Supplemental Web Appendix to "Quantile regression with Panel Data" by Bryan S. Graham, Jinyong Hahn, Alexandre Poirier and James L. Powell: Proofs of Theorems 8, 9 and 10

This Supplemental Web Appendix contains proofs of Theorems 8, 9 and 10 that appear in Appendix A of the main paper. All notation is as established in, and equation numbering continues in sequence with that of, the main paper.

Proof of Theorem 8

Using arguments analogous to those used in Appendix B of the main paper we can show that $\hat{\delta}(\tau)-\delta(\tau)$ has the linear representation

$$\begin{aligned}\sqrt{N}\left(\hat{\delta}(\tau)-\delta(\tau)\right) &= \left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{W}_i^{*\prime}\mathbf{W}_i^{*}\mathbf{1}(D_i=0)\right)^{-1} \\ &\quad\times\frac{1}{N}\sum_{i=1}^{N}\mathbf{W}_i^{*\prime}\mathbf{X}_i^{*}\mathbf{1}(D_i=0)\sqrt{N}\left(\widehat{Q}_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{X}_i)-Q_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{X}_i)\right) \\ &= \left(\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{w}_l^{*}\hat{p}_l\right)^{-1}\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}\hat{p}_l\sqrt{N}\left(\widehat{Q}_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)-Q_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)\right)\end{aligned} \tag{103}$$

with

$$\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{w}_l^{*}\hat{p}_l \xrightarrow{p} \sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{w}_l^{*}p_l=\mathbb{E}\left[\mathbf{W}^{*\prime}\mathbf{W}^{*}|D=0\right]\pi_0$$

and

$$\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}\hat{p}_l\sqrt{N}\left(\widehat{Q}_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)-Q_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)\right)\xrightarrow{D}\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}\sqrt{p_l}\mathbf{Z}_Q(\tau,\mathbf{x}_l),$$

which has an asymptotic covariance equal to

$$\begin{aligned}&\mathbb{E}\left[\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}\sqrt{p_l}\mathbf{Z}_Q(\tau,\mathbf{x}_l)\left(\sum_{l'=L+1}^{M}\mathbf{w}_{l'}^{*\prime}\mathbf{x}_{l'}^{*}\sqrt{p_{l'}}\mathbf{Z}_Q(\tau',\mathbf{x}_{l'})\right)'\right] \\ &=\sum_{l=L+1}^{M}\sum_{l'=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}(\min(\tau,\tau')-\tau\tau')\Lambda(\tau,\tau';\mathbf{x}_l)\cdot\mathbf{1}(l=l')\,\mathbf{x}_l^{*\prime}\mathbf{w}_l^{*}p_lp_{l'} \\ &=(\min(\tau,\tau')-\tau\tau')\mathbb{E}\left[\mathbf{W}^{*\prime}\mathbf{X}^{*}\Lambda(\tau,\tau';\mathbf{X})\mathbf{X}^{*\prime}\mathbf{W}^{*}|D=0\right]\pi_0.\end{aligned}$$

To derive the asymptotic distribution of $\sqrt{N}\left(\widehat{\beta}(\cdot;\cdot)-\beta(\cdot;\cdot)\right)$ we note that

$$\begin{aligned}\sqrt{N}\left(\widehat{\beta}(\tau;\mathbf{x}_l)-\beta(\tau;\mathbf{x}_l)\right) &= \mathbf{x}_l^{-1}\sqrt{N}\left(\widehat{Q}_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)-Q_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)\right)-\mathbf{x}_l^{-1}\mathbf{w}_l\sqrt{N}\left(\hat{\delta}(\tau)-\delta(\tau)\right) \\ &\xrightarrow{D}\mathbf{x}_l^{-1}\mathbf{Z}_Q(\tau,\mathbf{x}_l)-\mathbf{x}_l^{-1}\mathbf{w}_l\mathbf{Z}_\delta(\tau).\end{aligned} \tag{104}$$

$\mathbf{Z}_Q(\tau,\mathbf{x}_l)$ and $\mathbf{Z}_\delta(\tau)$ are independent processes, since they are computed using disjoint subpopulations (i.e., $\mathbf{x}_l$ for $l=1,\ldots,L$ are not used in the computation of $\hat{\delta}(\tau)$). Therefore, the asymptotic variance of (104) is the sum of the variance of its component terms.

