters the joint distribution of signals and valuations only through its effect on N_t, i.e.,

$$F_{SV}(S_t, V_t|N_t, Z_t, X_t, E_t) = F_{SV}(S_t, V_t|N_t, X_t, E_t),$$

[44] In practice we represent the set of bidders for neighboring tracts more parsimoniously with the number of such bidders.

[45] In our application we consider an extension allowing a random reserve price.

[46] As is standard in the literature, we assume that only bidders incurring the entry cost can submit a bid (see, e.g., Levin and Smith (1994), Li and Zheng (2009), Athey, Levin, and Seira (2011), Krasnokutskaya and Seim (2011), Gentry and Li (2014), or Bhattacharya, Roberts, and Sweeting (2014)). This assumption can be relaxed, allowing bidding with no signal, under an equilibrium selection rule specifying that firms indifferent to entry do not enter.

and that Z_t is independent of E_t conditional on X_t. We discuss the justification for this conditional independence assumption below. Finally, assume the multiplicatively separable structure

$$V_{it} = V_{it}^0 \, \lambda \left(X_t, E_t \right), \tag{A.1}$$

where the function λ is bounded and the random variables $\left(V_{1t}^0, \ldots, V_{n_t t}^0, S_{1t}, \ldots, S_{n_t t}\right)$ are independent of (X_t, E_t, Z_t) conditional on N_t.

We have placed no restriction on the dimension of E_t or the joint distribution of (X_t, E_t),[47] and the only requirement placed on the function λ is boundedness. Nonetheless, we can obtain the model of unobserved heterogeneity in the text by representing the random variable $\lambda(X_t, E_t)$ in terms of its quantiles conditional on X_t. In particular, given $X_t = x$, let $F_\lambda \left(\cdot | x \right)$ denote the CDF of the random variable $\lambda \left(x, E_t \right)$, and let

$$U_t = F_\lambda \left(\lambda(x, E_t) | x \right). \tag{A.2}$$

For $u \in [0, 1]$ define $F_\lambda^{-1} \left(u | x \right) = \inf \left\{ \lambda : F_\lambda \left(\lambda | x \right) \geq u \right\}$ and let

$$\Gamma \left(x, u \right) = F_\lambda^{-1} \left(u | x \right). \tag{A.3}$$

Combining (A.2) and (A.3), for each x we have $F_\lambda(\Gamma(x, U_t)|x) = U_t = F_\lambda(\lambda(x, E_t)|x)$, i.e.,

$$\Gamma(x, U_t) = \lambda(x, E_t).$$

By construction, Γ is weakly increasing in its second argument, and U_t is uniform $[0, 1]$ conditional on X_t. And because U_t is a measurable function of E_t conditional on X_t, U_t is independent of Z_t conditional on X_t.

[47] This may be important, as the nomination process by which tracts were offered for lease in our sample period suggests that all public information would affect selection into the sample. In that case, a tract with "undesirable" value of X_t may have been unlikely to be offered unless the value of E_t made the tract desirable. See, e.g., Hendricks, Porter, and Boudreau (1987) for a discussion of the nomination process.

Note that in this new representation of the model, the distribution of U_t does not vary with X_t, but its interpretation generally will. This is not a problem for many purposes motivating estimation of an auction model, where typically the key issue is controlling for auction-level heterogeneity, not assessing causal effects of covariates. Observe that because $\Gamma(x_t, u_t) = \lambda(x_t, e_t)$ for all t by construction, $\Gamma(X_t, U_t)$ fully characterizes the variation and dependence in valuations and bids that arises from the observables and unobservables. Likewise, controlling for the value of $\gamma(X_t, U_t)$ fully controls for the effects of auction observables and unobservables X_t, E_t on valuations, bids, and equilibrium first-order conditions. However, we will not be able to quantify the effect of a change in X_t (or one of its components) holding the unobservable fixed, since our U_t is redefined at every value of X_t.

A.2 Equilibrium

We henceforth use the representation of the model just derived. The set of firms $\mathcal{I}$, the rules of the game, the values of (X_t, U_t, Z_t), and the distribution $F_{SV}\left(S_t, V_t | N_t, X_t, U_t\right)$ are common knowledge among firms. We consider perfect Bayesian equilibrium in pure strategies.

