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Supplement for "Fixed-effect regressions on network data"

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FIXED-EFFECT REGRESSIONS ON NETWORK DATA

SUPPLEMENT

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Contents

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S.1 Proofs

PROOF OF LEMMA 1 (EXISTENCE)

The estimator is defined by the constraint minimization problem in (2.3). For convenience we express the constraint in quadratic form, $(\boldsymbol{a}'\boldsymbol{\iota}_n)^2 = 0$. By introducing the Lagrange multiplier $\lambda > 0$ we can write

$$\widehat{\boldsymbol{\alpha}} = \arg\min_{\boldsymbol{a}\in\mathbb{R}^n} \; (\boldsymbol{y} - \boldsymbol{B}\boldsymbol{a})'(\boldsymbol{y} - \boldsymbol{B}\boldsymbol{a}) + \lambda\,(\boldsymbol{a}'\boldsymbol{\iota}_n)^2\,.$$

Solving the corresponding first-order condition we obtain

$$\widehat{\boldsymbol{\alpha}} = (\boldsymbol{B}'\boldsymbol{B} + \lambda\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n')^{-1}\,\boldsymbol{B}'\boldsymbol{y}.$$

Here, the matrix $\boldsymbol{B}'\boldsymbol{B} + \lambda\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n'$ is invertible, because $\boldsymbol{L} = \boldsymbol{B}'\boldsymbol{B}$ only has a single zero eigenvalue (because we assume the graph to be connected) with eigenvector $\boldsymbol{\iota}_n$, so that adding $\lambda\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n'$ gives a non-degenerate matrix. The matrices $\boldsymbol{B}'\boldsymbol{B}$ and $\boldsymbol{\iota}_n\boldsymbol{\iota}_n'$ commute, and by properties of the Moore-Penrose inverse we thus have

$$(\boldsymbol{B}'\boldsymbol{B} + \lambda\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n')^{-1} = (\boldsymbol{B}'\boldsymbol{B})^{\dagger} + \lambda^{-1}\,(\boldsymbol{\iota}_n\boldsymbol{\iota}_n')^{\dagger}\,.$$

We furthermore have $(\boldsymbol{\iota}_n\boldsymbol{\iota}_n')^{\dagger} = n^{-2}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'$ and, because $\boldsymbol{B}\boldsymbol{\iota}_n = 0$, the contribution from $(\boldsymbol{\iota}_n\boldsymbol{\iota}_n')^{\dagger}$ drops out of the above formula for $\widehat{\boldsymbol{\alpha}}$, and we obtain $\widehat{\boldsymbol{\alpha}} = (\boldsymbol{B}'\boldsymbol{B})^{\dagger}\,\boldsymbol{B}'\boldsymbol{y}$. This concludes the proof. ■

PROOF OF THEOREM 1 (SAMPLING DISTRIBUTION)

As $\boldsymbol{y} = \boldsymbol{B}\boldsymbol{\alpha} + \boldsymbol{u}$, Lemma 1 gives

$$\widehat{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + (\boldsymbol{B}'\boldsymbol{B})^{\dagger}\,\boldsymbol{B}'\boldsymbol{u}.$$

Conditional on $\boldsymbol{B}$, $\boldsymbol{u} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2\,\boldsymbol{I}_n)$, and so

$$\widehat{\boldsymbol{\alpha}} \sim \mathcal{N}\left(\boldsymbol{\alpha},\; \sigma^2\,(\boldsymbol{B}'\boldsymbol{B})^{\dagger}\right),$$

where the variance expression follows from properties of the Moore-Penrose pseudoinverse. This concludes the proof. ■

PROOF OF COROLLARY 1 (INFERENCE)

The result follows from Theorem 1 by standard arguments on the F-statistic in linear regression models. Here, the degrees-of-freedom correction from $m-n$ to $m-(n-1)$ arises, because the projection matrix

$$\boldsymbol{I}_m - \boldsymbol{B}\left(\boldsymbol{B}'\boldsymbol{B}\right)^{\dagger}\boldsymbol{B}'$$

has rank $m-(n-1)$. Notice that although $\boldsymbol{B}$ has n columns, we have that rank $\boldsymbol{B} = (n-1)$. This concludes the proof. ■

PROOF OF THEOREMS 2 AND 5 (ZERO-ORDER BOUNDS)

There are no isolated vertices, because $\mathcal{G}$ is connected and $n > 2$. That is, $d_i > 0$ for all i, and so $\boldsymbol{D}$ is invertible. From Theorem 1 and the definition of the normalized Laplacian $\boldsymbol{S}$ we find

$$\mathrm{var}(\widehat{\boldsymbol{\alpha}}) = \sigma^2\, \boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{S}^{\dagger}\boldsymbol{D}^{-\frac{1}{2}}.$$

In the following we write $\boldsymbol{M}_1 \leq \boldsymbol{M}_2$ for symmetric matrices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ to indicate that $\boldsymbol{M}_2 - \boldsymbol{M}_1$ is positive semi-definite. We have $\boldsymbol{S}^{\dagger} \leq \lambda_2^{-1}\boldsymbol{I}_n$, because λ_2 is the smallest non-zero eigenvalue of $\boldsymbol{S}$. Therefore,

$$\mathrm{var}(\widehat{\boldsymbol{\alpha}}) \leq \frac{\sigma^2}{\lambda_2}\boldsymbol{D}^{-1}.$$

This result implies that, for any vector $\boldsymbol{v} \in \mathbb{R}^n$,

$$\mathrm{var}(\boldsymbol{v}'\widehat{\boldsymbol{\alpha}}) = \boldsymbol{v}'\mathrm{var}(\widehat{\boldsymbol{\alpha}})\,\boldsymbol{v} \leq \frac{\sigma^2}{\lambda_2}\boldsymbol{v}'\boldsymbol{D}^{-1}\boldsymbol{v} = \frac{\sigma^2}{\lambda_2}\boldsymbol{v}'\,\mathrm{diag}(d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1})\,\boldsymbol{v}.$$

The bound in Theorem 2 follows on setting $\boldsymbol{v} = \boldsymbol{e}_i$, the ith unit vector. The corresponding bound for the differences in Theorem 5 follows on setting $\boldsymbol{v} = \boldsymbol{e}_i - \boldsymbol{e}_j$ for $i \neq j$. This concludes the proof. ■

PROOF OF THEOREMS 3 AND 6 (FIRST-ORDER BOUNDS)

We first show that, if $\mathcal{G}$ is connected, then

$$0 \leq \left[\mathrm{var}(\widehat{\boldsymbol{\alpha}}) - \sigma^2\left(\boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1} - \frac{\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{D}^{-1}}{n} - \frac{\boldsymbol{D}^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'}{n}\right)\right] \leq \frac{\sigma^2}{\lambda_2}\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}. \tag{S.1}$$

Theorems 3 and 6 will then follow readily. First note that, because $\mathcal{G}$ is connected, we know that the zero eigenvalue of the Laplacian matrix $\boldsymbol{L}$ has multiplicity one, and the corresponding eigenvector is given by $\boldsymbol{\iota}$. The Moore-Penrose pseudoinverse of $\boldsymbol{L}$ therefore satisfies $\boldsymbol{L}^{\dagger}\boldsymbol{L} = \boldsymbol{I}_n - n^{-1}\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n'$, where the right hand side is the idempotent matrix that projects orthogonally to $\boldsymbol{\iota}_n$. Using that $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$ and solving this equation for $\boldsymbol{L}^{\dagger}$ gives

$$\boldsymbol{L}^{\dagger} = \boldsymbol{D}^{-1} + \boldsymbol{L}^{\dagger}\boldsymbol{A}\boldsymbol{D}^{-1} - n^{-1}\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{D}^{-1}. \tag{S.2}$$

The Laplacian is symmetric, and so transposition gives

$$\boldsymbol{L}^{\dagger} = \boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{L}^{\dagger} - n^{-1}\,\boldsymbol{D}^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'. \tag{S.3}$$

Replacing $\boldsymbol{L}^{\dagger}$ on the right-hand side of (S.2) by the expression for $\boldsymbol{L}^{\dagger}$ given by (S.3) yields

$$\boldsymbol{L}^{\dagger} = \boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{L}^{\dagger}\boldsymbol{A}\boldsymbol{D}^{-1} - n^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{D}^{-1} - n^{-1}\boldsymbol{D}^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n',$$

where we have also used the fact that $\boldsymbol{D}^{-1}\boldsymbol{A}\,\boldsymbol{\iota}_n = \boldsymbol{\iota}_n$. Re-arranging this equation allows us to write

$$\boldsymbol{L}^{\dagger} - \left(\boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1} - n^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{D}^{-1} - n^{-1}\boldsymbol{D}^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\right) = \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{L}^{\dagger}\boldsymbol{A}\boldsymbol{D}^{-1}.$$

Because $\boldsymbol{L} \geq 0$ and by the arguments in the preceding proof we also have the bounds

$$\boldsymbol{0} \leq \boldsymbol{L}^{\dagger} \leq \lambda_2^{-1}\boldsymbol{D}^{-1}.$$

Put together this yields

$$0 \leq \boldsymbol{L}^{\dagger} - \left(\boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1} - n^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{D}^{-1} - n^{-1}\boldsymbol{D}^{-1}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\right) \leq \lambda_2^{-1}\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1},$$

and multiplication with σ^2 gives the bounds stated in (S.1).

