Approximations to the Distribution of Conditional Moment Test Statisitics

Andrew Chesher
cemmap
and
University College London

Richard J. Smith
cemmap
I.F.S. and U.C.L.
and
University of Cambridge

Simon Peters
University of Manchester

Introduction
Objectives

Cross-Section/Short Panels.
Parametric Likelihood Framework.
Conditional Moment Tests.

- $O(T^{-1})$ Edgeworth-type approximations to OPG CM test distribution.

Introduction
Objectives

Cross-Section/Short Panels.
Parametric Likelihood Framework.
Conditional Moment Tests.

- $O(T^{-1})$ Edgeworth-type approximations to OPG CM test distribution.
- Analytic evidence of poor nominal chi-square asymptotic null distribution.

Introduction
Objectives

Cross-Section/Short Panels.
Parametric Likelihood Framework.
Conditional Moment Tests.

- $O(T^{-1})$ Edgeworth-type approximations to OPG CM test distribution.
- Analytic evidence of poor nominal chi-square asymptotic null distribution.
- Analytical comparison with FE CM test.

Introduction
Objectives

Cross-Section/Short Panels.
Parametric Likelihood Framework.
Conditional Moment Tests.

- $O(T^{-1})$ Edgeworth-type approximations to OPG CM test distribution.
- Analytic evidence of poor nominal chi-square asymptotic null distribution.
- Analytical comparison with FE CM test.
- Simulation experiments.

Introduction
Example

$X \sim N(\mu, \sigma^2)$. σ^2 known.

Invariance. Set $\sigma^2 = 1$.

Null hypothesis $H_0 : \mu = 0$.

FE Score Test Statistic.

$$T_{FE} = Tm_1^2.$$

OPG Score Test Statistic.

$$T_{OPG} = Tm_1^2 / m_2.$$

REMARK. m_i ith sample moment about the hypothesised mean.

Introduction
Example (Cont.)

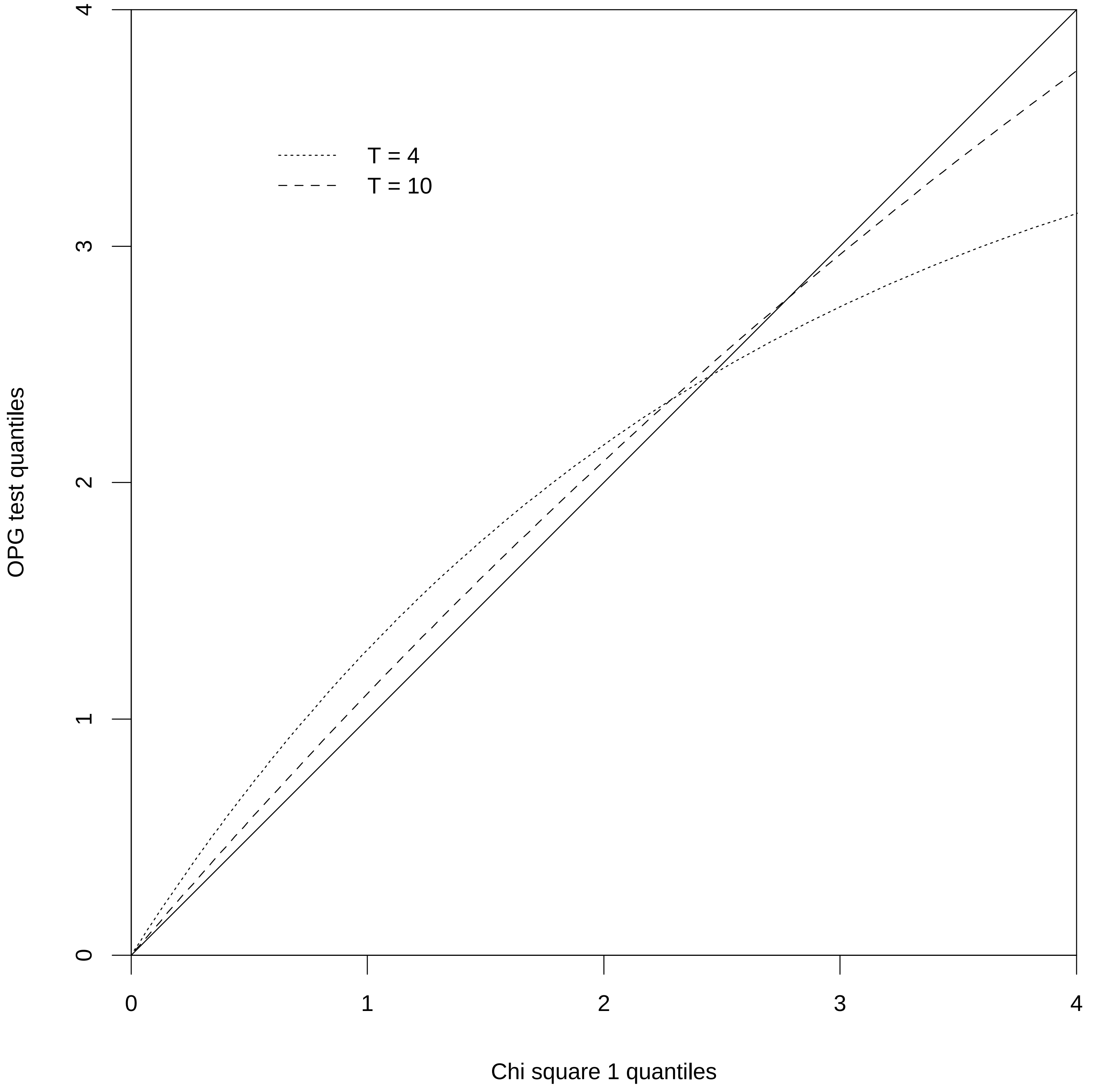
REMARK. T_{FE} exact $\chi^2(1)$.

REMARK. T_{OPG} exact $T\mathcal{B}(\frac{1}{2}, \frac{T-1}{2})$.

Exact and $O(T^{-2})$ approximate cumulants.

Cumulant	Exact	$O(T^{-2})$
κ_1	1	1
κ_2	$2 - \frac{6}{T+2}$	$2 - \frac{6}{T} + \frac{12}{T^2}$
κ_3	$8 - \frac{24(3T+2)}{(T+4)(T+2)}$	$8 - \frac{72}{T} + \frac{384}{T^2}$
κ_4	$48 - \frac{432(2T^3+8T^2+13T+12)}{(T+6)(T+4)(T+2)^2}$	$48 - \frac{864}{T} + \frac{8640}{T^2}$

$X \sim N(\mu, 1), H_0 : \mu = 0$



T = 4
T = 10
OPG test quantiles
0
1
2
3
4
Chi square 1 quantiles

Introduction
Outline

- CM tests as LM tests for additional conditional moments.

Introduction
Outline

- CM tests as LM tests for additional conditional moments.
- Stochastic approximations.

Introduction
Outline

- CM tests as LM tests for additional conditional moments.
- Stochastic approximations.
- Edgeworth approximation.

Introduction
Outline

- CM tests as LM tests for additional conditional moments.
- Stochastic approximations.
- Edgeworth approximation.
- Censored regression model. Omitted variables.

Conditional Moment Tests
Definitions

Data vector $x^T = (x_1, ..., x_T)$.

Independent sampling.

Density function $f_t(\cdot, \theta_0)$.

Fixed design conditional on z_t, $(t = 1, ..., T)$, $(T = 1, 2, ...)$.

Conditional Moment Tests
Asymptotics

x sampled conditional on z.

Fixed in repeated samples.

Define $T = dR$.

d: fixed d-point design in z.
R: number of design replications.
$R \to \infty$. Thus $T \to \infty$.

Conditional Moment Tests
MLE

$$\hat{\theta}_T = \arg\max_{\theta \in \Theta} \sum_{t=1}^{T} \log f_t(x_t, \theta).$$

Conditional Moment Tests
Misspecification Test

Test validity of

$$f^T(x^T, \theta_0) = \prod_{t=1}^{T} f_t(x_t, \theta_0).$$

Finite number of *implied* moment conditions.

$$E_{\theta_0}[m_t^i(x_t, \theta_0)] = 0, (i = 1, ..., m).$$

$E_{\theta_0}[\cdot]$: expectation w.r.t. $f^T(x^T, \theta_0)$.

REMARK: pure significance test.

Conditional Moment Tests

Chesher-Smith (1997)

Carrier function $h(\cdot)$.