The second stage of the game is identical to the first-price sealed bid auction with symmetric affiliated values studied by Milgrom and Weber (1982). Milgrom and Weber show existence and uniqueness of Bayes Nash equilibrium in such an auction and characterize the equilibrium strategies and payoffs. Bidder i's payoff in the auction stage can be written as a function of the commonly known (N_t, X_t, U_t) and the realized bidder signals S_t. As noted in the text, standard arguments ensure that multiplicative separability of valuations is inherited by equilibrium bids. This implies that equilibrium profits are multiplicatively separable in $\Gamma\left(X_t, U_t\right)$ as well.

Thus, bidder i's equilibrium payoff can be written as

$$\pi\left(S_{it}, S_{-it}, N_t, X_t, U_t\right) = \pi^0\left(S_{it}, S_{-it}, N_t\right)\Gamma\left(X_t, U_t\right).$$

Because Γ is weakly increasing in U_t, so is $\pi\left(S_{it}, S_{-it}, N_t, X_t, U_t\right)$. Further, we assume the usual case in which a bidder's expected equilibrium payoff $E\left[\pi^0\left(S_{it}, S_{-it}, N_t\right)|S_{it}, N_t\right]$ is strictly decreasing in N_t.[48]

Moving to the entry stage, firms make decisions based on the cost of entry and expected profit in the auction. Define

$$\bar{\pi}\left(N_t, X_t, U_t\right) = E\left[\pi\left(S_{it}, S_{-it}, N_t, X_t, U_t\right)|N_t, X_t, U_t\right].$$

Let $c_{it} = c_i\left(x_t\right)$. For firm i, entering when $n-1$ other firms will also enter implies expected profit

$$\bar{\pi}\left(n, X_t, U_t\right) - c_{it}.$$

Conditional on (X_t, Z_t, U_t), and given equilibrium beliefs about payoffs in the auction stage, the entry stage is then identical to the entry model of Berry (1992). Berry showed that there is a unique equilibrium number of entrants

$$n^E\left(X_t, Z_t, U_t\right) = \max_{0\leq n\leq I}\left\{n : \bar{\pi}\left(n, X_t, U_t\right) - c_{it} \geq 0\right\}.$$

Recall that (X_t, Z_t) determine the values of $\{c_{it}\}_{i\in\mathcal{I}}$. Thus, in any perfect Bayesian

[48]We know of no counterexample to strict monotonicity in N_t, but also no general proof. Non-monotonicity (within the relevant range of N_t) could lead to existence of multiple equilibria with different numbers of bidders. There are examples (e.g., Matthews (1984)) in which the winning bid counterintuitively decreases in the number of bidders over certain ranges. However, this does not imply non-monotonicity of the ex ante bidder profit, since raising the number of bidders from n to $n+1$ reduces a given bidder's ex ante probability of winning from $1/n$ to $1/(n+1)$. In addition, when there is a winner's curse, a bidder's expected value of winning the auction declines in n. See also Pinkse and Tan (2005) (especially the supplemental material).

equilibrium we have

$$N_t = n^E \left(X_t, Z_t, U_t \right).$$

Further, because

$$\bar{\pi} \left(N_t, X_t, U_t \right) = E \left[\pi^0 \left(S_{it}, S_{-it}, N_t \right) |, N_t \right] \Gamma \left(X_t, U_t \right),$$

the function n^E is weakly increasing in U_t.

A.3 The Instrument

Our instrument for bidder entry (an "entry shifter") Z_t is the number of neighbor firms. First consider the "exclusion" requirement (Assumption 5). We have assumed directly that Z_t is independent of (S_t, V_t^0) conditional on N_t, i.e., that X_t are the only observables directly affecting bidder valuations.[49] However we must verify that Z_t is also independent of U_t conditional on X_t. Recall that X_t includes the number of active neighbor leases and the identities of all bidders for each of those leases. A lease with three neighbor leases, for example, may have one, two, or three neighbor firms, depending on which bidders for the neighboring leases won those auctions. Given the number of neighbor leases and the bidders for each neighbor tract (i.e., conditional on X_t), the number of distinct winners reflects only random variation in bidders' signals at prior auctions. Recall that signals are assumed independent of tract-specific unobserved heterogeneity and independent across tracts. Thus, even in the case of spatially correlated tract-level unobservables E_t, the conditional independence requirement $Z_t \perp\!\!\!\perp U_t | X_t$ will hold.

Regarding the "relevance" requirement for the instrument Z_t,[50] observe that

[49]This could fail here if the number of neighbor firms had a direct effect on tract value (given X_t), e.g., by driving up costs of negotiating production from common pools.