To show Theorems 3 and 6 we calculate, for $i \neq j$,

$$\begin{aligned}
\boldsymbol{e}_i'\,\boldsymbol{D}^{-1}\,\boldsymbol{e}_i &= d_i^{-1}, & \boldsymbol{e}_i'\,\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\,\boldsymbol{e}_i &= d_i^{-1}h_i^{-1},\\
\boldsymbol{e}_i'\,\boldsymbol{D}^{-1}\,\boldsymbol{e}_j &= 0, & \boldsymbol{e}_i'\,\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\,\boldsymbol{e}_j &= d_i^{-1}d_j^{-1}d_{ij}h_{ij}^{-1},\\
\boldsymbol{e}_i'\,\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\,\boldsymbol{e}_i &= 0, & \boldsymbol{e}_i'\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{D}^{-1}\,\boldsymbol{e}_i = \boldsymbol{\iota}_n'\boldsymbol{D}^{-1}\,\boldsymbol{e}_i &= d_i^{-1},\\
\boldsymbol{e}_i'\,\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\,\boldsymbol{e}_j &= d_i^{-1}d_j^{-1}(\boldsymbol{A})_{ij}, & \boldsymbol{e}_i'\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{D}^{-1}\,\boldsymbol{e}_j = \boldsymbol{\iota}_n'\boldsymbol{D}^{-1}\,\boldsymbol{e}_j &= d_j^{-1}.
\end{aligned}$$

Combining these results with (S.1) gives the bounds on $\text{var}(\widehat{\alpha}_i) = \boldsymbol{e}_i'\text{var}(\widehat{\boldsymbol{\alpha}})\boldsymbol{e}_i$ and $\text{var}(\widehat{\alpha}_i - \widehat{\alpha}_j) = (\boldsymbol{e}_i - \boldsymbol{e}_j)'\text{var}(\widehat{\boldsymbol{\alpha}})(\boldsymbol{e}_i - \boldsymbol{e}_j)$ stated in the theorems and concludes the proof. ■

PROOF OF THEOREM 4 (GENERALIZED APPROXIMATION)

From the proof of Lemma 1, the least-squares estimator satisfies the first-order condition

$$\boldsymbol{L}\,\widehat{\boldsymbol{\alpha}} = \boldsymbol{B}'\boldsymbol{y}.$$

Using that $\boldsymbol{y} = \boldsymbol{B}\boldsymbol{\alpha} + \boldsymbol{u}$ and that $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$ this yields $\boldsymbol{D}^{1/2}\,(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) = \boldsymbol{D}^{-1/2}\boldsymbol{B}'\boldsymbol{u} + \boldsymbol{\epsilon}$, where

$$\boldsymbol{\epsilon} := \boldsymbol{D}^{-1/2}\boldsymbol{A}\,(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})\,.$$

Note that this is the vector version of the expression for $\sqrt{d_i}(\widehat{\alpha}_i - \alpha_i)$ as given in the theorem. From $\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} = (\boldsymbol{B}'\boldsymbol{B})^{\dagger}\boldsymbol{B}'\boldsymbol{u}$ it follows that $\mathbb{E}(\boldsymbol{\epsilon}) = \boldsymbol{0}$ while from the assumption that $\mathbb{E}(\boldsymbol{u}\boldsymbol{u}') \leq \overline{\sigma}^2\boldsymbol{I}_n$ we have that

$$\mathbb{E}(\boldsymbol{\epsilon}\boldsymbol{\epsilon}') = \boldsymbol{D}^{-1/2}\boldsymbol{A}(\boldsymbol{B}'\boldsymbol{B})^{\dagger}\boldsymbol{B}'\,\mathbb{E}\,(\boldsymbol{u}\boldsymbol{u}')\,\boldsymbol{B}(\boldsymbol{B}'\boldsymbol{B})^{\dagger}\boldsymbol{A}\boldsymbol{D}^{-1/2} = \overline{\sigma}^2\,\boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{L}^{\dagger}\boldsymbol{A}\boldsymbol{D}^{-1/2}.$$

As in the preceding proofs, we still have that $\boldsymbol{L}^{\dagger} \leq \lambda_2^{-1}\boldsymbol{D}^{-1}$, and so

$$\mathbb{E}(\boldsymbol{\epsilon}\boldsymbol{\epsilon}') \leq \overline{\sigma}^2\lambda_2^{-1}\boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1/2}.$$

From this we find

$$\mathbb{E}(\epsilon_i^2) \leq \frac{\overline{\sigma}^2}{\lambda_2\,h_i}.$$

Thus, if $\overline{\sigma}^2\lambda_2^{-1}h_i^{-1} \to 0$ as $n \to \infty$, then by Markov's inequality we have $\epsilon_i \to_p 0$. By the continuous mapping theorem we therefore have

$$\sqrt{d_i}\,(\widehat{\alpha}_i - \alpha_i) \to_p \frac{1}{\sqrt{d_i}}\sum_{j\in[i]} u_{ij}.$$

Moreover, if $\frac{1}{\sqrt{d_i}}\sum_{j\in[i]} u_{ij}$ is asymptotically normal, then so is $\sqrt{d_i}\,(\widehat{\alpha}_i - \alpha_i)$. This concludes the proof. ■

PROOF OF LEMMA 2 (VARIANCE OF COMPONENT ESTIMATORS)

Additional notation. Without loss of generality we relabel the elements of V such that

$$V_1 = \{1,\ldots,n_1\},\quad V_2 = \{n_1+1,\ldots,n_2\},\quad \ldots\quad V_q = \{n-n_q+1,\ldots,n\}.$$

We decompose $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2', \ldots, \boldsymbol{\beta}_q')'$ and $\widehat{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}_1', \widehat{\boldsymbol{\beta}}_2', \ldots, \widehat{\boldsymbol{\beta}}_q')'$, where each $\boldsymbol{\beta}_r$ and $\widehat{\boldsymbol{\beta}}_r$ are n_r column vectors. Note that the Laplacian matrix $\boldsymbol{L}_W$ is block-diagonal; moreover,

$$\boldsymbol{L}_W = \operatorname{diag}\left(\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_q\right),$$

where $\boldsymbol{L}_r$ is the Laplacian of the graph $\mathcal{G}_r$. We also decompose $\boldsymbol{A} = \boldsymbol{A}_W + \boldsymbol{A}_B$, where $\boldsymbol{A}_W$ and $\boldsymbol{A}_B$ are the adjacency matrix of $\mathcal{G}_W$ and $\mathcal{G}_B$, respectively, and write $\boldsymbol{D}_W = \operatorname{diag}\left(\boldsymbol{A}_W \boldsymbol{\iota}_n\right)$ and $\boldsymbol{D}_B = \operatorname{diag}\left(\boldsymbol{A}_B \boldsymbol{\iota}_n\right)$ for the corresponding degree matrices. We have $\boldsymbol{L}_W = \boldsymbol{D}_W - \boldsymbol{A}_W$ and $\boldsymbol{L}_B = \boldsymbol{D}_B - \boldsymbol{A}_B$. We also relabel the elements of E such that

$$E_B = \{1, \ldots, m_B\}, \qquad E_W = \{m_B + 1, \ldots, m\},$$

and correspondingly we decompose $\boldsymbol{B} = (\boldsymbol{B}_B', \boldsymbol{B}_W')'$, where $\boldsymbol{B}_B$ and $\boldsymbol{B}_W$ are $m_B \times n$ and $m_W \times n$ matrices, respectively, whose rows correspond to edges in $\mathcal{G}_B$ and $\mathcal{G}_W$, respectively. We then have $\boldsymbol{L} = \boldsymbol{B}'\boldsymbol{B} = \boldsymbol{B}_W'\boldsymbol{B}_W + \boldsymbol{B}_B'\boldsymbol{B}_B = \boldsymbol{L}_W + \boldsymbol{L}_B$.

