Lecture 6

Estimating the Degree of Activity of Jumps in High Frequency Financial Data

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based on joint work with Jean Jacod

References

- Estimating the Degree of Activity of Jumps in High Frequency Data, with Jean Jacod, forthcoming in the *Annals of Statistics*.

1. Introduction

- Different types of jumps
  - Large jumps, which are rather infrequent, are easy to pick out.
  - But visual inspection of most time series in finance does not provide clear evidence for either the presence or the absence of smaller, more frequent, jumps.
  - And even less so for the finer characteristics.

- For modelling purposes, one would like to infer the characteristics of X, that is, its drift, its volatility and its Lévy jump measure, from the observations.

  - When the time interval Δ_n goes to 0, it is well known that one can infer consistently the volatility, under very weak assumptions.

  - But such consistent inference is impossible for the drift or the Lévy measure, if the overall time of observation $[0, T]$ is kept fixed.

  - In fact, even in the unrealistic case where the whole path of X is observed over a fixed $[0, T]$, one can infer neither the drift nor the Lévy measure.

- One can however hope to be able to characterize the behavior of the Lévy measure near 0:
  - First whether it does not explode near 0, meaning that the number of jumps is finite;
  - Second, when the number of jumps is infinite, we would like to be able to say something about the concentration of small jumps.

2. Defining an Index of Jump Activity

- Suppose that we observe at time intervals Δ_n an asset price X which is an Itô semimartingale, that is

$$X_t \;=\; X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \text{JUMPS}$$

$$\text{JUMPS} = \int_0^t \int_{\{|x| \le 1\}} x(\mu - \nu)(ds, dx) + \int_0^t \int_{\{|x|>1\}} x\mu(ds, dx)$$

Here μ is the jump measure of X, and its predictable compensator ν can be factorized as

$$\nu(\omega; dt, dx) \;=\; dt \; F_t(\omega, dx).$$

- Consider the process $B(p)_t$ for the semimartingale X :

$$B(p)_t = \sum_{s \leq t} |\Delta X_s|^p$$

where $\Delta X_s = X_s - X_{s-}$ is the size of the jump at time s, if any.

- Define

$$I_t = \{p \geq 0 : B(p)_t < \infty\}.$$

- Necessarily, the (random) set I_t is of the form $[\beta_t, \infty)$ or (β_t, ∞) for some $\beta_t \leq 2$, and $2 \in I_t$ always, and $t \mapsto \beta_t$ is non-decreasing.

- We call $\beta_T(\omega)$ the jump activity index for the path $t \mapsto X_t(\omega)$ at time T.

- We define this index in analogy with the special case where X is a Lévy process:

  – Then $\beta_T(\omega) = \beta$ does not depend on (ω, T), and it is also the infimum of all $r \geq 0$ such that $\int_{\{|x| \leq 1\}} |x|^r F(dx) < \infty$, where F is the Lévy measure

  – This property shows that, for a Lévy process, the jump activity index coincides with the Blumenthal-Getoor index of the process.

  – In the further special case where X is a stable process, then β is also the stable index of the process.

- When X is a Lévy process, the index β is only a partial element of the whole Lévy measure F

- But this is the most informative knowledge one can draw about F from the observation of the path $t \mapsto X_t$ for all $t \leq T$, T finite.

- Things are very different when $T \to \infty$, though, since observing X over $[0, \infty)$ completely specifies F.

- However, β captures an essential qualitative feature of F, which is its level of activity: when β increases, the (small) jumps tend to become more and more frequent.

2.1. The Brownian Motion...

- Recall that the semimartingale X is only observed at times $i\Delta_n$, over $[0, T]$.

- The problem is made more challenging by the presence in X of a continuous, or Brownian, martingale part:
  - β characterizes the behavior of F near 0.
  - Hence it is natural to expect that the small increments of the process are going to be the ones that are most informative about β.

– But that is where the contribution from the continuous martingale part of the process is inexorably mixed with the contribution from the small jumps.

– We need to see through the continuous part of the semimartingale in order to say something about the number and concentration of small jumps.

3. Some Intuition

- Consider the special case $X = \sigma W + Y$, where Y is a β-stable process.