[50]This discussion is informal. We have specific "relevance" requirements in the identification

changes in the number of neighbor firms affects entry because for some combinations of (X_t, Z_t, U_t) the market will accommodate the $n+1$st entrant only if there is a potential bidder with low signal acquisition cost. For example, we will sometimes have two entrants because the market would support entry by a third (low cost) neighbor, but not by a third firm that is a (high cost) non-neighbor. Thus, larger values of Z_t will lead, all else equal, to weakly larger numbers of entrants. If the cost asymmetries are substantial, the effects of the instrument on participation will be substantial as well.

A.4 Truncation

In the OCS data we observe no information about (even existence of) leases offered for sale but attracting no bids. Given the tract nomination process in place during the sample period, this may not have been a frequent phenomenon. But existence of such leases could imply a form of selection on unobservables: leases attracting no bids would be those with relatively undesirable unobservables. Here we demonstrate that such selection is accommodated by interpreting the unobservable in our model as that conditioned on the event that the auction attracts at least one bid.

In our model, offered leases attracting no bids are those for which

$$\pi(1, x_t, u_t) \leq c_{it} \quad \forall i \in \mathcal{I}. \tag{A.4}$$

results and our instrument falls short of these by being discrete rather than continuous. As discussed in the identification analysis, discrete instruments may lead to reliance on functional form to interpolate between the points at which the index function $\gamma(x, \cdot)$ is identified.

Letting $\pi^{-1}(c; 1, x) = \sup\{u : \pi(1, x, u) < c\}$, we can rewrite (A.4) as[51]

$$u_t \leq \pi^{-1}\left(\min_{i \in \mathcal{I}} c_i(x_t); 1, x_t\right),$$

or, more simply,

$$u_t \leq \underline{u}(x_t).$$

Recalling that the definition of U_t already changes with each value of X_t, we obtain the original model by redefining U_t to denote the value of the unobservable conditional on truncation at $\underline{u}(X_t)$.[52]

B Proofs Omitted from the Text

Here we provide proofs omitted from the text. For convenience we restate the results being proved.

Corollary 1. Under Assumptions 1–6, the distribution of $U_t | X_t, N_t$ is identified.

Proof.

We can express $\Pr(U_t \leq u | X_t = x, N_t = n)$ as

$$F_{U|XN}(u|x, n) = \int F_{U|XZN}(u|x, z, n) \; d\zeta(z|x, n) \tag{B.1}$$

where $F_{U|XZN}$ is the distribution of $U_t | X_t, Z_t, N_t$ and ζ is the distribution of $Z_t | X_t, N_t$. Conditional on $N_t = n, Z_t = z$, and $X_t = x$, U_t is uniform on $[\tau_{n-1}(x, z), \tau_n(x, z)]$,

[51]Note that the threshold for attracting a single bidder varies depending on the presence of at least one neighbor tract (firm), but not with the number of neighbor firms Z_t.

[52]More formally, going back to the original formulation of the unobserved heterogeneity in terms of E_t, let $\tilde{\pi}(N_t, X_t, E_t)$ denote the expected second-stage payoff for a bidder facing $N_t - 1$ opponents given (X_t, E_t). Under the separable structure (A.1), we have $\tilde{\pi}(N_t, X_t.E_t) = \tilde{\pi}^0(N_t)\lambda(X_t, E_t)$. Given $X_t = x$, we have a zero-bidder auction when $\lambda(x, E_t) \leq \frac{\min_i c_i(x_t)}{\tilde{\pi}^0(1)}$. Thus, we modify the construction of U_t by letting $F_\lambda(\cdot|x)$ denote the CDF of $\lambda(x, E_t)$ conditional on $\lambda(x, E_t) > \frac{\min_i c_i(x_t)}{\tilde{\pi}^0(1)}$.

and by Theorem 1 the endpoints $\tau_{n-1}(x,z)$ and $\tau_n(x,z)$ are identified. So $F_{U|XZN}$ is known. Since ζ is directly observed, the result follows from (B.1). $\square$

Lemma 2. Under Assumptions 1–8, for all $n \geq \underline{n}$, all $(x,z) \in \mathbb{Y}(n)$, and all $(x',z') \in \mathbb{Y}(n)$, $\gamma(x, \tau_n(x,z)) - \gamma(x', \tau_{n-1}(x',z'))$ is identified.