Inverse expressions. Notice that, under the conventions from above, $\boldsymbol{P}$ is simply given by

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{\iota}_{n_1}' & 0 & \ldots & 0 \\ 0 & \boldsymbol{\iota}_{n_2}' & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \boldsymbol{\iota}_{n_q}' \end{pmatrix}.$$

We define the block-diagonal $n \times n$ matrix

$$\boldsymbol{M} = \boldsymbol{I}_n - \boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{P} = \begin{pmatrix} \boldsymbol{I}_{n_1} - n_1^{-1}\boldsymbol{\iota}_{n_1}\boldsymbol{\iota}_{n_1}' & 0 & \ldots & 0 \\ 0 & \boldsymbol{I}_{n_2} - n_2^{-1}\boldsymbol{\iota}_{n_2}\boldsymbol{\iota}_{n_2}' & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \boldsymbol{I}_{n_q} - n_q^{-1}\boldsymbol{\iota}_{n_q}\boldsymbol{\iota}_{n_q}' \end{pmatrix}.$$

Some useful relations are $\boldsymbol{H} = \boldsymbol{P}\boldsymbol{P}'$, $\boldsymbol{M}^2 = \boldsymbol{M}$, $\boldsymbol{P}\boldsymbol{M} = 0$, $\boldsymbol{P}'\boldsymbol{\iota}_q = \boldsymbol{\iota}_n$, $\boldsymbol{P}\boldsymbol{\iota}_n = \boldsymbol{H}\boldsymbol{\iota}_q$, and thus also $\boldsymbol{H}^{-1}\boldsymbol{P}\boldsymbol{\iota}_n = \boldsymbol{\iota}_q$. The various pseudo-inverses that appear in the following satisfy

$$\begin{aligned}
\boldsymbol{L}_W^{\dagger}\boldsymbol{L}_W &= \boldsymbol{M}, \\
\left(\boldsymbol{H}^{-1/2}\boldsymbol{L}_*\boldsymbol{H}^{-1/2}\right)^{\dagger}\left(\boldsymbol{H}^{-1/2}\boldsymbol{L}_*\boldsymbol{H}^{-1/2}\right) &= \boldsymbol{I}_q - n^{-1}\,\boldsymbol{H}^{1/2}\boldsymbol{\iota}_q\boldsymbol{\iota}_q'\boldsymbol{H}^{1/2}, \\
\left(\boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{L}_*\boldsymbol{H}^{-1}\boldsymbol{P}\right)^{\dagger}\left(\boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{L}_*\boldsymbol{H}^{-1}\boldsymbol{P}\right) &= \boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{P} - n^{-1}\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n', \\
\left(\boldsymbol{L}_W + \boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{L}_*\boldsymbol{H}^{-1}\boldsymbol{P}\right)^{\dagger}\left(\boldsymbol{L}_W + \boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{L}_*\boldsymbol{H}^{-1}\boldsymbol{P}\right) &= \boldsymbol{I}_n - n^{-1}\,\boldsymbol{\iota}_n\boldsymbol{\iota}_n',
\end{aligned} \tag{S.4}$$

where on the right-hand side always appears the projector orthogonal to the null-space of the respective matrix, e.g. we have $\boldsymbol{L}_W\boldsymbol{M} = \boldsymbol{L}_W$. Using that $\left(\boldsymbol{H}^{-1/2}\boldsymbol{P}\right)^{\dagger} = \boldsymbol{P}'\boldsymbol{H}^{-1/2}$ and the definition of $\boldsymbol{L}_*^{\text{inv}}$ we find that

$$\left(\boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{L}_*\boldsymbol{H}^{-1}\boldsymbol{P}\right)^{\dagger} = \boldsymbol{P}'\boldsymbol{H}^{-1/2}\left(\boldsymbol{H}^{-1/2}\boldsymbol{L}_*\boldsymbol{H}^{-1/2}\right)^{\dagger}\boldsymbol{H}^{-1/2}\boldsymbol{P} = \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}.$$

Proof of Lemma 2. We derive the result for $\widehat{\boldsymbol{\beta}}$ first. By applying Theorem 1 to each $\mathcal{G}_r$ separately we obtain

$$\widehat{\boldsymbol{\beta}}_r \sim \mathcal{N}(\boldsymbol{\beta}_r,\ \sigma^2\,\boldsymbol{L}_r^{\dagger}),$$

for $r = 1, \ldots, q$. Note that we do not rule out $n_r = 1$ (i.e., $\mathcal{G}_r$ may be a graph with one vertex and no edges), but in this case we simply have $\widehat{\boldsymbol{\beta}}_r = \boldsymbol{\beta}_r = \boldsymbol{L}_r = \boldsymbol{L}_r^{\dagger} = \boldsymbol{0}$, so the result for $\widehat{\boldsymbol{\beta}}_r$ holds trivially. Independence of the errors u_{ij} across observations implies independence of $\widehat{\boldsymbol{\beta}}_r$ and $\widehat{\boldsymbol{\beta}}_s$ for all $r \neq s$. We thus find $\widehat{\boldsymbol{\beta}} \sim \mathcal{N}(\boldsymbol{\beta},\ \sigma^2\,\boldsymbol{L}_W^{\dagger})$, which is the result in the theorem.

Now turn to $\widehat{\boldsymbol{\gamma}}$. Analogous to the proof of Lemma 1 we can write the minimization problem for $\widehat{\boldsymbol{\gamma}}$ as

$$\widehat{\boldsymbol{\gamma}} = \arg\min_{\boldsymbol{g}\in\mathbb{R}^q}\ (\boldsymbol{y} - \boldsymbol{B}_W\boldsymbol{\beta} - \boldsymbol{B}_B\boldsymbol{P}'\boldsymbol{g})'(\boldsymbol{y} - \boldsymbol{B}_W\boldsymbol{\beta} - \boldsymbol{B}_B\boldsymbol{P}'\boldsymbol{g}) + \lambda\,(\boldsymbol{g}'\boldsymbol{P}\boldsymbol{\iota}_n)^2\,,$$

where $\lambda > 0$ is a Lagrange multiplier. Solving the corresponding first-order condition gives

$$\begin{aligned}
\widehat{\boldsymbol{\gamma}} &= (\boldsymbol{P}\boldsymbol{B}_B'\boldsymbol{B}_B\boldsymbol{P}' + \lambda\,\boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}')^{-1}\,\boldsymbol{P}\boldsymbol{B}_B'(\boldsymbol{y} - \boldsymbol{B}_W\boldsymbol{\beta}) \\
&= (\boldsymbol{P}\boldsymbol{B}_B'\boldsymbol{B}_B\boldsymbol{P}' + \lambda\,\boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}')^{-1}\left[(\boldsymbol{P}\boldsymbol{B}_B'\boldsymbol{B}_B\boldsymbol{P}' + \lambda\,\boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}')\,\boldsymbol{\gamma} + \boldsymbol{P}\boldsymbol{B}_B'\boldsymbol{u}\right], \\
&= \boldsymbol{\gamma} + (\boldsymbol{P}\boldsymbol{B}_B'\boldsymbol{B}_B\boldsymbol{P}' + \lambda\,\boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}')^{-1}\,\boldsymbol{P}\boldsymbol{B}_B'\boldsymbol{u},
\end{aligned}$$

where in the second step we used the model $\boldsymbol{y} = \boldsymbol{B}_W \boldsymbol{\beta} + \boldsymbol{B}_B \boldsymbol{P}' \boldsymbol{\gamma} + \boldsymbol{u}$, and we added a term proportional to λ in the square brackets, which is zero due to the normalization of $\boldsymbol{\gamma}$, which can be written as $\boldsymbol{\iota}_n' \boldsymbol{P}' \boldsymbol{\gamma} = 0$. Notice that $\boldsymbol{P}\boldsymbol{B}_B' \boldsymbol{B}_B \boldsymbol{P}' = \boldsymbol{P}\boldsymbol{L}_B \boldsymbol{P}' = \boldsymbol{L}_*$. However, compared to the proof of Lemma 1 the difficulty is that, here, the matrices $\boldsymbol{L}_*$ and $\boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}'$ do not commute. To resolve this problem we rewrite the last result as

$$\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma} = \boldsymbol{H}^{-1/2} \left(\boldsymbol{H}^{-1/2} \boldsymbol{L}_* \boldsymbol{H}^{-1/2} + \lambda\, \boldsymbol{H}^{-1/2} \boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}' \boldsymbol{H}^{-1/2} \right)^{-1} \boldsymbol{H}^{-1/2} \boldsymbol{P}\boldsymbol{B}_B' \boldsymbol{u}.$$

