Shape Restrictions in the Analysis of Set Identifying Models

Conference on Shape Restrictions in Non- and Semi-parametric Estimation of Econometric Models
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Ordered outcome IV models

- A model for ordered discrete $Y \in \{1, 2, \ldots, M\}$ has:
  - structural equation:

$$Y = h(X, U) = \begin{cases} 1 & , \quad 0 \leq U \leq h_1(X) \\ 2 & , \quad h_1(X) < U \leq h_2(X) \\ \vdots & \quad \vdots \quad \vdots \quad \vdots \\ M & , \quad h_{M-1}(X) < U \leq 1 \end{cases}$$

  - U normalized $Unif(0,1)$, $\mathrm{supp}(X)$ independent of U, $U \perp\!\!\!\perp Z \in \mathcal{Z}$.
- Consider probability **distribution** functions:

$$F^0_{YX|Z}(y, x|z) \equiv \Pr_0[Y \leq y \wedge X \leq x | Z = z]$$

with $Z = z \in \mathcal{Z}$.

- The identified set $\mathcal{H}_0(\mathcal{Z})$ is the set of functions h for which there exist *admissible* $F_{UX|Z}$ delivering

$$F_{UX|Z}(h_m(x), x|z) = F^0_{YX|Z}(y, x|z) \quad \text{for all } y,\ x, \text{ and } z \in \mathcal{Z}$$

The identified set of functions h

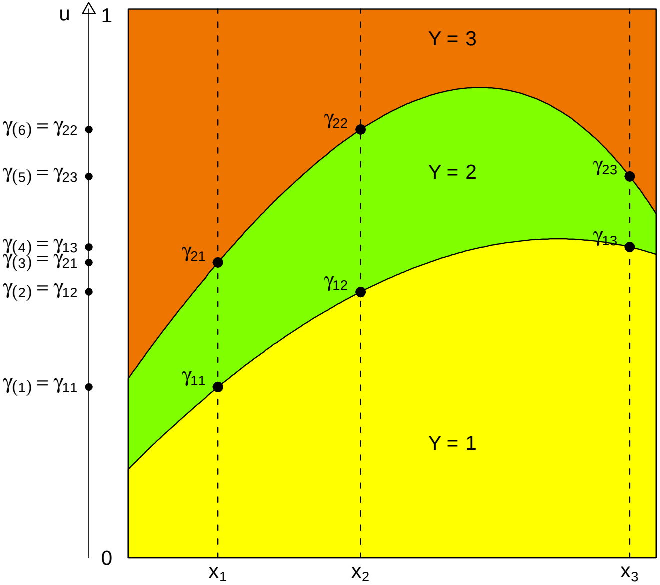
- The IV model set identifies h.
  - The set can be a complex object.
  - We consider the simplifications that shape restrictions bring.
- First consider discrete $X \in \{x_1, \ldots, x_K\}$.
- The structural function is characterized by $N = K(M-1)$ "parameters", γ.

$$\gamma_{mk} \equiv h_m(x_k) \qquad m \in \{1, \ldots, M-1\}, k \in \{1, \ldots, K\}.$$

- For binary Y the identified set comprises all h such that $\forall\, u \in (0,1]$:

$$\max_{z \in \mathcal{Z}} \Pr_0[Y < h(X, u)|z] < u \leq \min_{z \in \mathcal{Z}} \Pr_0[Y \leq h(X, u)|z]$$

- For any Y this is an outer region.
- The ordering of the γ_{mk}'s is key in calculating e.g. $\Pr_0[Y < h(X, u)|z]$.



u
1
0
Y = 3
Y = 2
Y = 1
$\gamma_{(6)} = \gamma_{22}$
$\gamma_{(5)} = \gamma_{23}$
$\gamma_{(4)} = \gamma_{13}$
$\gamma_{(3)} = \gamma_{21}$
$\gamma_{(2)} = \gamma_{12}$
$\gamma_{(1)} = \gamma_{11}$
γ_{22}
γ_{23}
γ_{13}
γ_{21}
γ_{12}
γ_{11}
x_1
x_2
x_3