- Any increment $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$ satisfies

$$\Delta_i^n X = \sigma \Delta_n^{1/2} W_1 + \Delta_n^{1/\beta} Y_1$$

(equality in law). Then:

  - Recalling $\beta < 2$ and $\Delta_n \to 0$, with a large probability $\Delta_i^n X$ is close to $\sigma \Delta_n^{1/2} W_1$ in law. Those increments give essentially no information on Y, and are of "order of magnitude" $\Delta_n^{1/2}$

  - However if Y has a "big" jump at time S, the corresponding increment is close to ΔY_S.

- Hence, one has to throw away all "small" increments. However, β is related to the behavior of F near 0, hence to the "very small" jumps of Y.

- In practice one uses only increments bigger than a cutoff level

$$\alpha\Delta_n^{\varpi} \qquad \text{for some } \varpi \in (0, 1/2).$$

- Asymptotically:

  – those increments are big because, since $\Delta_n^{1/2} << \Delta_n^{\varpi}$, the main contribution is due to Y.

  – those increments mostly contain a single "big" jump, of size of order at least Δ_n^{ϖ}.

  – we still get some information on small jumps, because $\Delta_n^{\varpi} \to 0$.

- The same heuristics works for Itô semimartingales.

- This leads to consider, for fixed $\varpi \in (0, 1/2)$ and $\alpha > 0$, the functionals

$$U(\varpi, \alpha, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} 1_{\{|\Delta_i^n X| > \alpha \Delta_n^{\varpi}\}}.$$

which simply counts the number of increments whose magnitude is greater than $\alpha \Delta_n^{\varpi}$.

- With $\varpi < 1/2$, we are retaining only those increments of X that are not predominantly made of contributions from the continuous part, which are $O_p(\Delta_n^{1/2})$.

4. Behavior of the Lévy Measure

- Our regularity assumption is that for some $\beta \in (0,2)$ and $\beta' \in [0, \beta/2)$, we have for all (ω, t):

$$F_t = F_t' + F_t'' + F_t''',$$

where F_t' is locally of the $\beta-$stable form

$$F_t'(dx) \;=\; \frac{1}{|x|^{1+\beta}} \left(a_t^{(+)} 1_{\{0<x\leq z_t^{(+)}\}} + a_t^{(-)} 1_{\{-z_t^{(-)}\leq x<0\}} \right) \; dx,$$

for some predictable non-negative processes $a_t^{(+)}, a_t^{(-)}, z_t^{(+)}$ and $z_t^{(-)}$.

- Any additional components F'' and F''' in the Lévy measure beyond the most active part F' must have jump activity indices (which are at most β' and $\beta/2$, respectively) that are sufficiently apart from the leading jump activity index β.

- For example, any process of the following form will satisfy the assumption

$$dX_t = b_t dt + \sigma_t dW_t + \delta_{t-} dY_t + \delta'_{t-} dY'_t$$

where:

  - δ and δ' are cadlag adapted processes
  - Y is $\beta-$stable
  - Y' is any Lévy process with jump activity index β' less that $\beta/2$.

5. Estimators of the Jump Activity Index

- We show that the functionals $U(\varpi, \alpha, \Delta_n)$ behave according to

$$\Delta_n^{\varpi \beta} U(\varpi, \alpha, \Delta_n)_t \xrightarrow{\mathbb{P}} \frac{\bar{A}_t}{\alpha^{\beta}}$$

where $\bar{A}_t = \frac{1}{\beta} \int_0^t \left(a_s^{(+)} + a_s^{(-)} \right) ds.$

- The dual dependence of the limit on β leads us to propose two different estimators, at each stage n.

- For the first one, fix $0 < \alpha < \alpha'$ and define

$$\widehat{\beta}_n(t, \varpi, \alpha, \alpha') \ = \ \frac{\log(U(\varpi, \alpha, \Delta_n)_t / U(\varpi, \alpha', \Delta_n)_t)}{\log(\alpha'/\alpha)},$$

- $\widehat{\beta}_n$ is constructed from a suitably scaled ratio of two Us evaluated on the same time scale Δ_n but at two levels of truncation of the increments, α and α'.

- Based on $\Delta_n^{\varpi\beta} U(\varpi, \alpha, \Delta_n)_t \xrightarrow{\mathbb{P}} \frac{\bar{A}_t}{\alpha^{\beta}}$, this will be consistent.