Now, the matrices $\boldsymbol{H}^{-1/2}\boldsymbol{L}_*\boldsymbol{H}^{-1/2}$ and $\boldsymbol{H}^{-1/2}\boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}'\boldsymbol{H}^{-1/2}$ commute, because the zero eigenvalue of $\boldsymbol{H}^{-1/2}\boldsymbol{L}_*\boldsymbol{H}^{-1/2}$ has multiplicity one (as we assume $\mathcal{G}_B$ to be connected) with eigenvector given by $\boldsymbol{H}^{-1/2}\boldsymbol{P}\boldsymbol{\iota}_n$, namely we have $\boldsymbol{L}_*\boldsymbol{H}^{-1}\boldsymbol{P}\boldsymbol{\iota}_n = \boldsymbol{L}_*\boldsymbol{\iota}_q = 0$. Here, we used $\boldsymbol{H}^{-1}\boldsymbol{P}\boldsymbol{\iota}_n = \boldsymbol{\iota}_q$, which follows from the definition of $\boldsymbol{H}$ and $\boldsymbol{P}$. We therefore have

$$\begin{aligned}\left(\boldsymbol{H}^{-1/2} \boldsymbol{L}_* \boldsymbol{H}^{-1/2} + \lambda\, \boldsymbol{H}^{-1/2} \boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}' \boldsymbol{H}^{-1/2} \right)^{-1} &= \left(\boldsymbol{H}^{-1/2} \boldsymbol{L}_* \boldsymbol{H}^{-1/2} \right)^{\dagger} + \frac{1}{\lambda} \left(\boldsymbol{H}^{-1/2} \boldsymbol{P}\boldsymbol{\iota}_n\boldsymbol{\iota}_n'\boldsymbol{P}' \boldsymbol{H}^{-1/2} \right)^{\dagger} \\ &= \boldsymbol{H}^{1/2} \boldsymbol{L}_*^{\text{inv}} \boldsymbol{H}^{1/2} + \frac{1}{\lambda\, n^2} \boldsymbol{H}^{1/2} \boldsymbol{\iota}_q \boldsymbol{\iota}_q' \boldsymbol{H}^{1/2},\end{aligned}$$

and the last term does not contribute to $\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$ because we have $\boldsymbol{\iota}_q'\boldsymbol{P}\boldsymbol{B}_B' = \boldsymbol{\iota}_n'\boldsymbol{B}_B' = 0$. We therefore have, independent from the choice of λ, that

$$\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma} = \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P}\boldsymbol{B}_B' \boldsymbol{u}.$$

Using $\mathbb{E}(\boldsymbol{u}\boldsymbol{u}') = \sigma^2 \boldsymbol{I}_m$ we thus find

$$\text{var}(\widehat{\boldsymbol{\gamma}}) = \sigma^2 \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P}\boldsymbol{B}_B' \boldsymbol{B}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} = \sigma^2 \boldsymbol{L}_*^{\text{inv}} \boldsymbol{L}_* \boldsymbol{L}_*^{\text{inv}} = \sigma^2 \boldsymbol{L}_*^{\text{inv}}.$$

Because $\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$ is a linear combination of the jointly normal errors it is also normally distributed, so we have $\widehat{\boldsymbol{\gamma}} \sim \mathcal{N}(\boldsymbol{\gamma},\ \sigma^2\, \boldsymbol{L}_*^{\text{inv}})$. This concludes the proof. ∎

PROOF OF THEOREM 7 (GRAPH PARTITIONING)

Throughout the proof we maintain the same notational conventions as for the proof of Lemma 2. Recall that the variance of $\widehat{\boldsymbol{\alpha}}$ is $\sigma^2\, \boldsymbol{L}^{\dagger}$. The variance of the infeasible estimator based on (5.4) equals $\sigma^2(\boldsymbol{L}_W^{\dagger} + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P})$. We show below that

$$-\boldsymbol{Z}_{\text{low}} - (\boldsymbol{Q} + \boldsymbol{Q}') \leq \boldsymbol{L}^{\dagger} - \left(\boldsymbol{L}_W^{\dagger} + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} \right) \leq \boldsymbol{Z}_{\text{up}} - (\boldsymbol{Q} + \boldsymbol{Q}'), \tag{S.5}$$

for matrices

$$\boldsymbol{Z}_{\text{low}} := \boldsymbol{L}_W^{\dagger}\boldsymbol{L}_B\boldsymbol{L}_W^{\dagger}, \qquad \boldsymbol{Z}_{\text{up}} := \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^{\dagger}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P},$$

and $\boldsymbol{Q} := \boldsymbol{L}_W^{\dagger}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}$. We also establish that

$$\boldsymbol{Z}_{\text{low}} \leq \kappa\boldsymbol{L}_W^{\dagger}, \qquad \boldsymbol{Z}_{\text{up}} \leq \kappa\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} = \kappa\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}, \tag{S.6}$$

and that

$$|\boldsymbol{v}'\boldsymbol{Q}\boldsymbol{v}| \leq \kappa^{1/2}\left(\boldsymbol{v}'\boldsymbol{L}_W^{\dagger}\boldsymbol{v}\right)^{1/2}\left(\boldsymbol{v}'\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{v}\right)^{1/2}, \tag{S.7}$$

for any $\boldsymbol{v} \in \mathbb{R}^n$. Combining these results yields that, for any $\boldsymbol{v} \in \mathbb{R}^n$,

$$\begin{aligned} &-\kappa\,\boldsymbol{v}'\boldsymbol{L}_W^{\dagger}\boldsymbol{v} - 2\kappa^{1/2}\left[\left(\boldsymbol{v}'\boldsymbol{L}_W^{\dagger}\boldsymbol{v}\right)\left(\boldsymbol{v}'\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\,\boldsymbol{v}\right)\right]^{1/2} \\ &\qquad \leq \boldsymbol{v}'\left(\boldsymbol{L}^{\dagger} - \boldsymbol{L}_W^{\dagger} - \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right)\boldsymbol{v} \leq \\ &\qquad\qquad \kappa\,\boldsymbol{v}'\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\,\boldsymbol{v} + 2\kappa^{1/2}\left[\left(\boldsymbol{v}'\boldsymbol{L}_W^{\dagger}\boldsymbol{v}\right)\left(\boldsymbol{v}'\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\,\boldsymbol{v}\right)\right]^{1/2}. \end{aligned}$$

By Lemma 2 this is the result of Theorem 7. It remains only to show (S.5), (S.6), and (S.7), which we do, in turn, next. ■

Proof of (S.5). Start with the upper bound. Because $\boldsymbol{L}_B \geq 0$ and $\boldsymbol{L}_W \geq 0$, it holds that

$$\begin{aligned} 0 \leq &\left(\boldsymbol{L}^{\dagger} - \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right)\boldsymbol{L}_B\left(\boldsymbol{L}^{\dagger} - \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right) \\ &+ \left(\boldsymbol{L}^{\dagger} - \boldsymbol{L}_W^{\dagger} + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^{\dagger}\right)\boldsymbol{L}_W\left(\boldsymbol{L}^{\dagger} - \boldsymbol{L}_W^{\dagger} + \boldsymbol{L}_W^{\dagger}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right). \end{aligned}$$

Expanding those terms, and using that $\boldsymbol{L}_W + \boldsymbol{L}_B = \boldsymbol{L}$, and $\boldsymbol{L}_W^{\dagger}\boldsymbol{L}_W = \boldsymbol{M}$, and $\boldsymbol{L}_W^{\dagger}\boldsymbol{L}_W\boldsymbol{L}_W^{\dagger} = \boldsymbol{L}_W^{\dagger}$, we obtain

$$\begin{aligned} 0 \leq\ &\boldsymbol{L}^{\dagger}\boldsymbol{L}\boldsymbol{L}^{\dagger} + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} - \boldsymbol{L}^{\dagger}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} - \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}^{\dagger} \\ &- \boldsymbol{M}\boldsymbol{L}^{\dagger} - \boldsymbol{L}^{\dagger}\boldsymbol{M} + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{M}\boldsymbol{L}^{\dagger} + \boldsymbol{L}^{\dagger}\boldsymbol{M}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} \\ &+ \boldsymbol{L}_W^{\dagger} - \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^{\dagger} - \boldsymbol{L}_W^{\dagger}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^{\dagger}\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}. \end{aligned}$$

Using that $\boldsymbol{L}^\dagger \boldsymbol{L} \boldsymbol{L}^\dagger = \boldsymbol{L}^\dagger$, and $\boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} = \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P}$, and

$$\begin{aligned} -\boldsymbol{L}^\dagger \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} + \boldsymbol{L}^\dagger \boldsymbol{M} \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} &= -\boldsymbol{L}^\dagger \left(\boldsymbol{I}_n - \boldsymbol{M} \right) \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} \\ &= -\boldsymbol{L}^\dagger \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{P} \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} \\ &= -\boldsymbol{L}^\dagger \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{P}, \end{aligned}$$

and also the transpose of the last result, we obtain

$$\begin{aligned} 0 \leq \boldsymbol{L}^\dagger + \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} - \left(\boldsymbol{M} + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{P} \right) \boldsymbol{L}^\dagger - \boldsymbol{L}^\dagger \left(\boldsymbol{M} + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{P} \right) \\ + \boldsymbol{L}_W^\dagger - \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} \boldsymbol{L}_B \boldsymbol{L}_W^\dagger - \boldsymbol{L}_W^\dagger \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} + \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} \boldsymbol{L}_B \boldsymbol{L}_W^\dagger \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P}. \end{aligned}$$

Because $\boldsymbol{M} + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{P} = \boldsymbol{I}_n$ we thus find

$$\boldsymbol{L}^\dagger \leq \boldsymbol{L}_W^\dagger + \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} - \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} \boldsymbol{L}_B \boldsymbol{L}_W^\dagger - \boldsymbol{L}_W^\dagger \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} + \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} \boldsymbol{L}_B \boldsymbol{L}_W^\dagger \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P},$$

which is the upper bound given in the lemma.