Proof of Theorem 9

Decomposing $\widehat{\bar{\beta}}^M(\tau)$ we see that

$$\sqrt{N}\left(\widehat{\bar{\beta}}^M(\tau)-\bar{\beta}^M(\tau)\right)=\sum_{l=1}^{L}\beta(\tau;\mathbf{x}_l)\sqrt{N}\left(\hat{q}_l^M-q_l^M\right) \tag{105}$$

$$+\sum_{l=1}^{L}\sqrt{N}\left(\hat{\beta}(\tau;\mathbf{x}_l)-\beta(\tau;\mathbf{x}_l)\right)\hat{q}_l^M. \tag{106}$$

By a result similar to that in (76) in the main text, term (105) converges to a mean zero Gaussian process with covariance equal to $\frac{\mathbb{C}\left(\beta(\tau,\mathbf{X}),\beta(\tau',\mathbf{X})|X\in\mathbb{X}^M\right)}{\Pr(\mathbf{X}\in\mathbb{X}^M)}$. Term ((106)) converges to

$$\sum_{l=1}^{L}\sqrt{N}\left(\hat{\beta}(\tau;\mathbf{x}_l)-\beta(\tau;\mathbf{x}_l)\right)\hat{q}_l^M\xrightarrow{D}\sum_{l=1}^{L}\mathbf{Z}(\tau,\mathbf{x}_l)q_l^M \tag{107}$$

which has a covariance kernel equal to

$$\begin{aligned}
&\mathbb{E}\left[\sum_{l=1}^{L}\sum_{l'=1}^{L}\mathbf{Z}(\tau,\mathbf{x}_l)\mathbf{Z}(\tau,\mathbf{x}_{l'})q_l^Mq_{l'}^M\right]\\
&=\mathbb{E}\left[\mathbf{Z}(\tau,\mathbf{x}_l)\mathbf{Z}(\tau,\mathbf{x}_{l'})\right]q_l^Mq_{l'}^M\\
&=\left(\min\left(\tau,\tau'\right)-\tau\tau'\right)\sum_{l=1}^{L}\sum_{l'=1}^{L}\frac{\mathbf{x}_l^{-1}\Lambda\left(\tau,\tau';\mathbf{x}_l\right)\mathbf{x}_l^{-1\prime}}{p_l}\cdot\mathbf{1}\left(l=l'\right)q_l^Mq_{l'}^M\\
&+\sum_{l=1}^{L}\sum_{l'=1}^{L}\mathbf{x}_l^{-1}\mathbf{w}_l\Sigma_\delta(\tau,\tau')\mathbf{w}_{l'}'\mathbf{x}_{l'}^{-1\prime}q_l^Mq_{l'}^M\\
&=\frac{\min\left(\tau,\tau'\right)-\tau\tau'}{\Pr\left(\mathbf{X}\in\mathbb{X}^M\right)}\mathbb{E}\left[\mathbf{X}^{-1}\Lambda(\tau,\tau',\mathbf{X})\mathbf{X}^{-1\prime}|\mathbf{X}\in\mathbb{X}^M\right]+\sum_{l=1}^{L}\mathbf{x}_l^{-1}\mathbf{w}_lq_l^M\Sigma_\delta(\tau,\tau')\sum_{l'=1}^{L}\mathbf{w}_{l'}'\mathbf{x}_{l'}^{-1\prime}q_{l'}^M\\
&=\Upsilon_1(\tau,\tau')+\Xi_0\Sigma_\delta(\tau,\tau')\Xi_0'.
\end{aligned}$$

Since terms (105) and (106) are uncorrelated, the asymptotic covariance of $\sqrt{N}\left(\widehat{\bar{\beta}}^M(\tau)-\bar{\beta}^M(\tau)\right)$ is equal to the sum of the covariances these two terms.

Proof of Theorem 10

We start by deriving the asymptotic distribution of the sample cumulative distribution function of $\widehat{\beta}_p(U;\mathbf{X})$ with U distributed uniformly on $[0,1]$ independently from $\mathbf{X}$, while conditioning on $\mathbf{X}\in\mathbb{X}^M$. The CDF estimand at $c \in \mathbb{R}$ is denoted as $F_{B_p|\mathbf{X}\in\mathbb{X}^M}(c)$ with estimator

$$\begin{aligned}\widehat{F}_{\widehat{\beta}_p(U;\mathbf{X})|\mathbf{X}\in\mathbb{X}^M}(c) &= \frac{\frac{1}{N}\sum_{i=1}^N \left[\int_0^1 \mathbf{1}(\widehat{\beta}_p(u,\mathbf{X}_i) \le c)\mathrm{d}u\right] \mathbf{1}(\mathbf{X}_i \in \mathbb{X}^M)}{\frac{1}{N}\sum_{i=1}^N \mathbf{1}(\mathbf{X}_i \in \mathbb{X}^M)} \\ &= \sum_{l=1}^L \left(\int_0^1 \mathbf{1}(\widehat{\beta}_p(u,\mathbf{x}_l) \le c)\mathrm{d}u\right) \widehat{q}_l^M. \end{aligned} \tag{108}$$