- Our second estimator is

$$\widehat{\beta}'_n(t,\varpi,\alpha) \;=\; \frac{\log(U(\varpi,\alpha,\Delta_n)_t/U(\varpi,\alpha,2\Delta_n)_t)}{\varpi\log 2}.$$

- $\widehat{\beta}'_n$ is constructed from a suitably scaled ratio of two Us evaluated at the same level of truncation α, but on two time scales, Δ_n and $2\Delta_n$.

- Based on $\Delta_n^{\varpi\beta} U(\varpi,\alpha,\Delta_n)_t \xrightarrow{\mathbb{P}} \frac{\bar{A}_t}{\alpha^{\beta}}$, this will be consistent.

- One could look at a third estimator obtained from two Us evaluated at two different rates of truncation ϖ and ϖ', but there does not appear to be immediate benefits from doing so.

- A more general class of estimators can be constructed from the truncated power variation functionals of order r:

$$U_r(\varpi, \alpha, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^r \mathbf{1}_{\{|\Delta_i^n X| > \alpha \Delta_n^{\varpi}\}}.$$

- Here we focus on $r = 0$, simply counting increments.

- While one could imagine looking at other (small) values of r, there does not appear to be immediate benefits from doing so in the present problem.

Theorem: Under regularity assumptions, both

$$\frac{\log(\alpha'/\alpha)}{\left(\frac{1}{U(\varpi,\alpha',\Delta_n)_t}-\frac{1}{U(\varpi,\alpha,\Delta_n)_t}\right)^{1/2}}\ \left(\widehat{\beta}_n(t,\varpi,\alpha,\alpha')-\beta\right)$$

$$\frac{\varpi\log 2}{\left(\frac{1}{U(\varpi,\alpha,2\Delta_n)_t}-\frac{1}{U(\varpi,\alpha,\Delta_n)_t}\right)^{1/2}}\ \left(\widehat{\beta}'_n(t,\varpi,\alpha)-\beta\right)$$

converge stably in law, in restriction to the set $\{\bar{A}_t > 0\}$, to a standard normal variable $\mathcal{N}(0,1)$ independent of X.

- The qualifier "in restriction to the set $\{\bar{A}_t > 0\}$" is essential in this statement.

  - On the (random) set $\{\bar{A}_t > 0\}$, the jump activity index is β.

  - On the complement set $\{\bar{A}_t = 0\}$, anything can happen: on that set, the number β has no meaning as a jump activity index for X on $[0, T]$.

- These results are model-free, because the drift and the volatility processes are totally unspecified apart from the regularity assumption on the Lévy measures F_t.

6. Simulation Results

- The data generating process is $dX_t/X_0 = \sigma_t dW_t + dY_t$

- Y is a pure jump process, $\beta-$stable or Compound Poisson ($\beta = 0$).

- Stochastic volatility $\sigma_t = v_t^{1/2}$

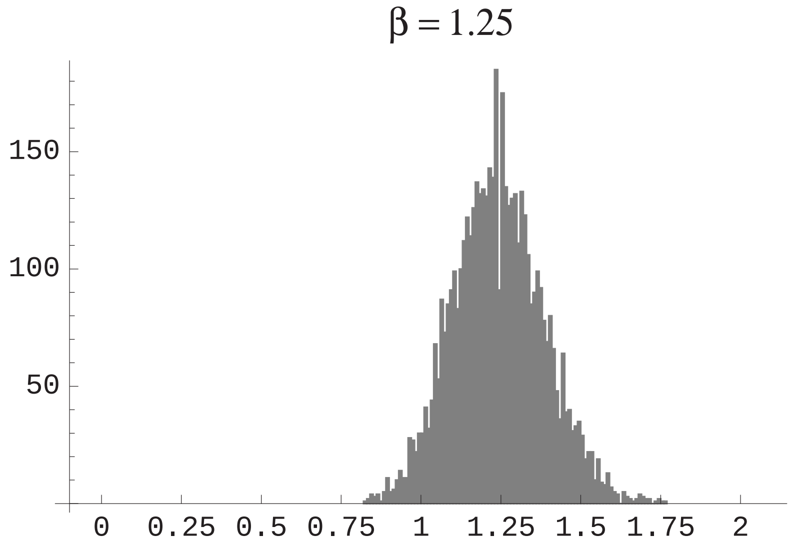
$$dv_t = \kappa(\eta - v_t)dt + \gamma v_t^{1/2} dB_t + dJ_t,$$