Geometry of the identified set

- The identified set for the $N = K(M-1)$ parameters γ is a subset of the unit N-cube.
- For each ordering, t, of the γ_{mk}'s there is a convex polyhedron $H_0^t(\mathcal{Z})$ defined by the intersection of linear half spaces.
- Two sets $H_0^t(\mathcal{Z})$ and $H_0^s(\mathcal{Z})$ only intersect on faces where elements of γ are equal.
- The identified set is the union of these sets.

$$H_0(\mathcal{Z}) = \bigcup_{\text{all } t} H_0^t(\mathcal{Z}).$$

- The number of admissible orderings of the γ_{mk}'s satisfying

$$\gamma_{mk} < \gamma_{nk} \qquad \text{for all } n < m \text{ and all } k$$

can be as large as:

$$\frac{((M-1)K)!}{((M-1)!)^K}$$

Example

- Here is an example with binary Y and $X \in \{x_1, x_2, x_3\}$.

$$Y = h(X,U) = \begin{cases} 1 \quad , & 0 \leq \quad U \quad \leq h_1(X) \\ 2 \quad , & h_1(X) < \quad U \quad \leq h_2(X) \end{cases}$$

- The parameters are γ_1, γ_2, γ_3.

$$\gamma_1 = h_1(x_1) \qquad \gamma_2 = h_1(x_2) \qquad \gamma_3 = h_1(x_3)$$

- $M = 2$, $K = 3$. The identified set is the union of $3! = 6$ convex sets.
- When instruments are good predictors some of the component sets may be empty.

2 3D pics here

Geometry of the identified set

- The number of admissible orderings of the γ_{mk}'s can be as large as:

$$\frac{((M-1)K)!}{((M-1)!)^K}$$

- This can be huge.

		K			
		2	3	4	5
	2	2	6	24	120
M	3	6	90	2,520	113,400
	4	20	1,680	369,600	168,168,000
	5	70	34,650	6,306,600	305,540,235,000

- Shape restrictions bring some simplification.

Shape restriction: complete separation

- Threshold functions are completely separated if

$$\min\{h_m(x_1), \ldots, h_m(x_K)\} \geq \max\{h_{m-1}(x_1), \ldots, h_{m-1}(x_K)\} \qquad \text{for all } m$$

- Must arise when the effect of X on Y is sufficiently weak.
- The number of admissible orderings of γ can be as large as:

$$L = (K!)^{M-1}$$

compared with:

$$\frac{((M-1)K)!}{((M-1)!)^K}$$

Shape restriction: monotonicity

- Threshold functions are monotone increasing in scalar x if

$$x_i > x_j \implies h_m(x_i) > h_m(x_j) \qquad \text{for all } m$$

- How many admissible arrangements?
- This is the number of ways a $(M-1) \times K$ array can be filled with $K(M-1)$ distinct values, ascending by rows and columns.
- For $M = 3$, $K = 3$ there is e.g.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix} \qquad \text{and so on...}$$

- These are rectangular Young Tableaux. The "hook length formula" gives the number of admissible orderings of γ.

$$\frac{((M-1)K)!}{\prod(\text{hook lengths})}$$

Shape restriction: monotonicity

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$$\begin{bmatrix} \boxed{1} & \boxed{3} & \boxed{5} \\ \boxed{2} & 4 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & \boxed{2} & \boxed{4} \\ 3 & \boxed{5} & 6 \end{bmatrix} \qquad \text{and so on...}$$

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$$\frac{((M-1)K)!}{\prod(\text{hook lengths})} = \frac{6.5.4.3.2.1}{4.3.2.3.2.1} = 5$$

Shape restriction: monotonicity

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- How many admissible arrangements?
- This is the number of ways a $(M-1) \times K$ array can be filled with $K(M-1)$ distinct values, ascending by rows and columns.
- For $M = 3$, $K = 3$ there is e.g.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix} \qquad \text{and so on...}$$