REMARK: scalar; $h(w) > 0$, $dh(w)/dw > 0$; $h(0) = dh(0)/dw = 1$.

Let $m_t(x_t, \theta) = (m_t^1(x_t, \theta), ..., m_t^m(x_t, \theta))'$, $(t = 1, ..., T)$.

Augmented density function

$$r_t(x_t, \varphi) = f_t(x_t, \theta)h(\lambda' m_t(x_t, \theta))/q_t(\varphi), (t = 1, ..., T).$$

REMARK: $\varphi = (\theta', \lambda')'$. $q_t(\varphi) = E_\theta[h(\lambda' m_t(x_t, \theta))]$.

Conditional Moment Tests
Chesher-Smith (1997) (Cont.)

$$\lambda = 0 \Leftrightarrow E_{\theta_0}[m_t(x_t, \theta_0)] = 0.$$

Score/LM test statistic for $\lambda = 0$ identical to CM test based on

$$E_{\theta_0}[m_t(x_t, \theta_0)] = 0.$$

Conditional Moment Tests

CM Test Statistic

r-vector $y_t(\theta)$.

$r = p + m$.

First p elements $\partial \log f_t(x_t, \theta)/\partial\theta$.
Last m elements $m_t(x_t, \theta)$.

Let $\hat{y}_T = T^{-1/2} \sum_{t=1}^{T} y_t(\hat{\theta}_T)$.

CM test statistic

$$\mathcal{S}_T = \hat{y}_T' \hat{G}_T^{-1} \hat{y}_T.$$

Conditional Moment Tests

CM Test Statistic (Cont.)

Under $f^T(x^T, \theta_0)$, $(T = 1, 2, ...)$,

$$\hat{G}_T \xrightarrow{p} G_0 = \lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} E_{\theta_0}[y_t(\theta_0) y_t(\theta_0)'].$$

G_0 positive definite.

Under $f^T(x^T, \theta_0)$, $(T = 1, 2, ...)$,

$$\mathcal{S}_T \xrightarrow{d} \chi^2_m.$$

REMARK: $\mathcal{S}_T$ invariant to $h(\cdot)$ choice.

REMARK: Approximations invariant to $h(\cdot)$ choice.

Conditional Moment Tests

CM Test Statistic (Cont.)

Choice of $\hat{G}_T$.

- Fully efficient (FE).

$$\hat{G}_T = T^{-1} \sum_{t=1}^{T} E_{\hat{\theta}_T}[y_t(\theta) y_t(\theta)'].$$

Conditional Moment Tests

CM Test Statistic (Cont.)

Choice of $\hat{G}_T$.

- Fully efficient (FE).

$$\hat{G}_T = T^{-1} \sum_{t=1}^{T} E_{\hat{\theta}_T}[y_t(\theta) y_t(\theta)'].$$

- Outer-Product of Gradients (OPG).

$$\hat{G}_T = T^{-1} \sum_{t=1}^{T} y_t(\hat{\theta}_T) y_t(\hat{\theta}_T)'.$$

REMARK: $T\mathcal{R}^2$ from $1 = y_t(\hat{\theta}_T)'\xi + v_t$, $(t = 1, ..., T)$.

REMARK: Classical score/LM statistics also included.

Stochastic Approximation
Derivatives

Require first four derivatives of $y_t(\theta)$.

Let

$$y_i^t = \frac{\partial \log r_t(x_t, \varphi_0)}{\partial \varphi_i}, \quad y_{ij}^t = \frac{\partial^2 \log r_t(x_t, \varphi_0)}{\partial \varphi_i \partial \varphi_j},$$

$$y_{ijk}^t = \frac{\partial^3 \log r_t(x_t, \varphi_0)}{\partial \varphi_i \partial \varphi_j \partial \varphi_k}, \quad y_{ijkl}^t = \frac{\partial^4 \log r_t(x_t, \varphi_0)}{\partial \varphi_i \partial \varphi_j \partial \varphi_k \partial \varphi_l}.$$

$(i, j, k, l = 1, ..., r)$, $(t = 1, ..., T)$.

Stochastic Approximation

Derivatives (Cont.)

Define

$$y_i = T^{-1/2}\sum_{t=1}^{T} y_i^t, \quad y_{ij} = T^{-1}\sum_{t=1}^{T} y_{ij}^t,$$
$$y_{ijk} = T^{-3/2}\sum_{t=1}^{T} y_{ijk}^t, \quad y_{ijkl} = T^{-2}\sum_{t=1}^{T} y_{ijkl}^t$$

$(i, j, k, l = 1, ..., r)$, $(T = 1, 2...)$.

REMARK: Cf. usual LR and W stochastic expansion components.

Stochastic Approximation
Conditional Cumulants

Conditional cumulants.

$$\kappa^t_{ij} = E_{\theta_0}[y^t_{ij}], \quad \kappa^t_{i,j} = E_{\theta_0}[y^t_i y^t_j],$$

$$\kappa^t_{ijk} = E_{\theta_0}[y^t_{ijk}], \quad \kappa^t_{i,jk} = E_{\theta_0}[y^t_i y^t_{jk}], \quad \kappa^t_{i,j,k} = E_{\theta_0}[y^t_i y^t_j y^t_k],$$

$$\kappa^t_{ijkl} = E_{\theta_0}[y^t_{ijkl}], \quad \kappa^t_{i,jkl} = E_{\theta_0}[y^t_i y^t_{jkl}],$$

$$\kappa^t_{ij,kl} = E_{\theta_0}[y^t_{ij} y^t_{kl}] - \kappa^t_{ij}\kappa^t_{kl}, \quad \kappa^t_{i,j,kl} = E_{\theta_0}[y^t_i y^t_j y^t_{kl}] - \kappa^t_{i,j}\kappa^t_{kl},$$

$$\kappa^t_{i,j,k,l} = E_{\theta_0}[y^t_i y^t_j y^t_k y^t_l] - \kappa^t_{i,j}\kappa^t_{k,l} - \kappa^t_{i,k}\kappa^t_{j,l} - \kappa^t_{i,l}\kappa^t_{j,k},$$

$(i, j, k, l = 1, ..., r)$, $(t = 1, ..., T)$.

Stochastic Approximation
Averaged Conditional Cumulants

Averaged conditional cumulants. E.g.

$$\kappa_{ij} = T^{-1} \sum_{t=1}^{T} \kappa_{ij}^{t}, \quad \kappa_{i,j} = T^{-1} \sum_{t=1}^{T} \kappa_{i,j}^{t}.$$

Other averaged conditional cumulants.

$$\kappa_{ijk}, \quad \kappa_{i,jk}, \quad \kappa_{i,j,k}, \quad \kappa_{ijkl},$$
$$\kappa_{i,jkl}, \quad \kappa_{ij,kl}, \quad \kappa_{i,j,kl}, \quad \kappa_{i,j,k,l},$$

$(T = 1, 2, ...)$.

Stochastic Approximation
Additional Cumulants

Plus

Additional averaged cumulant

$$\kappa_{i,j;k,l} = T^{-1} \sum_{t=1}^{T} \kappa_{i,j}^{t} \kappa_{k,l}^{t}.$$

REMARK: peculiar to Edgeworth expansion of OPG CM form.

REMARK: arises because conditioning variables $\{z_t\}_{t=1}^{T}$ present.

Stochastic Approximation
Additional Derivatives

CM test statistic

$$\mathcal{S}_T = \hat{g}^{ij}\hat{y}_i\hat{y}_j.$$

Let

$$g_{ab} = [T^{-1}\sum_{t=1}^{T} y_t(\theta_0)y_t(\theta_0)']_{ab}.$$

Additional derivatives.

$$g_{ab,k} = T^{-1/2}\sum_{t=1}^{T}\frac{\partial g_{ab}}{\partial \varphi_k} = T^{-3/2}\sum_{t=1}^{T}(y_a^t y_{bk}^t + y_{ak}^t y_b^t).$$

$$\begin{aligned} g_{ab,kl} &= T^{-1}\sum_{t=1}^{T}\frac{\partial^2 g_{ab}}{\partial \varphi_k \partial \varphi_l} \\ &= T^{-2}\sum_{t=1}^{T}(y_{al}^t y_{bk}^t + y_{ak}^t y_{bl}^t + y_a^t y_{bkl}^t + y_{akl}^t y_b^t). \end{aligned}$$

Stochastic Approximation

Stochastic Expansions

Let $\Delta\hat{\varphi}_{Ti} = \hat{\theta}_{Ti} - \theta_{0i}$, $(i = 1, ..., p)$, and 0, $(i = p + 1, ..., r)$.