Now turn to the lower bound. Introduce

$$\boldsymbol{\Delta} := \boldsymbol{M} \boldsymbol{L}_B \boldsymbol{M} + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{P} \boldsymbol{L}_B \boldsymbol{M} + \boldsymbol{M} \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{P}.$$

We then have

$$\boldsymbol{L} = \boldsymbol{L}_W + \boldsymbol{L}_B = \boldsymbol{L}_W + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{L}_* \boldsymbol{H}^{-1} \boldsymbol{P} + \boldsymbol{\Delta}.$$

Plugging this in the equality $\boldsymbol{L} \boldsymbol{L}^\dagger = \boldsymbol{I}_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'$ we obtain

$$\left(\boldsymbol{L}_W + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{L}_* \boldsymbol{H}^{-1} \boldsymbol{P} + \boldsymbol{\Delta} \right) \boldsymbol{L}^\dagger = \boldsymbol{I}_n - n^{-1} \, \boldsymbol{\iota}_n \boldsymbol{\iota}_n'.$$

Bringing $\boldsymbol{\Delta} \boldsymbol{L}^\dagger$ to the right-hand side, multiplying with $(\boldsymbol{L}_W + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{L}_* \boldsymbol{H}^{-1} \boldsymbol{P})^\dagger$ from the left, and using the last equality in (S.4), we obtain

$$\left(\boldsymbol{I}_n - n^{-1} \, \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \right) \boldsymbol{L}^\dagger = \left(\boldsymbol{L}_W + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{L}_* \boldsymbol{H}^{-1} \boldsymbol{P} \right)^\dagger \left(\boldsymbol{I}_n - \boldsymbol{\Delta} \boldsymbol{L}^\dagger - n^{-1} \, \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \right). \tag{S.8}$$

The matrices $\boldsymbol{L}_W$ and $\boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{L}_* \boldsymbol{H}^{-1} \boldsymbol{P}$ commute, and we therefore have

$$\left(\boldsymbol{L}_W + \boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{L}_* \boldsymbol{H}^{-1} \boldsymbol{P} \right)^\dagger = \boldsymbol{L}_W^\dagger + \left(\boldsymbol{P}' \boldsymbol{H}^{-1} \boldsymbol{L}_* \boldsymbol{H}^{-1} \boldsymbol{P} \right)^\dagger = \boldsymbol{L}_W^\dagger + \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P}.$$

Using this as well as $\boldsymbol{L}^\dagger \boldsymbol{\iota}_n = 0$ and $\left(\boldsymbol{L}_W + \boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{L}_*\boldsymbol{H}^{-1}\boldsymbol{P}\right)^\dagger \boldsymbol{\iota}_n = 0$ we find that the equation in (S.8) becomes

$$\boldsymbol{L}^\dagger = \boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} - \left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right)\boldsymbol{\Delta}\boldsymbol{L}^\dagger. \tag{S.9}$$

Taking the transpose of this last equation gives

$$\boldsymbol{L}^\dagger = \boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} - \boldsymbol{L}^\dagger\boldsymbol{\Delta}\left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right).$$

Replacing $\boldsymbol{L}^\dagger$ on the right-hand side of the last equation by the expression for $\boldsymbol{L}^\dagger$ in (S.9) we get

$$\boldsymbol{L}^\dagger = \boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} - \left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right)\left(\boldsymbol{\Delta} - \boldsymbol{\Delta}\boldsymbol{L}^\dagger\boldsymbol{\Delta}\right)\left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right).$$

Using the definition of $\boldsymbol{\Delta}$ we obtain

$$\boldsymbol{\Delta}\left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right) = \boldsymbol{M}\boldsymbol{L}_B\left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right) + \boldsymbol{P}'\boldsymbol{H}^{-1}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^\dagger,$$

and the last result on $\boldsymbol{L}^\dagger$ can therefore be rewritten as

$$\begin{aligned}\boldsymbol{L}^\dagger - \Big(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} - \boldsymbol{L}_W^\dagger\boldsymbol{L}_B\boldsymbol{L}_W^\dagger - \boldsymbol{L}_W^\dagger\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} - \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^\dagger\Big) \\ = \left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right)\boldsymbol{\Delta}\boldsymbol{L}^\dagger\boldsymbol{\Delta}\left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\right).\end{aligned}$$

Because $\boldsymbol{L}^\dagger \geq 0$ the last expression is positive semi-definite, which gives the lower bound on $\boldsymbol{L}^\dagger$ in the lemma. This concludes the proof. ■

Proof of (S.6). Let

$$\boldsymbol{\Lambda} := \text{diag}(\lambda_2^r : i \in \boldsymbol{V}, \text{ with } r \text{ such that } i \in V_r),$$

where we set $\lambda_2^r = 0$ if $n_r = 1$. Then $\boldsymbol{L}_W^\dagger \leq (\boldsymbol{\Lambda}\boldsymbol{D}_W)^\dagger$.[1] Also define the symmetrically normalized Laplacian of $\mathcal{G}_B$[2]

$$\boldsymbol{S}_B := \left(\boldsymbol{D}_B^\dagger\right)^{1/2}\boldsymbol{L}_B\left(\boldsymbol{D}_B^\dagger\right)^{1/2}.$$

[1]The diagonal matrix $\boldsymbol{\Lambda}\boldsymbol{D}_W$ has non-negative elements but may be non-invertible as, for $n_r = 1$, we have $\lambda_2^r d_i^W = 0$, with $i \in V_r$. We therefore write $(\boldsymbol{\Lambda}\boldsymbol{D}_W)^\dagger$ instead of just $(\boldsymbol{\Lambda}\boldsymbol{D}_W)^{-1}$.

[2]Again we write $\boldsymbol{D}_B^\dagger$ because we may have $d_i^B = 0$ for some $i \in V$.

From Chung (1997, Lemma 1.7) we know $\lambda_n(\boldsymbol{S}_B) \leq 2$, which can also be written as $\boldsymbol{S}_B \leq 2\boldsymbol{I}_n$. We have $\boldsymbol{L}_B = \boldsymbol{D}_B^{1/2}\boldsymbol{S}_B\boldsymbol{D}_B^{1/2}$, and thus find $\boldsymbol{L}_B \leq 2\boldsymbol{D}_B$.

The diagonal matrix $\boldsymbol{D}_B^{1/2}\left(\boldsymbol{\Lambda}\boldsymbol{D}_W\right)^\dagger\boldsymbol{D}_B^{1/2}$ has ith diagonal element equal to $d_i^B/(\lambda_2^r\, d_i^W)$ for $n_r > 1$, $i \in V_r$, and equal to zero otherwise. From the definition of κ in the main text we thus find

$$\boldsymbol{D}_B^{1/2}\left(\boldsymbol{\Lambda}\boldsymbol{D}_W\right)^\dagger\boldsymbol{D}_B^{1/2} \leq \frac{\kappa}{2}\boldsymbol{I}_n,$$

and therefore

$$\boldsymbol{D}_B^{1/2}\boldsymbol{L}_W^\dagger\boldsymbol{D}_B^{1/2} \leq \boldsymbol{D}_B^{1/2}\left(\boldsymbol{\Lambda}\boldsymbol{D}_W\right)^\dagger\boldsymbol{D}_B^{1/2} \leq \frac{\kappa}{2}\boldsymbol{I}_n. \tag{S.10}$$

The matrix $\boldsymbol{D}_B^{1/2}\boldsymbol{L}_W^\dagger\boldsymbol{D}_B^{1/2}$ is similar to $\left(\boldsymbol{L}_W^\dagger\right)^{1/2}\boldsymbol{D}_B\left(\boldsymbol{L}_W^\dagger\right)^{1/2}$, and so they share the same eigenvalues.[3] We therefore have that

$$\left(\boldsymbol{L}_W^\dagger\right)^{1/2}\boldsymbol{D}_B\left(\boldsymbol{L}_W^\dagger\right)^{1/2} \leq \frac{\kappa}{2}\boldsymbol{I}_n \tag{S.11}$$

holds.

Using the inequalities $\boldsymbol{S}_B \leq 2\boldsymbol{I}_n$ and $\boldsymbol{L}_B \leq 2\boldsymbol{D}_B$ along with (S.10) and (S.11) we obtain

$$\begin{aligned} \boldsymbol{L}_W^\dagger\boldsymbol{L}_B\boldsymbol{L}_W^\dagger &\leq 2\boldsymbol{L}_W^\dagger\boldsymbol{D}_B\boldsymbol{L}_W^\dagger \\ &= 2\left(\boldsymbol{L}_W^\dagger\right)^{1/2}\left(\boldsymbol{L}_W^\dagger\right)^{1/2}\boldsymbol{D}_B\left(\boldsymbol{L}_W^\dagger\right)^{1/2}\left(\boldsymbol{L}_W^\dagger\right)^{1/2} \\ &\leq \kappa\left(\boldsymbol{L}_W^\dagger\right)^{1/2}\left(\boldsymbol{L}_W^\dagger\right)^{1/2} \\ &\leq \kappa\boldsymbol{L}_W^\dagger, \end{aligned} \tag{S.12}$$

and

$$\begin{aligned} \boldsymbol{L}_B\boldsymbol{L}_W^\dagger\boldsymbol{L}_B &= \boldsymbol{D}_B^{1/2}\boldsymbol{S}_B^{1/2}\boldsymbol{S}_B^{1/2}\boldsymbol{D}_B^{1/2}\boldsymbol{L}_W^\dagger\boldsymbol{D}_B^{1/2}\boldsymbol{S}_B^{1/2}\boldsymbol{S}_B^{1/2}\boldsymbol{D}_B^{1/2} \\ &\leq \frac{\kappa}{2}\boldsymbol{D}_B^{1/2}\boldsymbol{S}_B^{1/2}\boldsymbol{S}_B\boldsymbol{S}_B^{1/2}\boldsymbol{D}_B^{1/2} \\ &\leq \kappa\boldsymbol{D}_B^{1/2}\boldsymbol{S}_B^{1/2}\boldsymbol{S}_B^{1/2}\boldsymbol{D}_B^{1/2} \\ &= \kappa\boldsymbol{L}_B. \end{aligned} \tag{S.13}$$

These yield the inequalities stated in (S.6). This concludes the proof. ■

[3]Two square matrices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are similar if there exists an invertible matrix $\boldsymbol{M}_3$ such that $\boldsymbol{M}_1 = \boldsymbol{M}_3^{-1}\boldsymbol{M}_2\boldsymbol{M}_3$. Two similar matrices have the same eigenvalues.