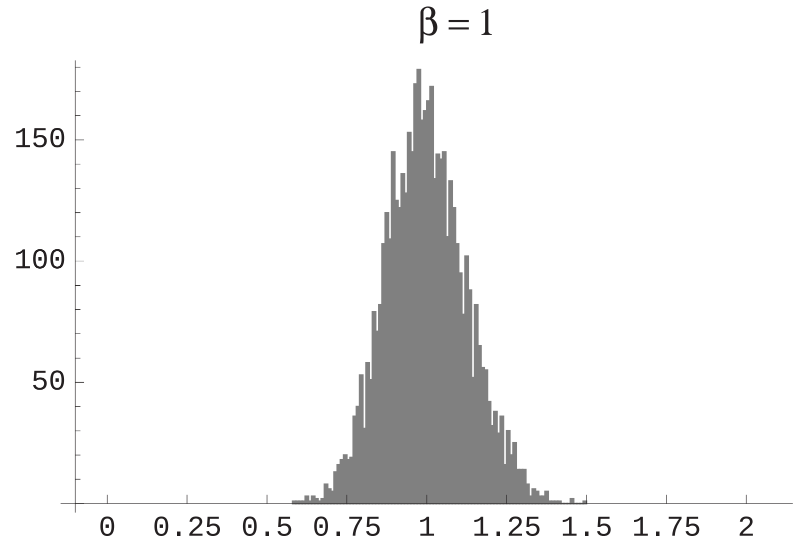
- Leverage effect: $E[dW_t dB_t] = \rho dt$, $\rho < 0$

- With jumps in volatility: J is a compound Poisson process with uniform jumps.

Simulations: $\beta = 1.25$ and $\beta = 1$

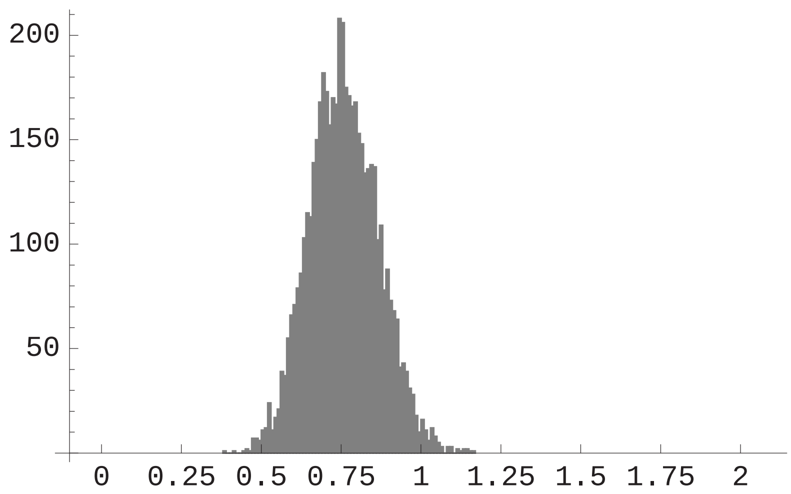
Estimator Based on Two Truncation Levels