- $\hat{g}^{ij}$.

$$\begin{aligned}\hat{g}^{ij} \quad = \quad & g^{ij} - g^{ia}g^{jb}g_{ab,k}(T^{1/2}\Delta\hat{\varphi}_{Tk}) \\ & + g^{ia}g_{ab,k}(T^{1/2}\Delta\hat{\varphi}_{Tk})g^{bc}g_{cd,l}(T^{1/2}\Delta\hat{\varphi}_{Tl})g^{dj} \\ & - \frac{1}{2}g^{ia}g_{ab,kl}(T^{1/2}\Delta\hat{\varphi}_{Tk})(T^{1/2}\Delta\hat{\varphi}_{Tl})g^{bj} + O_p(T^{-3/2}),\end{aligned}$$

$(i, j = 1, ..., r)$.

Stochastic Approximation

Stochastic Expansions

Let $\Delta\hat{\varphi}_{Ti} = \hat{\theta}_{Ti} - \theta_{0i}$, $(i = 1, ..., p)$, and 0, $(i = p + 1, ..., r)$.

- $\hat{g}^{ij}$.

$$\begin{aligned}\hat{g}^{ij} &= g^{ij} - g^{ia}g^{jb}g_{ab,k}(T^{1/2}\Delta\hat{\varphi}_{Tk}) \\ &\quad + g^{ia}g_{ab,k}(T^{1/2}\Delta\hat{\varphi}_{Tk})g^{bc}g_{cd,l}(T^{1/2}\Delta\hat{\varphi}_{Tl})g^{dj} \\ &\quad - \frac{1}{2}g^{ia}g_{ab,kl}(T^{1/2}\Delta\hat{\varphi}_{Tk})(T^{1/2}\Delta\hat{\varphi}_{Tl})g^{bj} + O_p(T^{-3/2}),\end{aligned}$$

$(i, j = 1, ..., r)$.

- $\hat{y}_i$.

$$\begin{aligned}\hat{y}_i &= y_i + y_{ij}(T^{1/2}\Delta\hat{\varphi}_{Tj}) + \frac{1}{2}y_{ijk}(T^{1/2}\Delta\hat{\varphi}_{Tj})(T^{1/2}\Delta\hat{\varphi}_{Tk}) \\ &\quad + \frac{1}{6}y_{ijkl}(T^{1/2}\Delta\hat{\varphi}_{Tj})(T^{1/2}\Delta\hat{\varphi}_{Tk})(T^{1/2}\Delta\hat{\varphi}_{Tl}) + O_p(T^{-3/2}),\end{aligned}$$

$(i = 1, ..., r)$.

Stochastic Approximation

Stochastic Expansions (Cont.)

- $\Delta\hat{\varphi}_{Ti}$. Successive back-substitution in $\hat{y}_i = 0$, $(i = 1, ..., p)$.

$$\begin{aligned} T^{1/2}\Delta\hat{\varphi}_{Ti} &= -[c_{iw}y_w] - \frac{1}{2}c_{ij}y_{jkl}[c_{kw}y_w][c_{lw}y_w] \\ &\quad -\frac{1}{2}c_{ij}y_{jkl}[c_{kw}y_w]c_{lm}y_{mno}[c_{mw}y_w][c_{ow}y_w] \\ &\quad +\frac{1}{6}c_{ij}y_{jklm}[c_{kw}y_w][c_{lw}y_w][c_{mw}y_w] + O_p(T^{-3/2}); \end{aligned}$$

$(i = 1, ..., r)$.

$$\begin{aligned} c_{ij} &= y^{ij}, (i, j = 1, ..., p) \\ &= 0 \text{ otherwise.} \end{aligned}$$

Stochastic Approximation

Stochastic Expansions (Cont.)

REMARK.

$$y_{ijkl} - E_{\theta_0}[y_{ijkl}] = O_p(T^{-3/2}), \quad g_{ij,kl} - E_{\theta_0}[g_{ij,kl}] = O_p(T^{-3/2}),$$

$(i, j, k, l = 1, ...r)$.

Replace terms $\{y_{ijkl}\}$ and $\{g_{ij,kl}\}$ by

$$\begin{aligned} E_{\theta_0}[y_{ijkl}] &= T^{-1}\kappa_{ijkl}, \\ E_{\theta_0}[g_{ij,kl}] &= T^{-1}(\kappa_{ik,jl} + \kappa_{il,jk} + \kappa_{i,jkl} + \kappa_{j,ikl} + \kappa_{i,l;j,k} + \kappa_{i,k;j,l}). \end{aligned}$$

Validity of the stochastic approximation unaffected to $O_p(T^{-3/2})$.
Likewise Edgeworth-type expansion to $o(T^{-1})$.

REMARK. Stochastic expansion for OPG CM form $\mathcal{S}_T$ involves $\{\kappa_{ij;kl}\}$.
Cf. FE CM statistic.

Stochastic Approximation

Stochastic Expansions (Cont.)

Combine stochastic expansions for $\hat{g}^{ij}$, $\hat{y}_i$ and $\Delta\hat{\varphi}_{Ti}$.

$$\mathcal{S}_T = O_p(1) + O_p(T^{-1/2}) + O_p(T^{-1}) + o_p(T^{-1}).$$

REMARK. Remainder $O_p(T^{-3/2})$.

Edgeworth Approximation

Edgeworth Series Distribution

Require Edgeworth density to $O(T^{-3/2})$ for

$$y_i = T^{-1/2} \sum_t y_i^t, \quad y_{ij} = T^{-1} \sum_t y_{ij}^t, \quad y_{ijk} = T^{-3/2} \sum_t y_{ijk}^t.$$

Plus

$$g_{ij} = T^{-1} \sum_t g_{ij}^t, \quad g_{ij,k} = T^{-3/2} \sum_t g_{ij,k}^t.$$

REMARK.

$$g_{ij}^t = y_i^t y_j^t, \quad g_{ij,k}^t = \partial g_{ij}^t / \partial \varphi_k = y_i^t y_{jk}^t + y_{ik}^t y_j^t,$$

$(t = 1, ..., T)$, $(T = 1, 2, ...)$.

Edgeworth Approximation

Edgeworth Series Distribution (Cont.)

Edgeworth series distribution $h(\{y_i\},\{y_{ij}\},\{y_{ijk}\},\{g_{ij}\},\{g_{ij,k}\})$

$$\exp(\sum(-1)^{r+s+t+u+v}(r,s,t,u,v)$$
$$\times\frac{(\partial/\partial y_i)^r}{r!}\frac{(\partial/\partial y_{jk})^s}{s!}\frac{(\partial/\partial y_{lmo})^t}{t!}\frac{(\partial/\partial g_{ij})^u}{u!}\frac{(\partial/\partial g_{kl,m})^v}{v!})$$
$$\times g(\{y_i\},\{y_{jk}\},\{y_{lmo}\},\{g_{ij}\},\{g_{kl,m}\}).$$

REMARK. Sum over $r+2s+3t+2u+3v\geq 3$.
Omit $t=1$ or $v=1$ when $r=s=u=0$.
REMARK. Truncate at $r+2s+3t+2u+3v=4$ for $O(T^{-3/2})$.
REMARK. $\kappa(r,s,t,u,v)$ cumulant associated with $(y_i)^r(y_{jk})^s(y_{lmo})^t(g_{ij})^u(g_{kl,m})^v$.

Edgeworth Approximation

Edgeworth Series Distribution (Cont.)

$$
\begin{aligned}
& g(\{y_i\}, \{y_{jk}\}, \{y_{lmo}\}, \{g_{ij}\}, \{g_{ij,k}\}) \\
&= (2\pi)^{-r/2} (\det \kappa_{i,j})^{-1/2} \exp\{-(1/2)\kappa^{i,j} y_i y_j\} \\
&\times \prod_{j,k} \delta(y_{jk} - \kappa_{jk}) \prod_{l,m,o} \delta(y_{lmo} - T^{-1/2}\kappa_{lmo}) \\
&\times \prod_{i,j} \delta(g_{ij} - \kappa_{i,j}) \prod_{k,l,m} \delta(g_{kl,m} - T^{-1/2}(\kappa_{k,lm} + \kappa_{l,km})).
\end{aligned}
$$

Dirac delta function $\delta(\cdot)$.
E.g. $\delta(y_{jk} - \kappa_{jk}) = 0$ if $y_{jk} \neq \kappa_{jk}$ and 1 if $y_{jk} = \kappa_{jk}$.