Proof of (S.7). Recall that

$$\boldsymbol{Q} = \boldsymbol{L}_W^{\dagger} \boldsymbol{L}_B \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P}.$$

Applying the Cauchy-Schwarz inequality $(\boldsymbol{x}'\boldsymbol{y})^2 \le (\boldsymbol{x}'\boldsymbol{x})(\boldsymbol{y}'\boldsymbol{y})$ with $\boldsymbol{x} = \boldsymbol{L}_B^{1/2} \boldsymbol{L}_W^{\dagger} \boldsymbol{v}$ and $\boldsymbol{y} = \boldsymbol{L}_B^{1/2} \boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P} \boldsymbol{v}$, and using (S.6) we find

$$|\boldsymbol{v}'\boldsymbol{Q}\boldsymbol{v}|^2 \le \kappa \left(\boldsymbol{v}' \boldsymbol{L}_W^{\dagger} \boldsymbol{v} \right) \left(\boldsymbol{v}' (\boldsymbol{P}' \boldsymbol{L}_*^{\text{inv}} \boldsymbol{P}) \boldsymbol{v} \right),$$

which gives (S.7). This concludes the proof. ■

PROOF OF THEOREM A.1 (SECOND-ORDER BOUND)

Proof of Theorem A.1. We start with the lower bound given in the theorem. Let $V_o := \{i\} \cup [i]$; then $n_o := |V_o| = 1 + d_i$. Without loss of generality we fix $i = 1$ and relabel the elements of V so that $V_o = \{1, 2, \ldots, 1 + d_i\}$. Let

$$\boldsymbol{L}_o := \begin{pmatrix} d_i & -\boldsymbol{\iota}'_{d_i} \\ -\boldsymbol{\iota}'_{d_i} & \boldsymbol{L}_{[i]} \end{pmatrix}, \qquad \boldsymbol{L}_{[i]} := \boldsymbol{D}_{[i]} - \boldsymbol{A}_{[i]},$$

using obvious notation for the $d_i \times d_i$ degree and adjacency matrices in the latter definition. Now, by the inversion formula for partitioned matrices,

$$\boldsymbol{L}_\circ^{-1} = \frac{1}{d_i - \boldsymbol{\iota}'_{d_i} \boldsymbol{L}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}} \begin{pmatrix} 1 & \boldsymbol{\iota}'_{d_i} \boldsymbol{L}_{[i]}^{-1} \\ \boldsymbol{L}_{[i]}^{-1} \boldsymbol{\iota}_{d_i} & \left[\frac{\boldsymbol{L}_{[i]} - d_i^{-1} \boldsymbol{\iota}_{d_i} \boldsymbol{\iota}'_{d_i}}{d_i - \boldsymbol{\iota}'_{d_i} \boldsymbol{L}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}} \right]^{-1} \end{pmatrix}.$$

Below we show that

$$0 \le \left\{ \text{var}(\widehat{\alpha}_i) - \frac{\sigma^2 \left[1 - \frac{2}{n} \left(1 + \boldsymbol{\iota}'_{d_i} \boldsymbol{L}_{[i]}^{-1} \boldsymbol{\iota}_{d_i} \right) \right]}{d_i - \boldsymbol{\iota}'_{d_i} \boldsymbol{L}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}} \right\} \le \frac{\sigma^2 \, \boldsymbol{\iota}'_{d_i} \boldsymbol{L}_{[i]}^{-1} (\boldsymbol{A}_{\circ\#}) \boldsymbol{D}_{\#}^{-1} (\boldsymbol{A}_{\circ\#})' \boldsymbol{L}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}}{\lambda_2 \left(d_i - \boldsymbol{\iota}'_{d_i} \boldsymbol{L}_{[i]}^{-1} \boldsymbol{\iota}_{d_i} \right)^2}, \tag{S.14}$$

where $\boldsymbol{L}_\circ$ is the upper left $n_\circ \times n_\circ$ block of $\boldsymbol{L}$, $\boldsymbol{A}_{\circ\#}$ is the upper right $n_\circ \times n_\#$ block of $\boldsymbol{A}$, and $\boldsymbol{D}_\#$ is the lower right $n_\# \times n_\#$ block of $\boldsymbol{D}$. To make further progress, note that the expansion

$$\boldsymbol{L}_{[i]}^{-1} = \sum_{q=0}^{\infty} \left(\boldsymbol{D}_{[i]}^{-1} \boldsymbol{A}_{[i]} \right)^q \boldsymbol{D}_{[i]}^{-1}$$

is convergent, because we have $\|\boldsymbol{D}_{[i]}^{-1}\boldsymbol{A}_{[i]}\|_\infty < 1$, where $\|.\|_\infty$ denotes the maximum absolute row sum matrix norm. We therefore have

$$\begin{aligned}\boldsymbol{\iota}'_{d_i}\boldsymbol{L}_{[i]}^{-1}\boldsymbol{\iota}_{d_i} &= \boldsymbol{\iota}'_{d_i}\boldsymbol{D}_{[i]}^{-1}\boldsymbol{\iota}_{d_i} + \boldsymbol{\iota}'_{d_i}\sum_{q=1}^{\infty}\left(\boldsymbol{D}_{[i]}^{-1}\boldsymbol{A}_{[i]}\right)^q\boldsymbol{D}_{[i]}^{-1}\boldsymbol{\iota}_{d_i}\\ &\geq \boldsymbol{\iota}'_{d_i}\boldsymbol{D}_{[i]}^{-1}\boldsymbol{\iota}_{d_i} = \sum_{j\in[i]} d_j^{-1}, \end{aligned} \tag{S.15}$$

where we used that $\boldsymbol{\iota}'_{d_i}\sum_{q=1}^{\infty}\left(\boldsymbol{D}_{[i]}^{-1}\boldsymbol{A}_{[i]}\right)^q\boldsymbol{D}_{[i]}^{-1}\boldsymbol{\iota}_{d_i} \geq 0$, because this is a product and sum of vector and matrices that all have non-negative entries. Define the $n_\circ \times n_\circ$ diagonal matrix $\underline{\boldsymbol{D}}_{[i]} = \operatorname{diag}(\underline{d}_{j,i} : j \in [i])$. We have

$$\boldsymbol{L}_{[i]} - \underline{\boldsymbol{D}}_{[i]} = \operatorname{diag}(\boldsymbol{A}_{[i]}\boldsymbol{\iota}_{d_i}) - \boldsymbol{A}_{[i]} \geq 0, \tag{S.16}$$

because $\operatorname{diag}(\boldsymbol{A}_{[i]}\boldsymbol{\iota}_{d_i}) - \boldsymbol{A}_{[i]}$ can be expressed as a sum of matrices of the form

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geq 0,$$

embedded into an $n_\circ \times n_\circ$ matrix. We therefore have $\boldsymbol{L}_{[i]}^{-1} \leq \underline{\boldsymbol{D}}_{[i]}^{-1}$, implying

$$\boldsymbol{\iota}'_{d_i}\boldsymbol{L}_{[i]}^{-1}\boldsymbol{\iota}_{d_i} \leq \boldsymbol{\iota}'_{d_i}\underline{\boldsymbol{D}}_{[i]}^{-1}\boldsymbol{\iota}_{d_i} = \sum_{j\in[i]} \underline{d}_{j,i}^{-1}. \tag{S.17}$$

Combining (S.14), (S.15) and (S.17) gives

$$\operatorname{var}(\widehat{\alpha}_i) \geq \frac{\sigma^2\left[1 - \frac{2}{n}\left(1 + \sum_{j\in[i]}\underline{d}_{j,i}^{-1}\right)\right]}{\sum_{j\in[i]}\left(1 - d_j^{-1}\right)} = \frac{\sigma^2}{d_i(1-h_i^{-1})}\left(1 - \frac{2}{n} - \frac{2}{n}\frac{d_i}{\underline{h}_i}\right),$$

which is the lower bound stated in the theorem.