Proof. For n^* as defined in Assumption 8, take $n \leq n^*$ and let $x(n)$, $z(n)$, and $\hat{z}(n)$ be as in part (i) of Assumption 8 so that

$$\begin{aligned} \underline{n}(x(n), z(n)) &= n \\ \underline{n}(x(n), \hat{z}(n)) &= n+1. \end{aligned} \tag{B.2}$$

Since $(x(n), z(n)) \in \mathbb{Y}(n)$ and $(x(n), \hat{z}(n)) \in \mathbb{Y}(n+1)$, Lemma 1 implies identification of

$$\gamma(x', \tau_{n-1}(x',z')) - \gamma(x(n), \tau_{n-1}(x(n), z(n))) \tag{B.3}$$

and

$$\gamma(x'', \tau_n(x'',z'')) - \gamma(x(n), \tau_n(x(n), \hat{z}(n))) \tag{B.4}$$

for all $(x',z') \in \mathbb{Y}(n)$ and $(x'',z'') \in \mathbb{Y}(n+1)$. By (4) and (B.2),

$$\tau_{n-1}(x(n), z(n)) = 0 = \tau_n(x(n), \hat{z}(n)),$$

so subtracting (B.3) from (B.4) yields identification of

$$\gamma(x'', \tau_n(x'',z'')) - \gamma(x', \tau_{n-1}(x',z')) \tag{B.5}$$

for all $(x'',z'') \in \mathbb{Y}(n+1)$ and $(x',z') \in \mathbb{Y}(n)$. By Assumption 7, there exists some (x'',z'') that is in both $\mathbb{Y}(n+1)$ and $\mathbb{Y}(n)$. The claim then follows from Lemma 1. A symmetric argument applies for $n > n^*$.[53] $\square$

[53]Note that the arguments used to show Lemmas 1 and 2 will often imply several forms of overiden-

Lemma 3. Let Assumptions 1–8 hold. Then for all $n \geq \underline{n}$ and all $(x, z) \in \mathbb{Y}(n)$, the values of $\gamma(x, \tau_{n-1}(x, z))$ and $\gamma(x, \tau_n(x, z))$ are identified.

Proof. We proceed by induction, starting with $n = \underline{n}$. By the normalization (1), $\gamma(x^0, 0) = 0$, where for some z we have $(x^0, z) \in \mathbb{Y}(\underline{n})$. Lemma 1 then implies identification of $\gamma(x, \tau_{\underline{n}-1}(x, z))$ for all $(x, z) \in \mathbb{Y}(\underline{n})$. Lemma 2 then implies identification of $\gamma(x, \tau_{\underline{n}}(x, z))$ for all $(x, z) \in \mathbb{Y}(\underline{n})$. Now take any $n > \underline{n}$ and suppose that $\gamma(x, \tau_{n-1}(x, z))$ is known for all $(x, z) \in \mathbb{Y}(n-1)$. By Assumption 7 there exists a point $(\tilde{x}, \tilde{z})$ in $\mathbb{Y}(n-1) \cap \mathbb{Y}(n)$. Since we have already identified $\gamma(\tilde{x}, \tau_{n-1}(\tilde{x}, \tilde{z}))$, by Lemma 1 we also know the value of $\gamma(x, \tau_{n-1}(x, z))$ for all (x, z) in $\mathbb{Y}(n)$. By Lemma 2, this implies identification of $\gamma(x, \tau_n(x, z))$ for all (x, z) in $\mathbb{Y}(n)$. □

Lemma 4. Under Assumptions 1–9, $\tau_{n-1}(X_t, Z_t)$ is continuous in Z_t on the pre-image of $(0, 1)$.

Proof. Fix $n, x,$ and z such that $\tau_{n-1}(x, z) \in (0, 1)$. Let $\tau = \tau_{n-1}(x, z)$ and let $\nu > 0$ be sufficiently small that $\tau + \nu < 1$ and $\tau - \nu > 0$. We show that for any such ν there exists $\epsilon > 0$ such that for every z' satisfying $\|z' - z\| < \epsilon$ we have $\tau_{n-1}(x, z') \in (\tau - \nu, \tau + \nu)$. Let $\delta = \nu/2$. By the definition of $\tau_{n-1}(x, z)$ and weak monotonicity of η in U_t, $\eta(x, z, \tau - \delta) < n$. So by Assumption 10 there exists $\epsilon_1 > 0$ such that for any z' satisfying $\|z' - z\| < \epsilon_1$, $\eta(x, z', \tau') < n$ for some $\tau' \in (\tau - 2\delta, \tau)$. Similarly, because $\eta(x, z, \tau + \delta) \geq n$, Assumption 10 ensures that there exists $\epsilon_2 > 0$ such that for any z' satisfying $\|z' - z\| < \epsilon_2$, $\eta(x, z', \tau'') \geq n$ for some $\tau'' \in (\tau, \tau + 2\delta)$. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we have shown that for any z' satisfying $\|z' - z\| < \epsilon$, $\eta(x, z', \tau') < n$ for some $\tau' \in (\tau - \nu, \tau)$ while $\eta(x, z', \tau'') \geq n$ for some $\tau'' \in (\tau, \tau + \nu)$.