Edgeworth Approximation

Characteristic Function

Let $q = 2\imath s/(1 - 2\imath s)$ and $\imath = \sqrt{-1}$.

Characteristic function.

$$\begin{aligned} \varphi_{\mathcal{S}_T}(s) &= \int \ldots \int \exp(\imath s \mathcal{S}_T) h(\{y_i\}, \{y_{ij}\}, \{y_{ijk}\}, \{g_{ij}\}, \{g_{ij,k}\}) \\ &\quad \times dy_i dy_{ij} dy_{ijk} dg_{ij} dg_{ij,k} + O(T^{-3/2}). \end{aligned}$$

REMARK. Integrate w.r.t $\{y_{ij}\}$, $\{y_{ijk}\}$, $\{g_{ij}\}$, $\{g_{ij,k}\}$. Then $\{y_i\}$.

Let

$$\begin{aligned} a_{ij} &= \kappa^{i,j}, (i, j = 1, ..., p) \\ &= 0 \text{ otherwise.} \end{aligned}$$

and

$$m_{ij} = \kappa^{i,j} - a_{ij}, (i, j = 1, ..., r).$$

Edgeworth Approximation

Characteristic Function (Cont.)

Characteristic function.

$$\varphi_{\mathcal{S}_T}(s) = (1 - 2\imath s)^{-\frac{m}{2}} \left(1 + \frac{1}{24T}(\mathcal{B}_1 q + \mathcal{B}_2 q^2 + \mathcal{B}_3 q^3)\right) + O(T^{-3/2}).$$

Edgeworth Approximation

Characteristic Function (Cont.)

$$
\begin{aligned}
\mathcal{B}_1 &= 3(\kappa_{ijk}+2\kappa_{i,jk})a_{ij}m_{kp}(\kappa_{pqr}+2\kappa_{pq,r})a_{qr}-12(\kappa_{ijk}+2\kappa_{i,jk})a_{ij}m_{kp}\kappa_{p,q,r}m_{qr}\\
&\quad +12\kappa_{i,j,k}m_{ij}\kappa^{k,p}\kappa_{p,q,r}m_{qr}+12\kappa_{i,j,k,l}a_{ij}m_{kl}-6\kappa_{ij,k}a_{ip}a_{jq}m_{kr}\kappa_{pq,r}\\
&\quad -6\kappa_{i,j,k}a_{ip}a_{jq}m_{kr}\kappa_{p,q,r}+12\kappa_{i,j,k}m_{ip}m_{jq}\kappa^{k,r}\kappa_{p,q,r}\\
&\quad +24\kappa_{i,j;k,l}a_{ik}m_{jl}+12\kappa_{i,j;k,l}a_{ij}m_{kl},\\
\mathcal{B}_2 &= 24\kappa_{i,j,k}m_{ij}m_{kp}\kappa_{p,q,r}m_{qr}-12(\kappa_{ijk}+2\kappa_{i,jk})a_{ij}m_{kp}\kappa_{p,q,r}m_{qr}-6\kappa_{i,j,k,l}m_{ij}m_{kl}\\
&\quad +12\kappa_{i,j,k}m_{ij}a_{kp}\kappa_{p,q,r}m_{qr}+24\kappa_{i,j,k}m_{ip}m_{jq}\kappa^{k,r}\kappa_{p,q,r}\\
&\quad -6\kappa_{i,j;k,l}m_{ij}m_{kl}-12\kappa_{i,j;k,l}m_{ik}m_{jl},\\
\mathcal{B}_3 &= 12\kappa_{i,j,k}m_{ij}m_{kp}\kappa_{p,q,r}m_{qr}+8\kappa_{i,j,k}m_{ip}m_{jq}m_{kr}\kappa_{p,q,r}.
\end{aligned}
$$

REMARK. i.i.d. case

$$
\begin{aligned}
\mathcal{B}_1 &: \kappa_{i,j;k,l}a_{ij}m_{kl}=mp;\\
\mathcal{B}_2 &: \kappa_{i,j;k,l}m_{ik}m_{jl}=\frac{1}{2}m(m+2).
\end{aligned}
$$

Edgeworth Approximation
Characteristic Function (Cont.)

FE CM Test.

Replace $\{\mathcal{B}_i\}$by

$$\begin{aligned}
\mathcal{A}_1 &= 3(\kappa_{ijk} + 2\kappa_{i,jk})a_{ij}m_{kp}(\kappa_{pqr} + 2\kappa_{pq,r})a_{qr} - 6(\kappa_{ijk} + 2\kappa_{i,jk})a_{ij}a_{kp}\kappa_{p,q,r}m_{qr} \\
&\quad +6(\kappa_{i,jk} - \kappa_{i,j,k})m_{ip}a_{jq}a_{kr}(\kappa_{pqr} + 2\kappa_{pq,r}) - 6(\kappa_{i,j,k,l} + \kappa_{i,j,kl})a_{ij}m_{kl}, \\
\mathcal{A}_2 &= -3\kappa_{i,j,k}m_{ij}a_{kp}\kappa_{p,q,r}m_{qr} + 6(\kappa_{ijk} + 2\kappa_{i,jk})a_{ij}m_{kp}\kappa_{p,q,r}m_{qr} + 3\kappa_{i,j,k,l}m_{ij}m_{kl} \\
&\quad -6\kappa_{i,j,k}m_{ip}m_{jq}a_{kr}\kappa_{p,q,r}, \\
\mathcal{A}_3 &= 3\kappa_{i,j,k}m_{ij}m_{kp}\kappa_{p,q,r}m_{qr} + 2\kappa_{i,j,k}m_{ip}m_{jq}m_{kr}\kappa_{p,q,r}.
\end{aligned}$$

REMARK. Typically coefficients of $\{\mathcal{B}_i\}$ cumulants greater than $\{\mathcal{A}_i\}$. E.g. $\mathcal{B}_3 = 4\mathcal{A}_3$. Affects $O(n^{-1})$ skewness/kurtosis expressions. Deviations from $\chi^2(m)$ greater for OPG than FE.

Edgeworth Approximation
Cumulants

$O(T^{-1})$ OPG CM cumulants.

$$\begin{aligned}
\kappa_1^{\mathcal{S}_T} &= m + \frac{1}{12T}\mathcal{B}_1, \\
\kappa_2^{\mathcal{S}_T} &= 2m + \frac{1}{3T}(\mathcal{B}_1 + \mathcal{B}_2), \\
\kappa_3^{\mathcal{S}_T} &= 8m + \frac{2}{T}(\mathcal{B}_1 + 2\mathcal{B}_2 + \mathcal{B}_3), \\
\kappa_4^{\mathcal{S}_T} &= 48m + \frac{16}{T}(\mathcal{B}_1 + 3\mathcal{B}_2 + 3\mathcal{B}_3)
\end{aligned}$$

REMARK. Remainders $O(T^{-3/2})$.

REMARK. First term $\chi^2(m)$.

Edgeworth Approximation
C.D.F.

Let $F_{\mathcal{S}_T}(x) = \mathcal{P}\{\mathcal{S}_T \leq x\}$.

Fourier inversion.

$$\begin{aligned} F_{\mathcal{S}_T}(x) &= P_m(x) + \frac{1}{24T}[(\mathcal{B}_2 - \mathcal{B}_1 - \mathcal{B}_3)P_m(x) + (3\mathcal{B}_3 - 2\mathcal{B}_2 + \mathcal{B}_1)P_{m+2}(x) \\ &\quad +(\mathcal{B}_2 - 3\mathcal{B}_3)P_{m+4}(x) + \mathcal{B}_3 P_{m+6}(x)] + O(T^{-3/2}). \end{aligned}$$

REMARK. $\mathcal{P}_r(x) = \mathcal{P}\{\chi^2(r) \leq x\}$.

Edgeworth Approximation
C.D.F. (Cont.)

Alternative formulation.

$$F_{\mathcal{S}_T}(x) = P_m(x) - \frac{1}{12T}\beta(x)p_m(x) + O(T^{-3/2}).$$

REMARK. $p_m(\cdot)$ p.d.f. of $\chi^2(m)$.

REMARK. $\beta(x) = \sum_{i=1}^{3} \beta_i x^i$ and

$$\beta_1 = \frac{(\mathcal{B}_3 - \mathcal{B}_2 + \mathcal{B}_1)}{m}, \beta_2 = \frac{(\mathcal{B}_2 - 2\mathcal{B}_3)}{m(m+2)}, \beta_3 = \frac{\mathcal{B}_3}{m(m+2)(m+4)}.$$

REMARK. Enables investigation of monotonicity properties.