To show the upper bound, consider the the graph $\widetilde{\mathcal{G}} := (V, \widetilde{E})$, with $\widetilde{E} := E \setminus [i] \times [i]$. That is, we construct $\widetilde{G}$ by deleting all edges between neighbors of i from $\mathcal{G}$. Note that $\widetilde{\mathcal{G}}$ is still connected, because all vertices in $[i]$ are connected through i. Let $\widetilde{\boldsymbol{\alpha}}$ be the estimator for $\boldsymbol{\alpha}$ obtained for $\widetilde{\mathcal{G}}$, in the same way that $\widehat{\boldsymbol{\alpha}}$ was obtained for $\mathcal{G}$. Let $\widetilde{\boldsymbol{L}}$ be the Laplacian matrix of $\widetilde{\mathcal{G}}$. Analogous to (S.16) we have $\widetilde{\boldsymbol{L}} \leq \boldsymbol{L}$, and therefore $\widetilde{\boldsymbol{L}}^\dagger \geq \boldsymbol{L}^\dagger$. The result (S.14) holds for any connected graph, and so can equally be applied to $\widetilde{\mathcal{G}}$, we only need to replace $\mathrm{var}(\widehat{\alpha}_i)$ by $\mathrm{var}(\widetilde{\alpha}_i)$ and $\boldsymbol{L}$ by $\widetilde{\boldsymbol{L}}$. The matrices $\boldsymbol{A}_{\circ\#}$ and $\boldsymbol{D}_{\#}^{-1}$ are identical for $\widetilde{\mathcal{G}}$ and $\mathcal{G}$. However, for $\widetilde{\mathcal{G}}$ we find $\widetilde{\boldsymbol{D}}_{[i]} = \underline{\boldsymbol{D}}_{[i]}$, because the degree of vertex j is given by $\underline{d}_{j,i}$, and we have $\widetilde{\boldsymbol{A}}_{[i]} = 0$, because there are no edges that connect elements in $[i]$. We thus have $\widetilde{\boldsymbol{L}}_{[i]} = \widetilde{\boldsymbol{D}}_{[i]} - \widetilde{\boldsymbol{A}}_{[i]} = \underline{\boldsymbol{D}}_{[i]}$. Therefore,

$$\mathrm{var}(\widehat{\alpha}_i) \le \mathrm{var}(\widetilde{\alpha}_i) \le \frac{\sigma^2 \left[1 - \frac{2}{n}\left(1 + \boldsymbol{\iota}_{d_i}' \underline{\boldsymbol{D}}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}\right)\right]}{d_i - \boldsymbol{\iota}_{d_i}' \underline{\boldsymbol{D}}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}} + \frac{\sigma^2\, \boldsymbol{\iota}_{d_i}' \underline{\boldsymbol{D}}_{[i]}^{-1} (\boldsymbol{A}_{\circ\#}) \boldsymbol{D}_{\#}^{-1} (\boldsymbol{A}_{\circ\#})' \underline{\boldsymbol{D}}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}}{\lambda_2 \left(d_i - \boldsymbol{\iota}_{d_i}' \underline{\boldsymbol{D}}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}\right)^2},$$

and evaluating the right-hand side of the last inequality gives the upper bound on $\mathrm{var}(\widehat{\alpha}_i)$ in the theorem. This concludes the proof. ■

Proof of (S.14). We prove the following more general result. Let $\mathcal{G}$ be connected. Choose $V_\circ \subset V$ with $0 < |V_\circ| < n$, and let $V_\# = V \setminus V_\circ$. Let $n_\circ = |V_\circ|$ and $n_\# = n - n_\circ$. Relabel the elements in V such that $V_\circ = \{1, 2, \ldots, n_\circ\}$. Let $\widehat{\boldsymbol{\alpha}}_\circ = (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{n_\circ})'$, $\boldsymbol{L}_\circ$ be the upper left $n_\circ \times n_\circ$ block of $\boldsymbol{L}$, $\boldsymbol{A}_{\circ\#}$ be the upper right $n_\circ \times n_\#$ block of $\boldsymbol{A}$, and $\boldsymbol{D}_\#$ be the lower right $n_\# \times n_\#$ block of $\boldsymbol{D}$. Then,

$$0 \le \left[\mathrm{var}(\widehat{\boldsymbol{\alpha}}_\circ) - \sigma^2 \left(\boldsymbol{L}_\circ^{-1} - \frac{\boldsymbol{\iota}_{n_\circ} \boldsymbol{\iota}_{n_\circ}' \boldsymbol{L}_\circ^{-1} + \boldsymbol{L}_\circ^{-1} \boldsymbol{\iota}_{n_\circ} \boldsymbol{\iota}_{n_\circ}'}{n}\right)\right] \le \frac{\sigma^2}{\lambda_2} \boldsymbol{L}_\circ^{-1} (\boldsymbol{A}_{\circ\#}) \boldsymbol{D}_\#^{-1} (\boldsymbol{A}_{\circ\#})' \boldsymbol{L}_\circ^{-1}$$

holds.

To show the result, define the $n \times n$ matrices

$$\boldsymbol{L}_{\mathrm{b}} := \begin{pmatrix} \boldsymbol{L}_\circ & 0 \\ 0 & \boldsymbol{L}_\# \end{pmatrix}, \qquad \boldsymbol{A}_{\mathrm{b}} := \begin{pmatrix} 0 & \boldsymbol{A}_{\circ\#} \\ (\boldsymbol{A}_{\circ\#})' & 0 \end{pmatrix},$$

with obvious definition of $\boldsymbol{L}_\#$ such that $\boldsymbol{L} = \boldsymbol{L}_{\mathrm{b}} - \boldsymbol{A}_{\mathrm{b}}$. Because the graph is connected the pseudo-inverse $\boldsymbol{L}^\dagger$ satisfies $\boldsymbol{L}^\dagger \boldsymbol{L} = \boldsymbol{I}_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'$. Plugging $\boldsymbol{L} = \boldsymbol{L}_{\mathrm{b}} - \boldsymbol{A}_{\mathrm{b}}$ into this expression we obtain

$$\boldsymbol{L}^\dagger = \boldsymbol{L}_{\mathrm{b}}^{-1} \left(\boldsymbol{I}_n + \boldsymbol{A}_{\mathrm{b}} \boldsymbol{L}^\dagger - n^{-1}\, \boldsymbol{\iota}_n \boldsymbol{\iota}_n'\right).$$

Using the transposed of this last equation to replace $\boldsymbol{L}^\dagger = (\boldsymbol{L}^\dagger)'$ on the right-hand side of that same equation we obtain

$$\begin{aligned} \boldsymbol{L}^\dagger &= \boldsymbol{L}_{\mathrm{b}}^{-1} + \boldsymbol{L}_{\mathrm{b}}^{-1} \boldsymbol{A}_{\mathrm{b}} \boldsymbol{L}_{\mathrm{b}}^{-1} + \boldsymbol{L}_{\mathrm{b}}^{-1} \boldsymbol{A}_{\mathrm{b}} \boldsymbol{L}^\dagger \boldsymbol{A}_{\mathrm{b}} \boldsymbol{L}_{\mathrm{b}}^{-1} - n^{-1}\, \boldsymbol{L}_{\mathrm{b}}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' - n^{-1}\, \boldsymbol{L}_{\mathrm{b}}^{-1} \boldsymbol{A}_{\mathrm{b}} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \boldsymbol{L}_{\mathrm{b}}^{-1} \\ &= \boldsymbol{L}_{\mathrm{b}}^{-1} + \boldsymbol{L}_{\mathrm{b}}^{-1} \boldsymbol{A}_{\mathrm{b}} \boldsymbol{L}_{\mathrm{b}}^{-1} + \boldsymbol{L}_{\mathrm{b}}^{-1} \boldsymbol{A}_{\mathrm{b}} \boldsymbol{L}^\dagger \boldsymbol{A}_{\mathrm{b}} \boldsymbol{L}_{\mathrm{b}}^{-1} - n^{-1}\, \boldsymbol{L}_{\mathrm{b}}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' - n^{-1}\, \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \boldsymbol{L}_{\mathrm{b}}^{-1}, \end{aligned}$$

where in the last step we have used that $\boldsymbol{L}_{\rm b}^{-1}\boldsymbol{A}_{\rm b}\boldsymbol{\iota}_n = \boldsymbol{\iota}_n$, which follows from $0 = \boldsymbol{L}\,\boldsymbol{\iota}_n = (\boldsymbol{L}_{\rm b} - \boldsymbol{A}_{\rm b})\boldsymbol{\iota}_n$. Evaluating the last result for the upper left $n_\circ \times n_\circ$ block gives

$$(\boldsymbol{L}^\dagger)_\circ = \boldsymbol{L}_\circ^{-1} + \boldsymbol{L}_\circ^{-1}(\boldsymbol{A}_{\circ\#})(\boldsymbol{L}^\dagger)_\#(\boldsymbol{A}_{\circ\#})'\boldsymbol{L}_\circ^{-1} - n^{-1}\,\boldsymbol{L}_\circ^{-1}\boldsymbol{\iota}_{n_\circ}\boldsymbol{\iota}_{n_\circ}' - n^{-1}\,\boldsymbol{\iota}_{n_\circ}\boldsymbol{\iota}_{n_\circ}'\boldsymbol{L}_\circ^{-1},$$

with obvious definition of $(\boldsymbol{L}^\dagger)_\#$. We obtain the result searched for for $\mathrm{var}(\widehat{\boldsymbol{\alpha}}_\circ) = \sigma^2(\boldsymbol{L}^\dagger)_\circ$ by also using $0 \leq (\boldsymbol{L}^\dagger)_\# \leq \lambda_2^{-1}\boldsymbol{D}_\#^{-1}$. This concludes the proof. ■