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tification. For example, Lemma 1 implies overidentification of $\gamma(x', \tau_m(x', z')) - \gamma(x, \tau_m(x, z))$ for any m which is both smaller than $\min\{\overline{n}(x', z'), \overline{n}(x, z)\}$ and larger than $\max\{\underline{n}(x', z'), \underline{n}(x, z)\}$. And while Assumption 7 ensures only that there exist one $(\tilde{x}, \tilde{z}) \in \mathbb{Y}(n)$ that is also in $\mathbb{Y}(n-1)$, when there is more than one such pair the proof of Lemma 2 will provide multiple ways of constructing the same value of a given difference $\gamma(x'', \tau_n(x'', z'')) - \gamma(x', \tau_{n-1}(x', z'))$. Finally, in practice there may often be more than one value of n^* satisfying Assumption 8, resulting in some duplication in the differences identified in the two halves of the proof of Lemma 2.

At such z', $\tau_{n-1}(x, z')$ must lie in $[\tau', \tau'']$. $\square$

Corollary 2. Under Assumptions 1–11,

(a) the hypothesis of equilibrium bidding in the affiliated values model is testable;

(b) the affiliated private values model is identified;

(c) the hypothesis of private values is testable against the alternative of common values;

(d) if the ex post value of the good to the winner is observed, the hypothesis of equilibrium bidding in the pure common values is testable.

Proof. Parts (a) and (c) follow immediately from Theorem 3 and the results in HHS. And because with private values we have $w(S_{it}; N_t, X_t, U_t) = S_{it} = V_{it}$, part (b) follows immediately from Theorem 3. To show part (d), first observed that under the assumption of pure common values directly observe the joint distribution of $\bar{V}_t, B_{1t}, \ldots, B_{N_t t}$ conditional on X_t, N_t, where $\bar{V}_t$ denotes the pure common value. We therefore know the expectation

$$E_{\bar{V}_t}\left[\bar{V}_t \middle| B_{it}, \max_{j \neq i} B_{jt} = B_{it}, X_t, N_t\right]$$

as well as

$$E_{B_{it}}\left[E_{\bar{V}_t}\left[\bar{V}_t \middle| B_{it}, \max_{j \neq i} B_{jt} = B_{it}, X_t, N_t\right] \middle| X_t, N_t\right]. \tag{B.6}$$

By strict monotonicity of equilibrium bidding, the latter is equal to

$$E_{S_{it}}\left[E_{\bar{V}_t}\left[\bar{V}_t \middle| S_{it}, \max_{j \neq i} S_{jt} = S_{it}, X_t, N_t\right] \middle| X_t, N_t\right].$$

By the law of iterated expectations, this is equal to

$$E_{S_{it}}\left[E_{U_t}\left[E_{\bar{V}_t}\left[\bar{V}_t \middle| S_{it}, \max_{j \neq i} S_{jt} = S_{it}, X_t, U_t, N_t\right] \middle| X_t, N_t\right]\right]$$

i.e.,

$$E_{S_{it},U_t}\left[w\left(S_{it}; N_t, X_t, U_t\right) \middle| X_t, N_t\right]. \tag{B.7}$$

By Theorem 3, the distribution of $w\left(S_{it}; N_t, X_t, U_t\right)$ conditional on X_t, U_t, N_t is identified without using observation of $\bar{V}_t$. Since the distribution of $U_t|X_t, N_t$ is also known (Corollary 1), Theorem 3 implies that the value of (B.7) is also identified without observation of $\bar{V}_t$. Thus we obtain the testable restriction that the value of (B.6) implied directly by the joint distribution of $\bar{V}_t, B_{1t}, \ldots, B_{N_t t}|X_t, N_t$ be equal to the value of (B.7) implied by the bidding data and the restrictions implied by equilibrium bidding. □