Edgeworth Approximation

C.D.F. (Cont.)

Also

$$F_{\mathcal{S}_T}(x) = P_m\left(x - \frac{1}{12T}\beta(x)\right) + O(T^{-3/2}).$$

REMARK. Basis for Cornish-Fisher inverse expansion.

Let c_α denote $O(1)$ approximating $\chi^2(m)$ α-quantile.

Corresponding CM statistic $\mathcal{S}_T$ critical value c^*_α to $O(T^{-3/2})$.

$$c^*_\alpha - \frac{1}{12T}\beta(c^*_\alpha) = c_\alpha$$

Approximated by

$$c^*_\alpha = c_\alpha + \frac{1}{12T}\beta(c_\alpha).$$

Censored Regression

Design

Censored regression

$$y = \max[0, N(\beta_0 + \beta_1 z, \sigma^2)].$$

REMARK. Set $\sigma^2 = 1$.
Intercept β_0 adjusted for expected 25% or 50% censoring.

REMARK. Slope β_1 set for expected latent regression model $\mathcal{R}^2$ of 0.75.

Censored Regression

Covariate Design

Sample size $d = 50$.

- Design 1. Expected $U(0,1)$ order statistics.
- Design 2. Expected $N(0,1)$ order statistics.
- Design 3. Expected log normal $\Lambda(0, 0.3)$ order statistics.
- Design 4. Replace Design 1 lowest and highest values by -1 and $+2$.

REMARK. Designs 1, 2 and 4 symmetric. Design 3 asymmetric.
REMARK. Designs 3 and 4 points of relatively high leverage.

Censored Regression
Omitted Variable Design

- W_1: squared Design 2 covariate.

Censored Regression

Omitted Variable Design

- W_1: squared Design 2 covariate.
- W_2: $i^2/50$, $(i = 1, ..., 50)$.

Censored Regression

Simulation Experiments

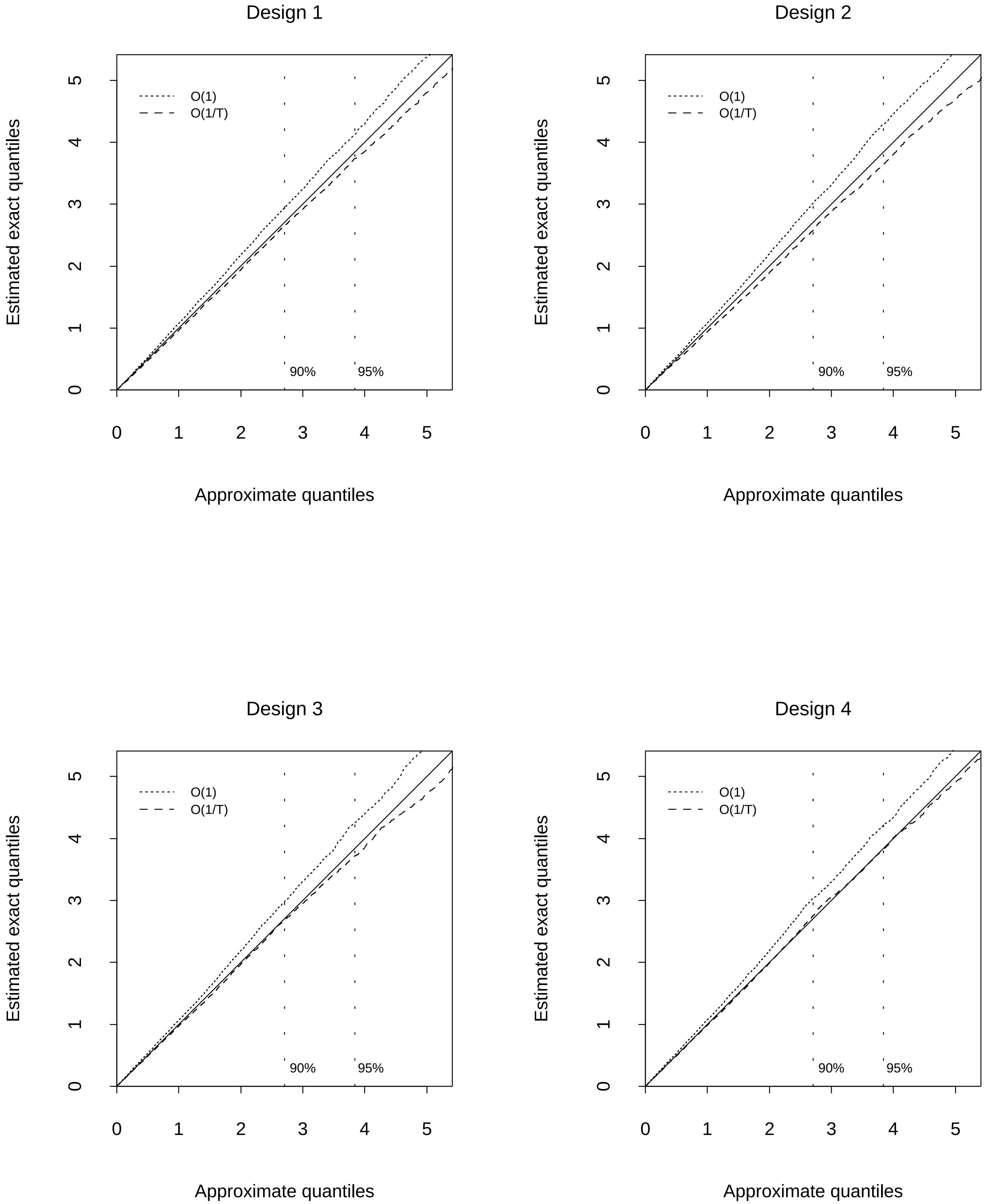
10000 replications.

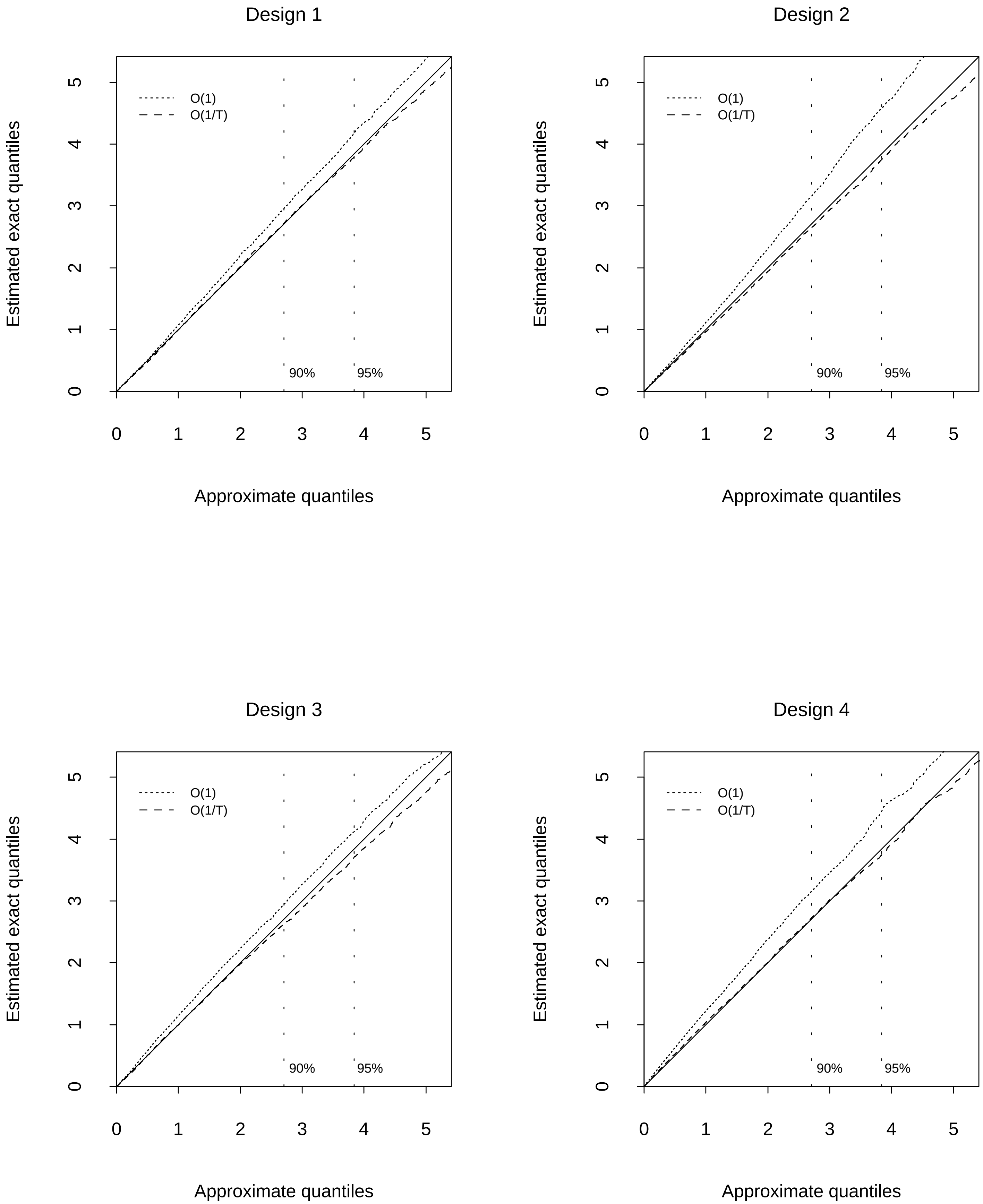
Design replication. $R = 5$ and 10.