- Applying the "hook length formula" the number of admissible orderings of γ is as large as:

$$\frac{((M-1)K)!}{\prod_{m=1}^{M-1} \prod_{k=1}^{K} (M+K-m-k)}$$

compared with

$$\frac{((M-1)K)!}{((M-1)!)^K}$$

Numbers of admissible orderings

		K			
M	Restriction	2	3	4	5
2	None	2	6	24	120
	MO	1	1	1	1
	CS	2	6	24	120
3	None	6	90	2,520	113,400
	MO	2	5	14	42
	CS	4	36	576	14,400
4	None	20	1,680	369,600	168,168,000
	MO	5	42	462	6,006
	CS	8	216	13,824	1,728,000
5	None	70	34,650	6,306,300	305,540,235,000
	MO	14	462	24,024	1,662,804
	CS	16	1,296	331,776	207,360,000

MO: monotonicity CS: complete separation

Shape restrictions: single peak

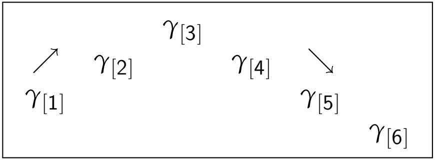
- For binary Y, under a monotonicity restriction there is one ordering.
- Define ordered values of scalar X:

$$x_{[1]}, \ldots, x_{[K]}$$

  and let $\gamma_{[i]} \equiv h_1(x_{[i]})$.
- Under the single peak restriction the sequence:

$$\gamma_{[1]}, \gamma_{[2]}, \ldots, \gamma_{[K]}$$

  rises to a peak and then falls, e.g.



Shape restrictions: single peak

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$$\gamma_{[1]}, \gamma_{[2]}, \ldots, \gamma_{[K]}$$

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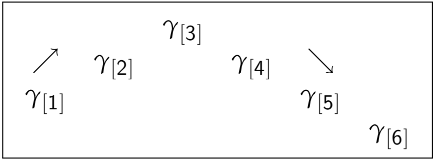


- There are as many as 2^{K-1}admissible orderings with a single peak.

Numbers of admissible orderings, binary Y, scalar X

	Restriction	
K	None	Single peak
2	2	2
3	6	4
4	24	8
5	120	16
6	720	32
7	5,040	64
8	40,320	128
9	362,880	256
10	3,628,800	512

Continuous X?

- Even with the shape restrictions considered the identified set remains complex for large M or K.
- It is useful to have a way of dealing effectively with continuous endogenous variables, so consider the monotone index restriction.
- Now consider binary Y with

$$Y = h(X\theta, U) \qquad U \perp\!\!\!\perp Z$$

- Consider the shape restrictions: h is strictly monotone in $X\theta$.
- Then the identified set for θ and h can be obtained quite straightforwardly.

Monotone index restriction

- For binary Y the identified set of values of θ comprises all values of θ for which $\exists$ a monotone function h such that $\forall\, u \in (0,1)$:

$$\max_{z\in\mathcal{Z}} \Pr_0[Y < h(X\theta, u)|z] < u \leq \min_{z\in\mathcal{Z}} \Pr_0[Y \leq h(X\theta, u)|z]$$

- Using the threshold crossing representation:

$$Y = h(X, U) = \begin{cases} 1 \quad , & 0 \leq \quad U \quad \leq h(X\theta) \\ 2 \quad , & h(X\theta) < \quad U \quad \leq 1 \end{cases}$$

this can be written:

$$\max_{z\in\mathcal{Z}} \Pr_0[Y = 1 \wedge h(X\theta) < u|z] < u \leq \min_{z\in\mathcal{Z}} (1 - \Pr_0[Y = 2 \wedge h(X\theta) \geq u)|z])$$