C Monte Carlo Simulations

We present a small Monte Carlo study in order to evaluate the behavior of our estimation procedure for different auction models in moderately sized samples. For each exercise we draw simulated samples with 3000 auctions and observed characteristics $X_t = (1, X_{1t}, X_{2t})'$, where $X_{jt} \sim U[0,1]$ for $j = 1, 2$, and $Z_t \sim U[0,1]$. Unobserved auction heterogeneity is given by $U_t \sim U[0,1]$. Entry is modeled with an ordered probit: with $\Phi^{-1}(\cdot)$ denoting the inverse of the normal CDF and $\lceil \cdot \rceil$ the ceiling function, we specify $N_t = \max\{2, \lceil X_t'\lambda_x + \lambda_z Z_t + \Phi^{-1}(U_t) \rceil\}$. Auction heterogeneity shifts values multiplicatively as

$$V_{it} = \exp\left(X_t'\gamma_x + \gamma_u U_t\right) V_{it}^0, \tag{C.1}$$

implying that equilibrium bidding follows

$$\log\left(B_{it}\right) = X_t'\gamma_x + \gamma_u U_t + \log\left(B_{it}^0\right). \tag{C.2}$$

We consider two private values models, both with lognormal valuations.[54] The first is an independent private values (IPV) model with $\log(V_{it}^0) \sim_{iid} N(0,1)$. The second is an affiliated private values (APV) model where valuations have covariance matrix Σ, which has ones on the diagonal and 0.5 in off-diagonal positions. We perform 600 Monte Carlo replications for each design.

We omit results for the entry model, as the ordered probit estimates are of course close to their true values and precisely estimated. Table 9 shows estimates of the parameters of the index function $\gamma(x,u)$ for each design, while the estimated correlation coefficients are presented in Table 10. Although these parameters have been estimated in the sieve QMLE step, we again see very good performance in terms of both the location and precision of the estimates.

Table 9: Index Parameter Estimates

	true	**mean**	**std**	$\mathbf{Q_{0.1}}$	$\mathbf{Q_{0.5}}$	$\mathbf{Q_{0.9}}$
			IPV Model			
γ_{x_1}	1	0.9926	0.033	0.9513	0.9921	1.0333
γ_{x_2}	1	0.9947	0.0359	0.947	0.9945	1.0397
γ_u	1	0.957	0.063	0.8686	0.9610	1.04
			APV Model			
γ_{x_1}	1	1.0213	0.0538	0.9443	1.0180	1.0955
γ_{x_2}	1	1.0186	0.0553	0.9488	1.0288	1.0960
γ_u	1	1.0297	0.0901	0.8982	1.0367	1.1372

Finally, in Figure 8 we show the true marginal CDF of bidders' log homogenized valuations with the pointwise 5th and 95th percentile values of our estimated distri-

[54] Note that log-normal values do not imply log-normal bids.

Table 10: Gaussian Copula Correlation Estimates

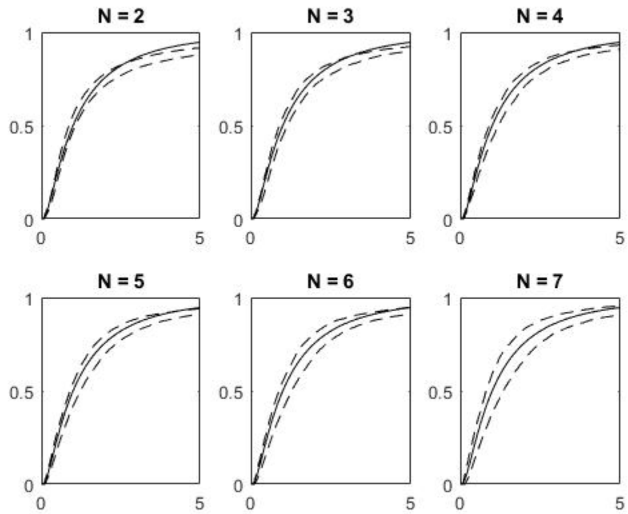
n	$\bar{\mathbf{T}}_{\mathbf{n}}$	**true** ρ	**mean** $\hat{\rho}$	**std**	$\mathbf{Q_{0.1}}$	$\mathbf{Q_{0.5}}$	$\mathbf{Q_{0.9}}$
			IPV Model				
2	626.6	0	0.009	0.047	-0.057	0.011	0.069
3	872.2	0	0.004	0.024	-0.025	0.005	0.033
4	875.5	0	0.003	0.016	-0.016	0.003	0.024
5	473.2	0	0.002	0.017	-0.020	0.002	0.023
6	132.7	0	-0.002	0.023	-0.032	-0.004	0.028
7	19.8	0	-0.007	0.050	-0.069	-0.015	0.060
			APV Model				
2	626.0	0.5	0.486	0.047	0.448	0.487	0.532
3	874.1	0.5	0.470	0.040	0.445	0.472	0.498
4	873.6	0.5	0.478	0.039	0.457	0.480	0.506
5	472.4	0.5	0.478	0.040	0.451	0.482	0.507
6	134.0	0.5	0.483	0.055	0.429	0.482	0.546
7	19.9	0.5	0.476	0.119	0.329	0.481	0.629

Notes: Column $\bar{T}_n$ reports the average number of auctions in each 'number of bidders' bin.