$T = 250$ and 500.

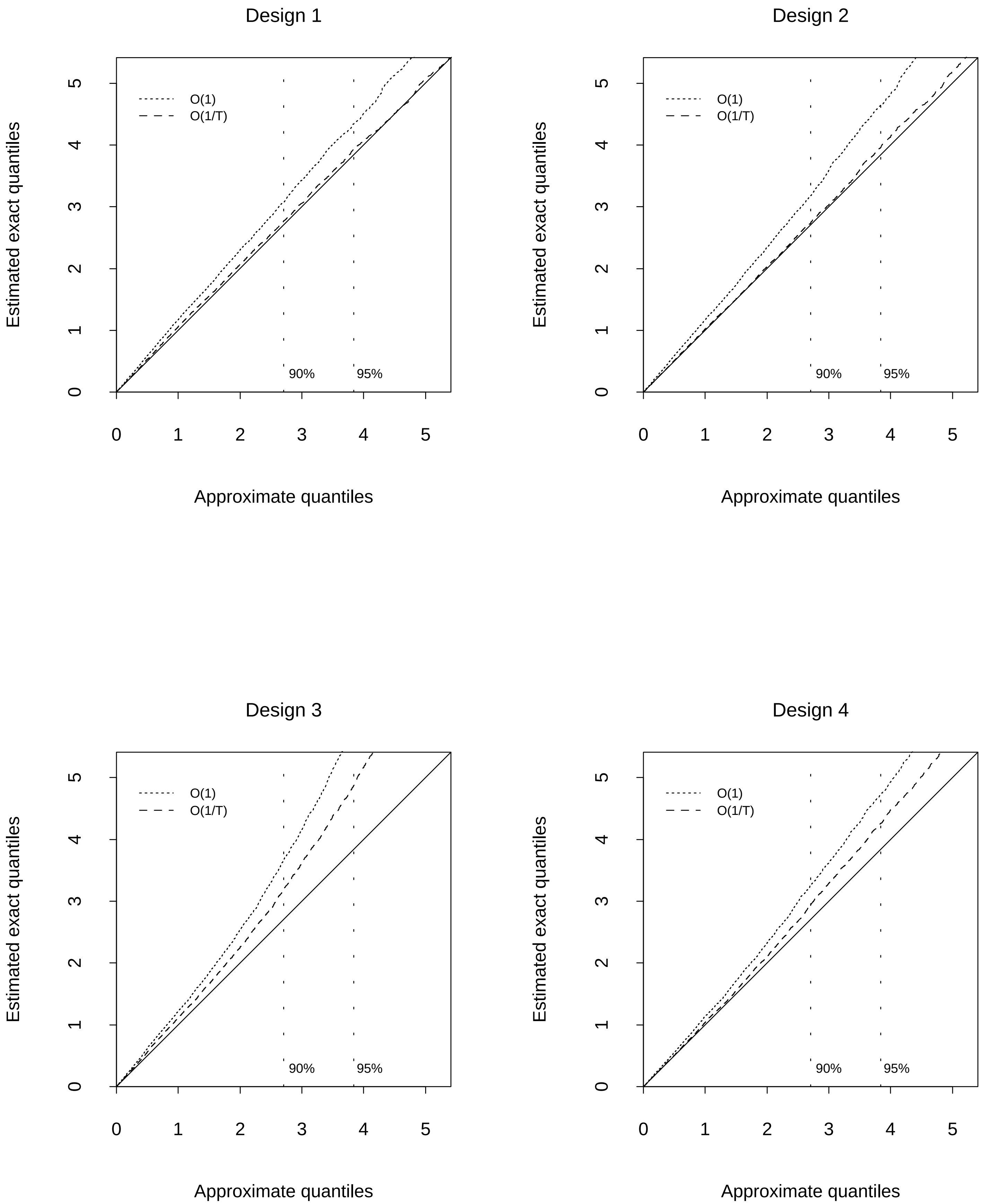
QQ plots. $T = 250$.

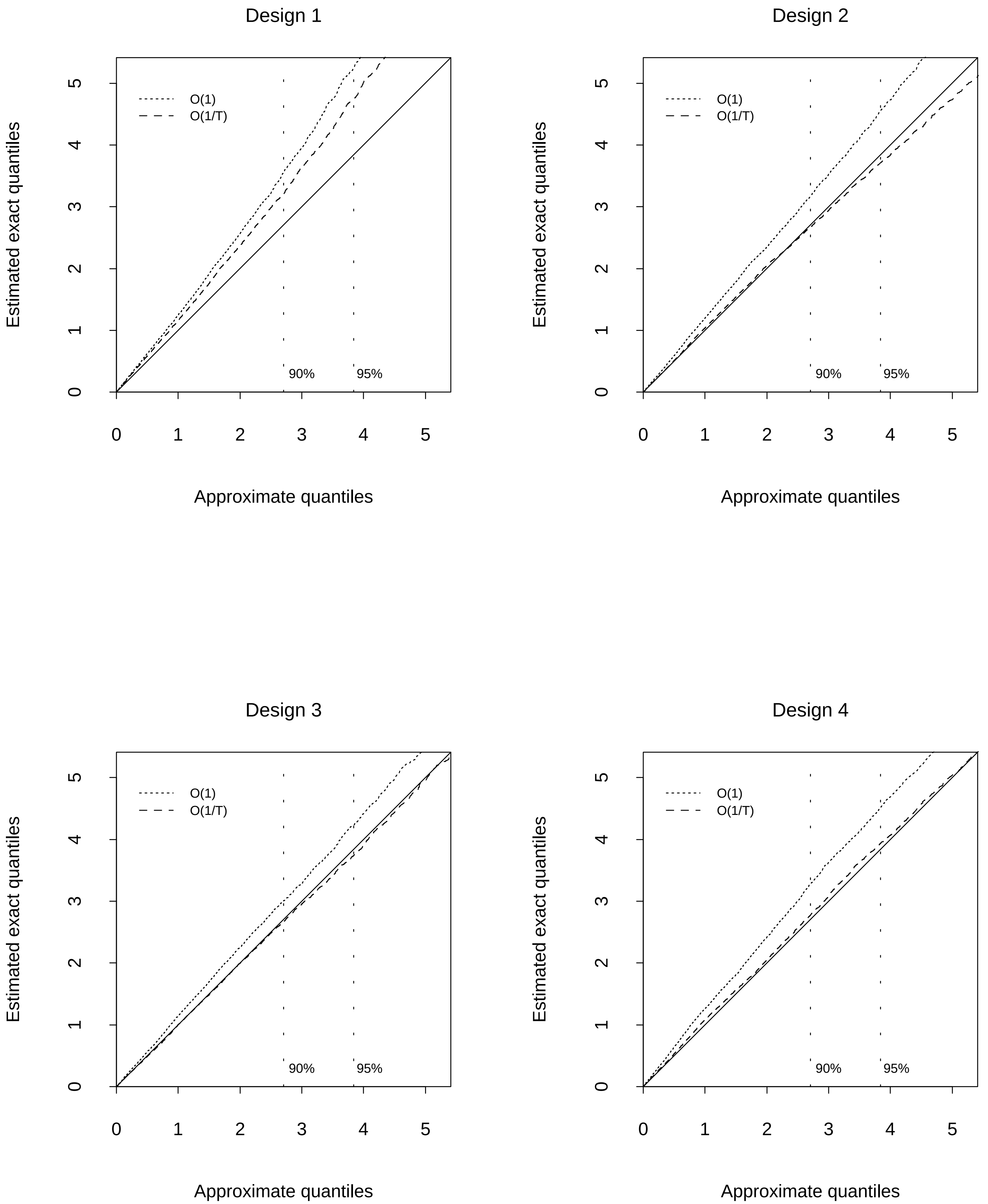
REMARK. Clear regression design effects.