Monotone index restriction

- With h monotone increasing (decreasing):

$$\max_{z\in\mathcal{Z}} \Pr_0[Y = 1 \wedge X\theta \underset{(>)}{<} h^{-1}(u)|z]$$

$$< u \leq$$

$$\min_{z\in\mathcal{Z}} \left(1 - \Pr_0[Y = 2 \wedge X\theta \underset{(\leq)}{\geq} h^{-1}(u)|z]\right)$$

- Define $\sigma \equiv h^{-1}(u) \implies u = h(\sigma)$:

$$\max_{z\in\mathcal{Z}} \Pr_0[Y = 1 \wedge X\theta \underset{(>)}{<} \sigma|z]$$

$$< h(\sigma) \leq$$

$$\min_{z\in\mathcal{Z}} \left(1 - \Pr_0[Y = 2 \wedge X\theta \underset{(\leq)}{\geq} \sigma|z]\right)$$

Monotone index restriction

- For each θ we ask is there a monotone function $h(\sigma)$ lying between the increasing functions:

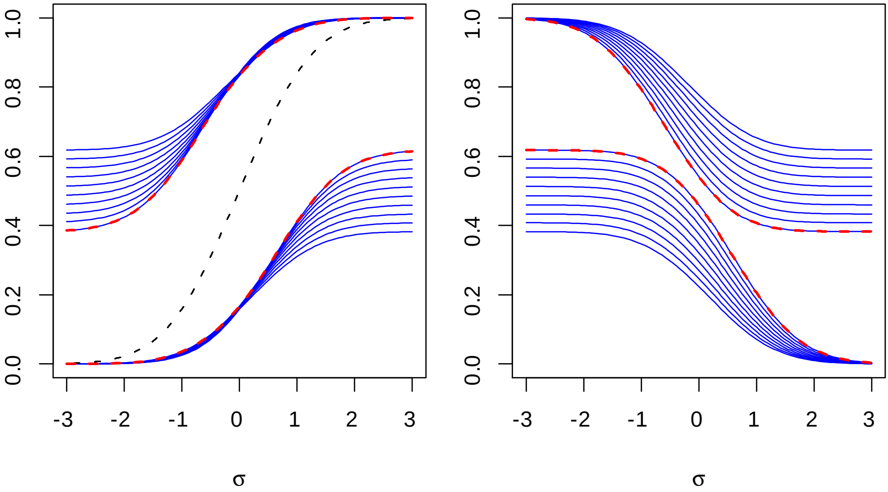
$$\max_{z\in\mathcal{Z}} \Pr_0[Y = 1 \wedge X\theta < \sigma | z] \text{ and } \min_{z\in\mathcal{Z}} (1 - \Pr_0[Y = 2 \wedge X\theta \geq \sigma | z])$$

or between the decreasing functions

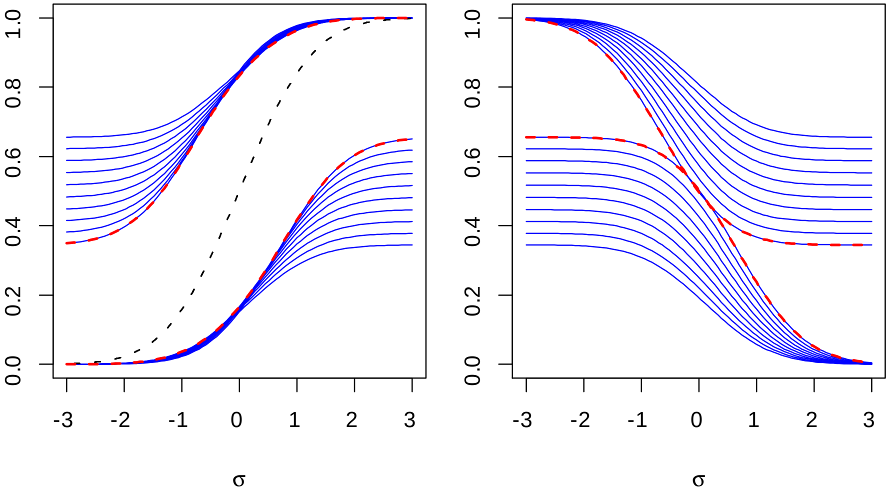
$$\max_{z\in\mathcal{Z}} \Pr_0[Y = 1 \wedge X\theta > \sigma | z] \text{ and } \min_{z\in\mathcal{Z}} (1 - \Pr_0[Y = 2 \wedge X\theta \leq \sigma | z])$$