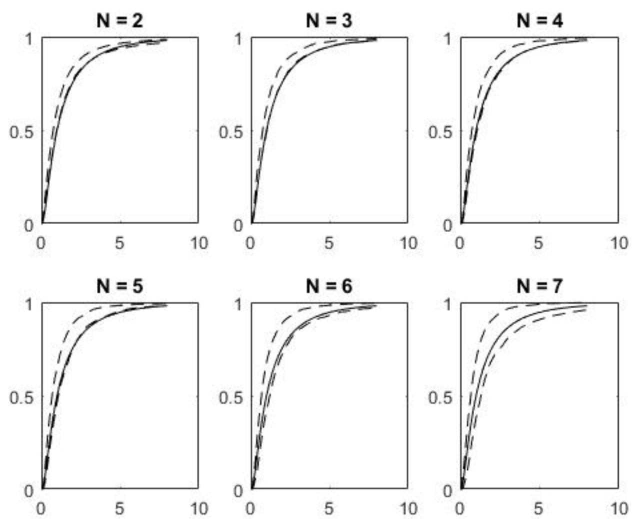
bution. The results exhibit some bias in the smallest samples—e.g., for $n = 2$ where a typical sample included about 1250 bids. Even in this case the estimates appear to be quite good, and the estimates become substantially better as sample sizes grow.

Figure 8: Estimated Distribution of Homogenized Valuations

IPV Model



APV Model



D Additional Results and Tables

D.1 Semi-nonparametric Entry Model

Table 11 displays the results of the parametric part of the entry model when estimated using the semi-nonparametric specification (see section 6.2). Figure 9 shows the estimated cumulative distribution function alongside the normal distribution used in the baseline model. Because the interpretation of the linear component of the model changes as we change H, there is no need for the estimated parameters or CDF to agree, even when the more restrictive model is correctly specified. Nonetheless, we see that the estimated parameters and distribution are actually quite similar.

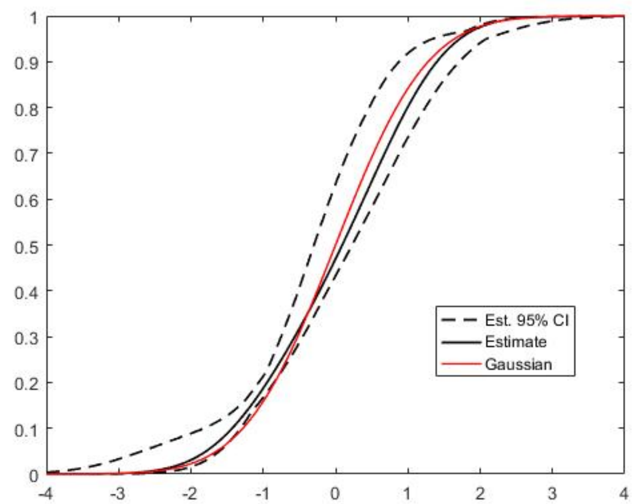
Table 11: Semi-nonparametric participation estimates

		Est.	**Std. err.**
# active leases	X	-0.240	0.053
# firms that bid for neighbors		-0.063	0.034
re-offered tract		-0.281	0.035
# neighbors drilled		-0.018	0.032
# neighbor hits		0.145	0.034
Time controls		Sale year dummies	
# neighbor firms	Z	0.109	0.052

Notes: Estimates for the semi-nonparametric participation model. Dependent variable is the number of bidders. Standard errors computed by bootstrap.