S.2 Component estimators from graph partitioning

Here we strenghten the result of Theorem 7 by showing that the estimator $\widehat{\boldsymbol{\alpha}}$ is close to the (infeasible) estimator $\widehat{\boldsymbol{\beta}} + \boldsymbol{P}'\widehat{\boldsymbol{\gamma}}$ when κ is small. We also provide a corresponding result for the feasible version $\widehat{\boldsymbol{\beta}} + \boldsymbol{P}'\widetilde{\boldsymbol{\gamma}}$, where

$$\widetilde{\boldsymbol{\gamma}} := \arg\min_{\boldsymbol{g}\in\mathbb{R}^q} \sum_{(i,j)\in E_B} \Big(y_{ij} - (\widehat{\beta}_i + g_{r(i)}) + (\widehat{\beta}_j + g_{r(j)})\Big)^2 \qquad \text{s.t.} \ \sum_{r=1}^{q} n_r\, g_r = 0.$$

Our focus in the main text is on the infeasible estimator. This is so because we use it as a devise to analyze the variance of $\widehat{\boldsymbol{\alpha}}$, and $\widehat{\boldsymbol{\gamma}}$ is independent of $\widehat{\boldsymbol{\beta}}$ while its feasible version is clearly not. If an alternative estimator to $\widehat{\boldsymbol{\alpha}}$ is desired, $\widehat{\boldsymbol{\beta}} + \boldsymbol{P}'\widetilde{\boldsymbol{\gamma}}$ will obviously be of interest. Note, however, that $\mathrm{var}(\widehat{\boldsymbol{\alpha}}_i) \leq \mathrm{var}(\widehat{\boldsymbol{\beta}} + \boldsymbol{P}'\widetilde{\boldsymbol{\gamma}})$ by the Gauss-Markov theorem (this, in fact, yields the upper bound given in (S.5)).

The following theorem is the main result of this section.

Theorem S.1. *Let $\mathcal{G}$ and $\mathcal{G}_1, \ldots, \mathcal{G}_q$ be connected. For $i \in V$ define $r_i, R_i \in \mathbb{R}$ by*

$$\widehat{\alpha}_i = \widehat{\beta}_i + \widetilde{\gamma}_{r(i)} + r_i, \qquad\qquad \widehat{\alpha}_i = \widehat{\beta}_i + \widehat{\gamma}_{r(i)} + r_i + R_i.$$

We then have

$$\mathbb{E}(r_i^2) \ \leq\ \kappa\ \Big[\mathrm{var}(\widehat{\beta}_i) + \mathrm{var}(\widehat{\gamma}_{r(i)})\Big], \qquad\qquad \mathbb{E}(R_i^2) \ \leq\ \kappa\,\mathrm{var}(\widehat{\gamma}_{r(i)}).$$

The theorem shows that, if κ is small, then the differences between $\widehat{\alpha}_i$ and $\widehat{\beta}_i + \widetilde{\gamma}_{r(i)}$, and between $\widehat{\alpha}_i$ and $\widehat{\beta}_i + \widehat{\gamma}_{r(i)}$, are both small compared to the stochastic variability of $\widehat{\beta}_i$ and $\widehat{\gamma}_{r(i)}$

themselves. Thus, the result of Theorem 7 generalizes from the variances to the estimators themselves.

The result (and its proof) also immediately extends to a setting as in Theorem 4, where the errors u_{ij} can be non-normal, heteroscedastic, or correlated. One only needs to replace $\mathrm{var}(\widehat{\beta}_i)$ by $\overline{\sigma}^2(\boldsymbol{L}_W^\dagger)_{ii}$ and $\mathrm{var}(\widehat{\gamma}_{r(i)})$ by $\overline{\sigma}^2(\boldsymbol{L}^{\mathrm{inv}})_{rr}$, where $\overline{\sigma}^2$ is a bound on the largest eigenvalue of $\mathbb{E}\left(\boldsymbol{u}\boldsymbol{u}'\right)$.

Proof of Theorem S.1. In vector notation the estimator decompositions reads

$$\widehat{\boldsymbol{\alpha}} = \widehat{\boldsymbol{\beta}} + \boldsymbol{P}'\widetilde{\boldsymbol{\gamma}} + \boldsymbol{r}, \qquad \widehat{\boldsymbol{\alpha}} = \widehat{\boldsymbol{\beta}} + \boldsymbol{P}'\widehat{\boldsymbol{\gamma}} + \boldsymbol{r} + \boldsymbol{R}.$$

Analogous to the proof of Lemma 1 and Lemma 2 above we can use the first-order conditions of their respective minimization problem to obtain explicit formulas for $\widehat{\boldsymbol{\beta}}$, $\widetilde{\boldsymbol{\gamma}}$ and $\widehat{\boldsymbol{\gamma}}$. We thus find

$$\boldsymbol{r} = \boldsymbol{C}_1\boldsymbol{Y}, \qquad \boldsymbol{C}_1 = \left(\boldsymbol{L}^\dagger\boldsymbol{B}' - \boldsymbol{L}_W^\dagger\boldsymbol{B}_W' - \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\boldsymbol{B}_B' + \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^\dagger\boldsymbol{B}_W'\right),$$

and

$$\boldsymbol{r} + \boldsymbol{R} = \boldsymbol{C}_2\boldsymbol{Y} + \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{\beta}, \qquad \boldsymbol{C}_2 = \left(\boldsymbol{L}^\dagger\boldsymbol{B}' - \boldsymbol{L}_W^\dagger\boldsymbol{B}_W' - \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\boldsymbol{B}_B'\right).$$

It is easy to verify that $\boldsymbol{C}_1\boldsymbol{B} = 0$ and $\boldsymbol{C}_2\boldsymbol{B}\boldsymbol{\alpha} + \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{\beta} = 0$, and therefore

$$\boldsymbol{r} = \boldsymbol{C}_1\boldsymbol{U}, \qquad \boldsymbol{r} + \boldsymbol{R} = \boldsymbol{C}_2\boldsymbol{U}.$$

Using this we find

$$\begin{aligned}
\sigma^{-2}\mathbb{E}\left(\boldsymbol{r}\boldsymbol{r}'\right) &= \boldsymbol{C}_1\boldsymbol{C}_1' \\
&= -\boldsymbol{L}^\dagger + \boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P} + \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^\dagger\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P} \\
&\qquad - \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^\dagger - \boldsymbol{L}_W^\dagger\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P} \\
&\leq \boldsymbol{L}_W^\dagger\boldsymbol{L}_B\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^\dagger\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P} \\
&\leq \kappa\left(\boldsymbol{L}_W^\dagger + \boldsymbol{P}'\boldsymbol{L}_*^{\mathrm{inv}}\boldsymbol{P}\right) = \kappa\left[\mathrm{var}\big(\widehat{\boldsymbol{\beta}}\big) + \mathrm{var}\left(\boldsymbol{P}'\widehat{\boldsymbol{\gamma}}\right)\right],
\end{aligned}$$

where in the second to last inequality we used the lower bound for $\boldsymbol{L}^\dagger$ in (S.5) above, and in the last inequality we used results from the proof of Theorem 7. We have thus shown the result for $\mathbb{E}(r_i^2)$ in the theorem. Similarly we find

$$\begin{aligned}\sigma^{-2}\mathbb{E}\left(\boldsymbol{R}\boldsymbol{R}'\right) &= (\boldsymbol{C}_1 - \boldsymbol{C}_2)(\boldsymbol{C}_1 - \boldsymbol{C}_2)' \\ &= \boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P}\boldsymbol{L}_B\boldsymbol{L}_W^\dagger\boldsymbol{L}_B\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} \\ &\leq \kappa\boldsymbol{P}'\boldsymbol{L}_*^{\text{inv}}\boldsymbol{P} \leq \kappa\ \text{var}\left(\boldsymbol{P}'\widehat{\boldsymbol{\gamma}}^{\text{inf}}\right),\end{aligned}$$

which implies the result for $\mathbb{E}(R_i^2)$ in the theorem. This concludes the proof. ■

References

Chung, F. R. K. (1997). *Spectral Graph Theory.* Volume 92 of CBMS Regional Conference Series in Mathematics, American Mathematical Society.