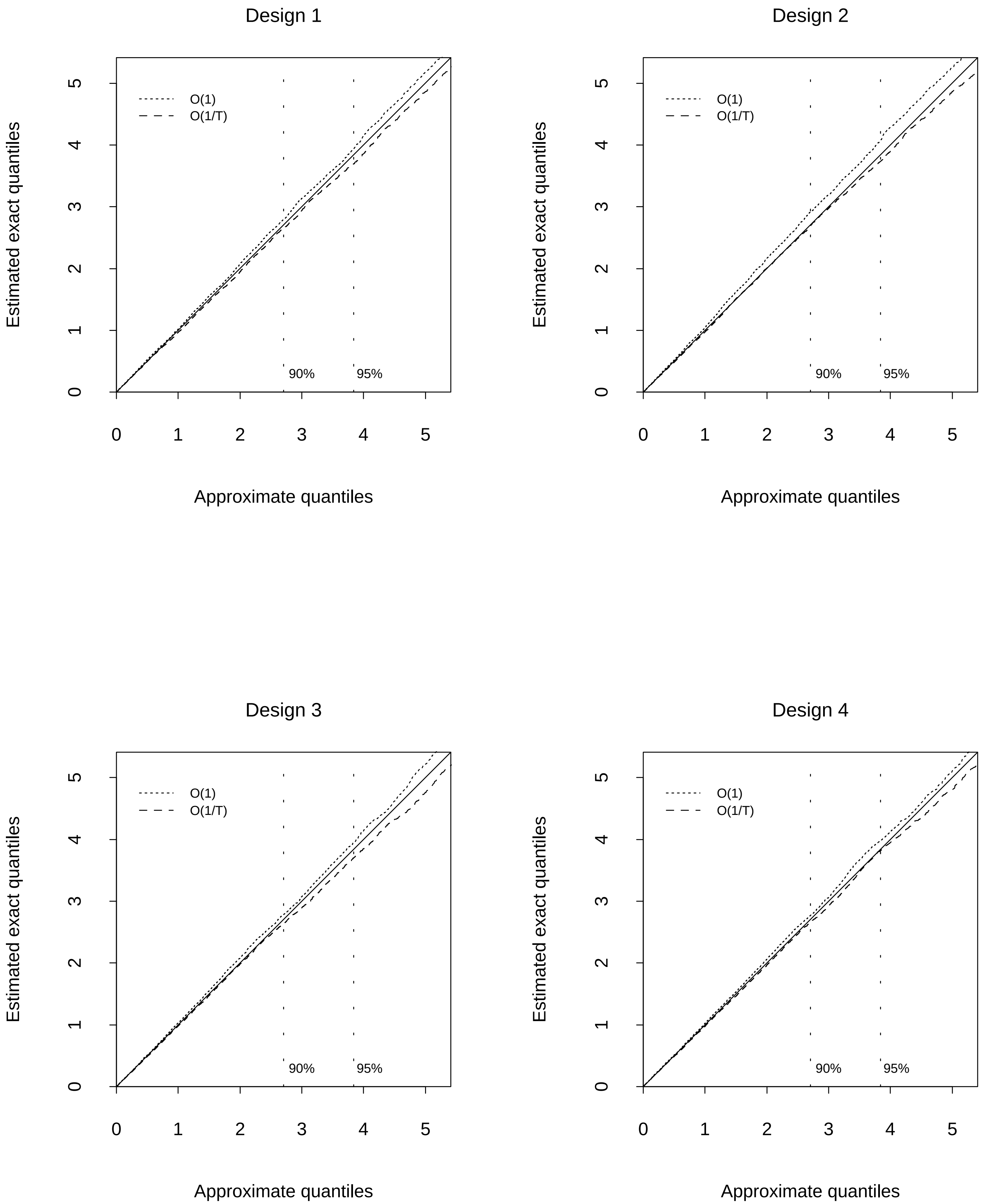

Design 1
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
Design 2
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
Design 3
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
Design 4
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles

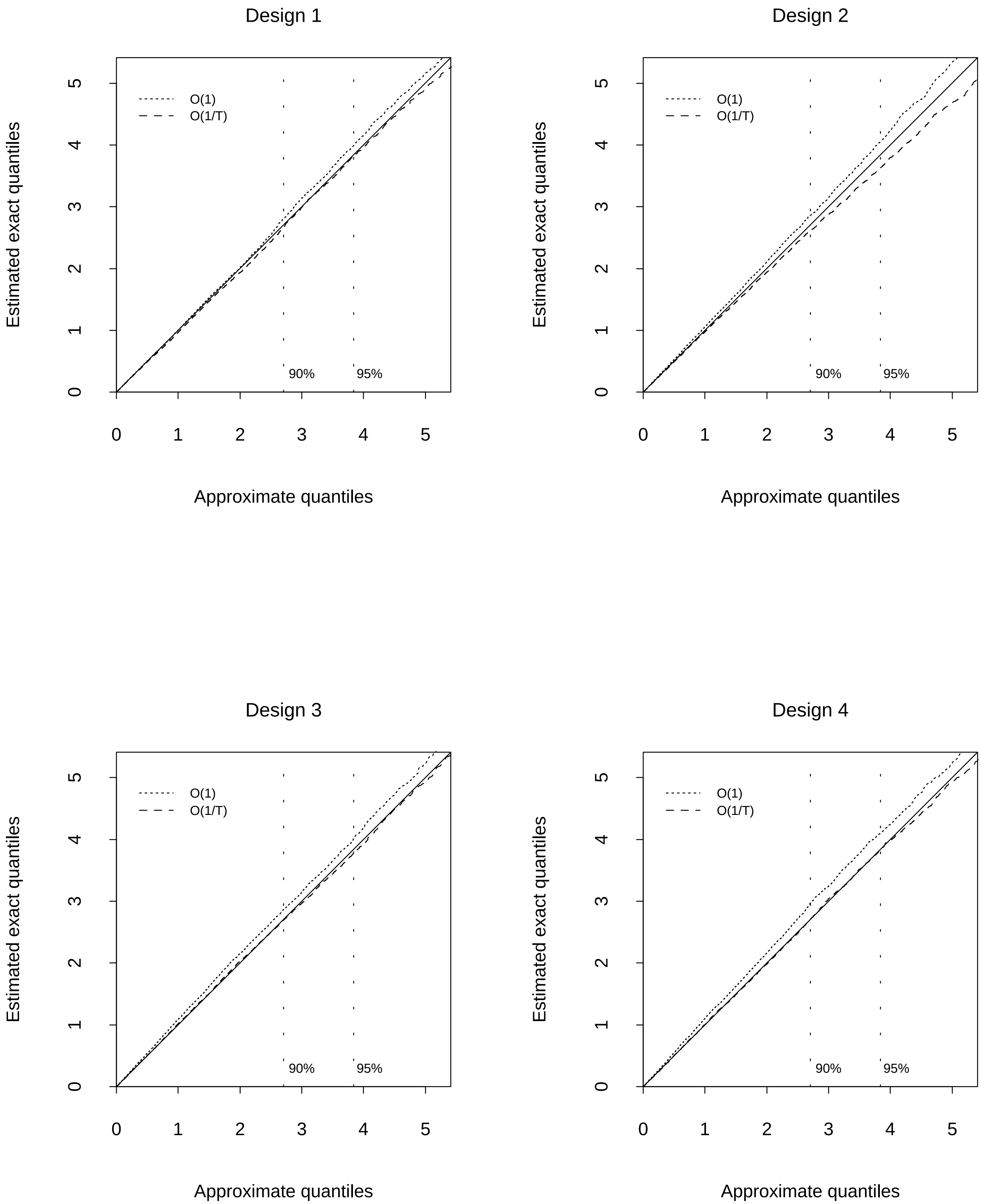


Design 1
Design 2
Design 3
Design 4
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
0
1
2
3
4
5