The integration over $u \in (0,1)$ can be done exactly since $\widehat{\beta}_p(u,\mathbf{x}_l)$ is piecewise linear for each $l \in \{1,\dots,L\}$ with finitely many pieces. We can express $\widehat{F}_{\widehat{\beta}_p(U;\mathbf{X})|\mathbf{X}\in\mathbb{X}^M}(c)$ as the sum of two terms:

$$\widehat{F}_{\widehat{\beta}_p(U;\mathbf{X})|\mathbf{X}\in\mathbb{X}^M}(c) - F_{B_p|\mathbf{X}\in\mathbb{X}^M}(c) = \sum_{l=1}^L \left(\left[\int_0^1 \mathbf{1}(\widehat{\beta}_p(u,\mathbf{x}_l) \le c)\mathrm{d}u\right] - \left[\int_0^1 \mathbf{1}(\beta_p(u,\mathbf{x}_l) \le c)\mathrm{d}u\right]\right)\widehat{q}_l^M \tag{109}$$

$$+ \sum_{l=1}^L \left[\int_0^1 \mathbf{1}(\beta_p(u,\mathbf{x}_l) \le c)\mathrm{d}u\right]\left(\widehat{q}_l^M - q_l^M\right). \tag{110}$$

We will show that these two terms both converge in uniformly over $c \in \mathbb{R}$. For term (109), we have that $\sqrt{N}\left(\hat{\beta}_p(\tau;\mathbf{x}_l) - \hat{\beta}_p(\tau;\mathbf{x}_l)\right) \xrightarrow{D} \left(\mathbf{Z}(\tau,\mathbf{x}_l)\right)_p = \mathbf{Z}_p(\tau,\mathbf{x}_l)$ over $\tau \in (0,1)$ and all $l = 1,\dots,L$, and $(\cdot)_p$ denotes the p^{th} element of the vector. By the same argument use to show (79) in Appendix B, we have

$$\begin{aligned}&\sqrt{N}\left(\int_0^1 \mathbf{1}(\hat{\beta}_p(u;\mathbf{x}_l) \le c)\mathrm{d}u - \int_0^1 \mathbf{1}(\beta_p(u;\mathbf{x}_l) \le c)\mathrm{d}u\right) \\ &= \sqrt{N}\left(\hat{\beta}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l);\mathbf{x}_l) - \hat{\beta}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l);\mathbf{x}_l)\right) f_{B_p|\mathbf{X}}(c|\mathbf{x}_l) + o_p(1) \\ &\xrightarrow{D} \mathbf{Z}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l),\mathbf{x}_l) f_{B_p|\mathbf{X}}(c|\mathbf{x}_l).\end{aligned}$$

This convergence is uniform in $c \in \mathbb{R}$ since $F_{B_p|\mathbf{X}}(c|\mathbf{x}_l)$ ranges between 0 and 1, and uniform in $\mathbf{x}_l$ since its support is finite. Therefore,

$$\sum_{l=1}^L \sqrt{N}\left(\int_0^1 \mathbf{1}(\hat{\beta}_p(u;\mathbf{x}_l) \le c)\mathrm{d}u - \int_0^1 \mathbf{1}(\beta_p(u;\mathbf{x}_l) \le c)\mathrm{d}u\right)\hat{q}_l^M \xrightarrow{D} \sum_{l=1}^L \mathbf{Z}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l),\mathbf{x}_l) f_{B_p|\mathbf{X}}(c|\mathbf{x}_l) q_l^M \tag{111}$$

for $c \in \mathbb{R}$. Also, (110) will converge over $c \in \mathbb{R}$ to a Gaussian process $\mathbf{Z}_{2p}(c)$ with asymptotic covariance of

$$\mathbb{E}\left[\mathbf{Z}_{2p}(c)\mathbf{Z}_{2p}(c')'\right] = \frac{\mathbb{C}\left(F_{B_p|\mathbf{X}}(c|\mathbf{X}), F_{B_p|\mathbf{X}}(c'|\mathbf{X})|\mathbf{X} \in \mathbb{X}^M\right)}{\Pr\left(\mathbf{X} \in \mathbb{X}^M\right)}.$$

Note that $\mathbf{Z}_{2p}(c)$ and $\sum_{l=1}^L \mathbf{Z}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l),\mathbf{x}_l) f_{B_p|\mathbf{X}}(c|\mathbf{x}_l) q_l^M$ are uncorrelated since the variation in the latter is conditional on $\mathbf{X}$ while that in the former depends on $\mathbf{X}$ only. Therefore,

$$\sqrt{N}\left(\widehat{F}_{\widehat{\beta}_p(U;\mathbf{X})|\mathbf{X}\in\mathbb{X}^M}(c) - F_{B_p|\mathbf{X}\in\mathbb{X}^M}(c)\right) \xrightarrow{D} \sum_{l=1}^L \mathbf{Z}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l),\mathbf{x}_l) f_{B_p|\mathbf{X}}(c|\mathbf{x}_l) q_l^M + \mathbf{Z}_{2p}(c)$$

for $c \in \mathbb{R}$.