The results from the participation model that matter for our purposes are only the estimates of the bounds $\tau_n(x, z)$ and $\tau_{n-1}(x, z)$ on the unobserved heterogeneity. In Table 12 we compare the intervals generated by the ordered probit and semi-nonparametric model. The first two columns report the median length of the interval $\tau_n(x, z) - \tau_{n-1}(x, z)$ for different values of n. The third column reports the median

Figure 9: SNP vs. Normal



Notes: 95% CI computed by bootstrap.

length of the intersection between these intervals. In the case of perfect agreement, the lengths reported in all three columns would be identical; otherwise the maximal intersection is the smaller of the two intervals in the previous columns. The results indicate that the lengths of the estimated intervals are very similar across the two specifications. We see nontrivial disagreement in lengths only for the top bin covering $n \in \{11, \ldots, 18\}$, which leads to imperfections in the locations of the other intervals that are roughly equally spread over lower values of n.

Table 12: Comparison of UH bounds

n	Median $\tau_n(x,z) - \tau_{n-1}(x,z)$ Ordered Probit	SNP	Median interval intersection
1	0.428	0.417	0.417
2	0.210	0.204	0.196
3	0.120	0.119	0.109
4	0.092	0.096	0.087
5	0.063	0.067	0.060
6	0.052	0.055	0.042
7	0.048	0.051	0.032
8	0.033	0.036	0.018
9	0.026	0.029	0.017
10	0.027	0.031	0.023
11 - 18	0.086	0.107	0.086

D.2 Time Varying Coefficients

Tables 13 and 14 present parameter estimates for the alternative specification discussed in section 6.1.

Table 13: Entry Model w/Time Varying Coeffs

		Estimate (std err)	
$(t \in 50's)\times$	X		
# active leases		-0.037	0.0256
# firms that bid for neighbors		-0.0228	0.0369
re-offered tract		-0.0201	0.021
# neighbors drilled		-0.0437	0.0312
# neighbor hits		-0.0125	0.0279
time trend (year)		0.0333	0.0251
$(t \in 60's)\times$			
# active leases		-0.0125	0.0567
# firms that bid for neighbors		-0.0874	0.0377
re-offered tract		-0.2113	0.0237
# neighbors drilled		-0.1069	0.0562
# neighbor hits		0.1109	0.0379
time trend (year)		0.0808	0.0367
$(t \in 70's)\times$			
# active leases		-0.208	0.0437
# firms that bid for neighbors		-0.0838	0.0341
re-offered tract		-0.2134	0.0222
# neighbors drilled		0.0332	0.0476
# neighbor hits		0.0544	0.0306
time trend (year)		0.0777	0.041
$(t \in 80's)\times$			
# active leases		-0.1995	0.0688
# firms that bid for neighbors		0.056	0.0395
re-offered tract		-0.0841	0.025
# neighbors drilled		0.0377	0.0559
# neighbor hits		0.0768	0.0375
time trend (year)		-0.1756	0.0387
$(t \in 50's)\times$ # neighbor firms	Z	0.0036	0.0369
$(t \in 60's)\times$ # neighbor firms		-0.0119	0.047
$(t \in 70's)\times$ # neighbor firms		0.1241	0.0405
$(t \in 80's)\times$ # neighbor firms		0.1433	0.0575

Table 14: Index Function with Time-Varying Coefficients

		Estimate (std err)	
$(t \in 50's)\times$	X		
# active leases		-0.0086	0.0368
# firms that bid for neighbors		-0.0455	0.0376
re-offered tract		-0.0619	0.2404
# neighbors drilled		0.0801	0.1774
# neighbor hits		0.2408	0.2626
time trend (year)		0.3003	0.0521
$(t \in 60's)\times$			
# active leases		-0.0146	0.0466
# firms that bid for neighbors		-0.0133	0.0202
re-offered tract		-0.4708	0.0983
# neighbors drilled		0.0188	0.0568
# neighbor hits		0.2389	0.0563
time trend (year)		0.0856	0.0101
$(t \in 70's)\times$			
# active leases		0.0339	0.0246
# firms that bid for neighbors		0.0304	0.007
re-offered tract		-0.707	0.0802
# neighbors drilled		-0.0076	0.0221
# neighbor hits		-0.0674	0.0247
time trend (year)		0.1001	0.0051
$(t \in 80's)\times$			
# active leases		0.0094	0.0382
# firms that bid for neighbors		0.052	0.0148
re-offered tract		-0.1978	0.0956
# neighbors drilled		0.1073	0.0306
# neighbor hits		-0.0355	0.0404
time trend (year)		0.0464	0.0053
Unobserved heterogeneity	U		
$U \times (t \in 50's)$		1.3749	0.4094
$U \times (t \in 60's)$		0.8804	0.3963
$U \times (t \in 70's)$		1.0042	0.3801
$U \times (t \in 80's)$		1.4519	0.4022

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