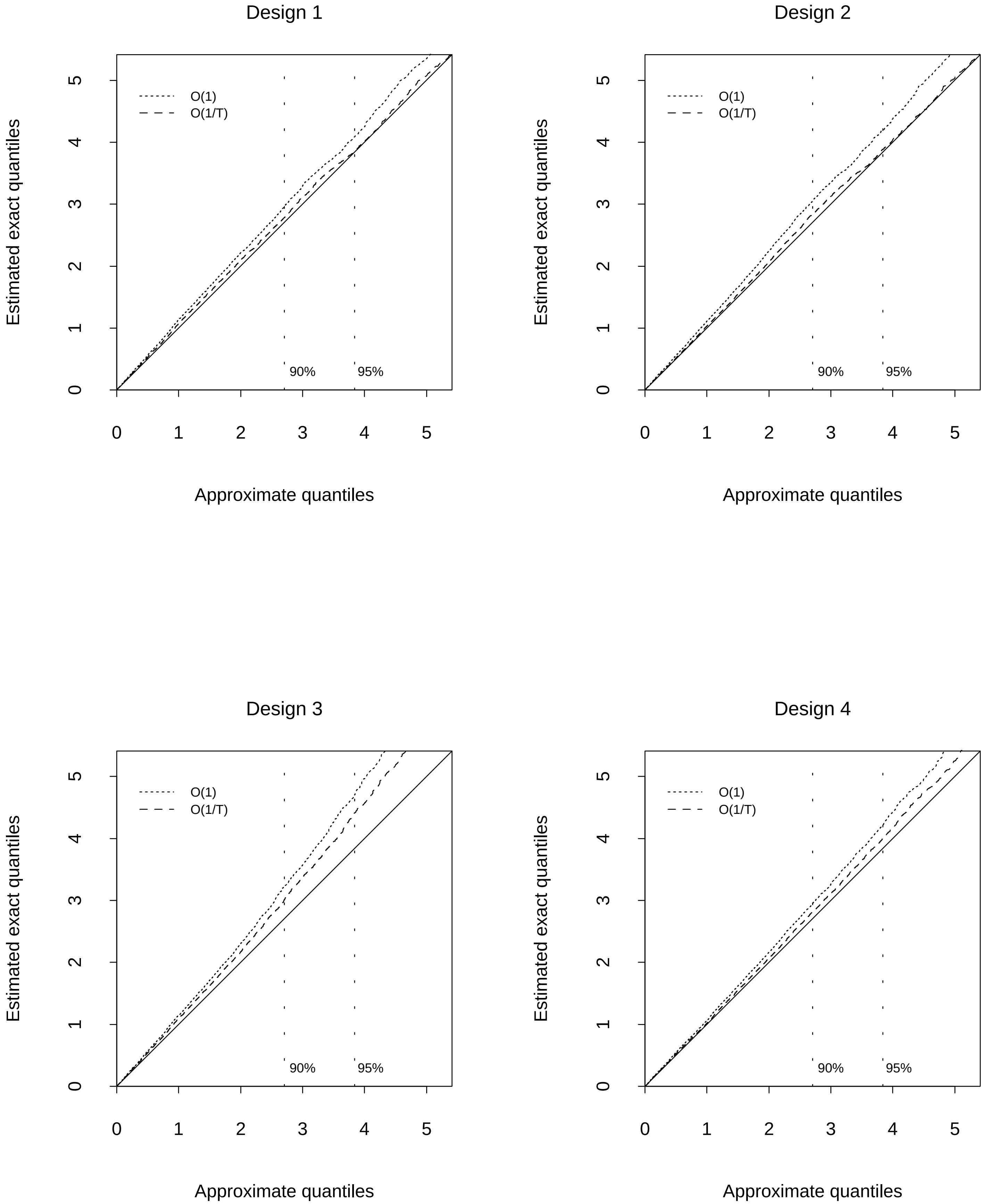

Design 1
Design 2
Design 3
Design 4
Estimated exact quantiles
Approximate quantiles
O(1)
O(1/T)
90%
95%
0
1
2
3
4
5

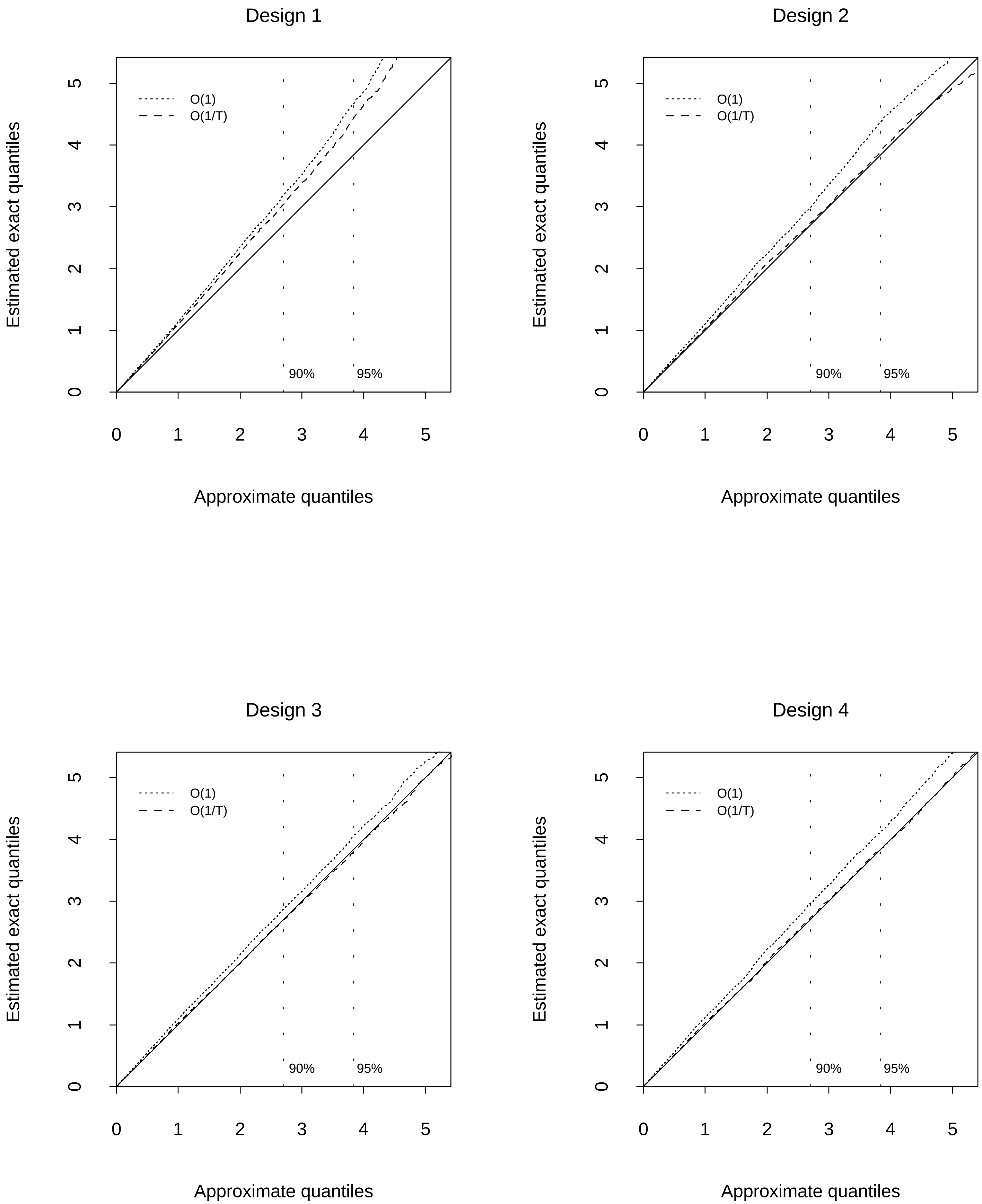

Design 1
Design 2
Design 3
Design 4
O(1)
O(1/T)
90%
95%
0
1
2
3
4
5
Estimated exact quantiles
Approximate quantiles


Design 1
Design 2
Design 3
Design 4
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
0
1
2
3
4
5



Design 1
Design 2
Design 3
Design 4
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
0
1
2
3
4
5


Design 1
Design 2
Design 3
Design 4
Estimated exact quantiles
Approximate quantiles
O(1)
O(1/T)
90%
95%
0
1
2
3
4
5



Design 1
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
0
1
2
3
4
5
Design 2
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
Design 3
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles
Design 4
O(1)
O(1/T)
90%
95%
Estimated exact quantiles
Approximate quantiles