  - YES - then θ is in the identified set associated with all monotone functions $h(\sigma)$ that lie between the bounding functions.
  - NO - then θ is not in the identified set.

Bounding functions admitting both increasing and decreasing h



Bounding functions admitting only increasing h



Monotone index restriction

- For each θ we ask is there a monotone function h lying between the increasing functions:

$$\max_{z \in \mathcal{Z}} \text{Pr}_0[Y = 1 \wedge X\theta < \sigma | z] \text{ and } \min_{z \in \mathcal{Z}} (1 - \text{Pr}_0[Y = 2 \wedge X\theta \geq \sigma | z])$$

or between the decreasing functions

$$\max_{z \in \mathcal{Z}} \text{Pr}_0[Y = 1 \wedge X\theta > \sigma | z] \text{ and } \min_{z \in \mathcal{Z}} (1 - \text{Pr}_0[Y = 2 \wedge X\theta \leq \sigma | z])$$

- Define

$$\begin{aligned}
\overline{b}_0^{inc}(\sigma, \theta, \mathcal{Z}) &\equiv \max_{z \in \mathcal{Z}} \text{Pr}_0[Y = 1 \wedge X\theta < \sigma | z] \\
\underline{b}_0^{inc}(\sigma, \theta, \mathcal{Z}) &\equiv \min_{z \in \mathcal{Z}} (1 - \text{Pr}_0[Y = 2 \wedge X\theta \geq \sigma | z]) \\
\\
\overline{b}_0^{dec}(\sigma, \theta, \mathcal{Z}) &\equiv \max_{z \in \mathcal{Z}} \text{Pr}_0[Y = 1 \wedge X\theta > \sigma | z] \\
\underline{b}_0^{dec}(\sigma, \theta, \mathcal{Z}) &\equiv \min_{z \in \mathcal{Z}} (1 - \text{Pr}_0[Y = 2 \wedge X\theta \leq \sigma | z])
\end{aligned}$$

Monotone index restriction

- For each θ we ask is there a monotone function $h(\sigma)$ lying between the increasing functions:

$$\overline{b}_0^{inc}(\sigma, \theta, \mathcal{Z}) \text{ and } \underline{b}_0^{inc}(\sigma, \theta, \mathcal{Z})$$

  or between the decreasing functions

$$\overline{b}_0^{dec}(\sigma, \theta, \mathcal{Z}) \text{ and } \underline{b}_0^{dec}(\sigma, \theta, \mathcal{Z})$$

- The identified set of values of θ is:

$$\left\{\theta : \overline{b}_0^{inc}(\sigma, \theta, \mathcal{Z}) \geq \underline{b}_0^{inc}(\sigma, \theta, \mathcal{Z}) \quad \forall \sigma\right\} \cup \left\{\theta : \overline{b}_0^{dec}(\sigma, \theta, \mathcal{Z}) \geq \underline{b}_0^{dec}(\sigma, \theta, \mathcal{Z}) \quad \forall \sigma\right\}$$

Remarks

- Identified sets for nonparametrically specified functions are complex.
- Shape restrictions bring simplification.
- But when the support of outcomes is rich, semiparametric and parametric restrictions will typically be required.
- Monotone index restriction:
  - allows fast computation of identified sets for index coefficients.
  - can be used for $M > 2$ applying the proposed method to $M - 1$ binary indicators, but it generally delivers an outer region.
- Further results in CeMMAP Working Papers: CWP23/09, CWP 27/09, CWP 11/10, all under revision to be found at:

www.cemmap.com