Using the same invertibility argument as in (82), we see that

$$\sqrt{N}\left(\hat{\beta}_p^M(\tau)-\beta_p^M(\tau)\right)\overset{D}{\to}\frac{\sum_{l=1}^{L}\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l),\mathbf{x}_l)f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l)q_l^M+\mathbf{Z}_{2p}(\beta_p^M(\tau))}{f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau))}$$
$$=\mathbf{Z}_{\beta_p}(\tau) \tag{112}$$

uniformly over $\tau\in(0,1)$.

To conclude this proof, we evaluate $\mathbb{E}\left[\mathbf{Z}_{\beta_p}(\tau)\,\mathbf{Z}_{\beta_p}(\tau')'\right]$, the asymptotic covariance of ((112)):

$$\begin{aligned}\mathbb{E}\left[\mathbf{Z}_{\beta_p}(\tau)\,\mathbf{Z}_{\beta_p}(\tau')'\right]&=\frac{\sum_{l=1}^{L}\sum_{l'=1}^{L}f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l)f_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'})q_l^Mq_{l'}^M}{f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau))f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau'))}\\&\times\mathbb{E}\left[\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l),\mathbf{x}_l)\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}),\mathbf{x}_{l'})\right]\\&+\frac{\mathbb{E}\left[\mathbf{Z}_{2p}(\beta_p^M(\tau))\mathbf{Z}_{2p}(\beta_p^M(\tau'))\right]}{f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau))f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau'))},\end{aligned}$$

where

$$\begin{aligned}&\mathbb{E}\left[\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l),\mathbf{x}_l)\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}),\mathbf{x}_{l'})\right]\\&=\left(\min\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l),F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'})\right)-F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l)F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'})\right)\\&\times e_p'\frac{\mathbf{x}_l^{-1}\Lambda\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l),F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'});\mathbf{x}_l\right)\mathbf{x}_l^{-1\prime}}{p_l}e_p\cdot\mathbf{1}\left(l=l'\right)\\&+e_p'\mathbf{x}_l^{-1}\mathbf{w}_l\Sigma_\delta(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l),F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}))\mathbf{w}_{l'}'\mathbf{x}_{l'}^{-1\prime}e_p,\end{aligned}$$

and

$$\begin{aligned}&\sum_{l=1}^{L}\sum_{l'=1}^{L}f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l)q_l^M\mathbb{E}\left[\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l),\mathbf{x}_l)\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}),\mathbf{x}_{l'})\right]f_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'})q_{l'}^M\\&=\mathbb{E}\Bigg[\frac{\left(\min\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}),F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X})\right)-F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X})F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X})\right)}{\Pr\left(\mathbf{X}\in\mathbb{X}^M\right)}\\&\times e_p'\;\mathbf{X}^{-1}\Lambda\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}),F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X});\mathbf{X}\right)\mathbf{X}^{-1\prime}e_pf_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X})f_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X})\Big|\,\mathbf{X}\in\mathbb{X}^M\Bigg]\\&+e_p'\mathbb{E}\Big[f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X})f_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\tilde{\mathbf{X}})\mathbf{X}^{-1}\mathbf{W}\Sigma_\delta(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}),F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\tilde{\mathbf{X}}))\\&\times\tilde{\mathbf{W}}'\tilde{\mathbf{X}}^{-1\prime}|\mathbf{X}\in\mathbb{X}^M,\tilde{\mathbf{X}}\in\mathbb{X}^M\Big]e_p\\&=\Upsilon_3(\tau,\tau')+\Upsilon_4(\tau,\tau'),\end{aligned} \tag{113}$$

with $\tilde{\mathbf{X}}$ is an independent copy of $\mathbf{X}$. Finally,

$$\begin{aligned}\mathbb{E}\left[\mathbf{Z}_{2p}(\beta_p^M(\tau))\mathbf{Z}_{2p}(\beta_p^M(\tau'))\right]&=\frac{\mathbb{C}\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}),F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X})|\mathbf{X}\in\mathbb{X}^M\right)}{\Pr\left(\mathbf{X}\in\mathbb{X}^M\right)}\\&=\Upsilon_2(\tau,\tau').\end{aligned}$$