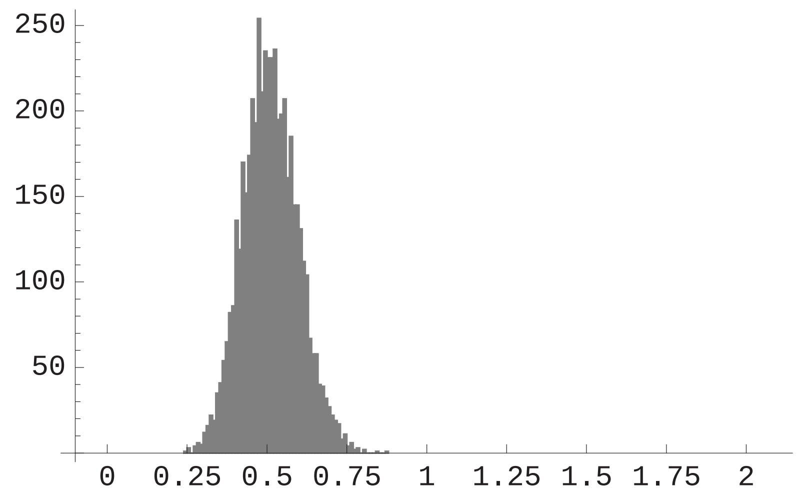


Simulations: $\beta = 0.75$ and $\beta = 0.5$

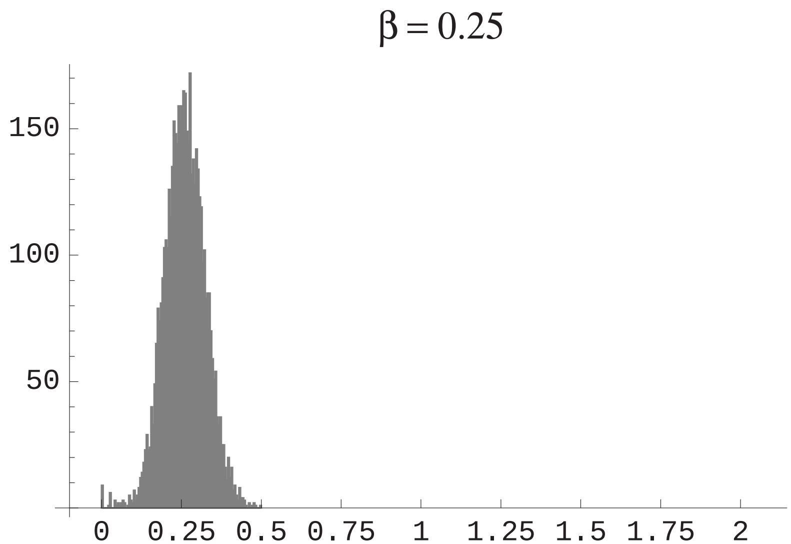
$\beta = 0.75$

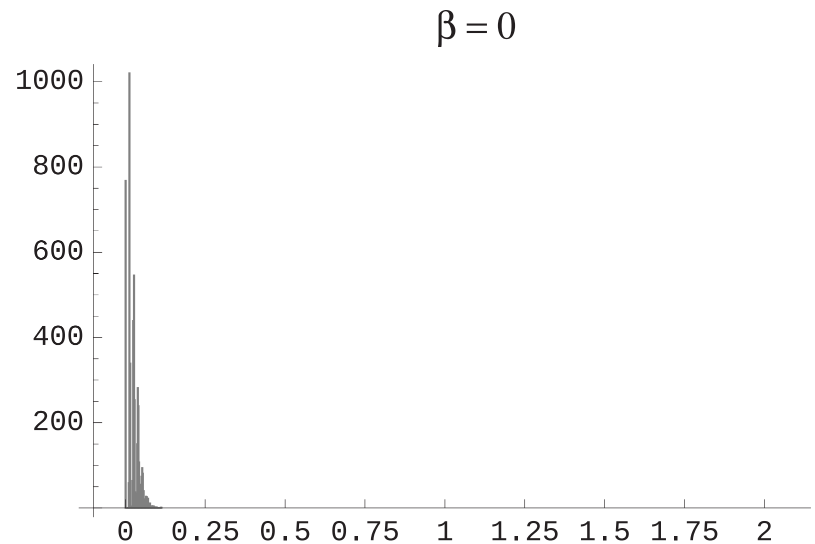


$\beta = 0.5$

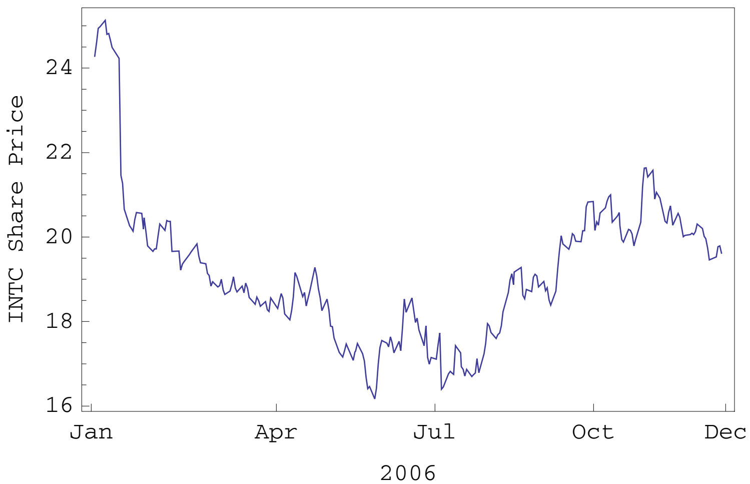


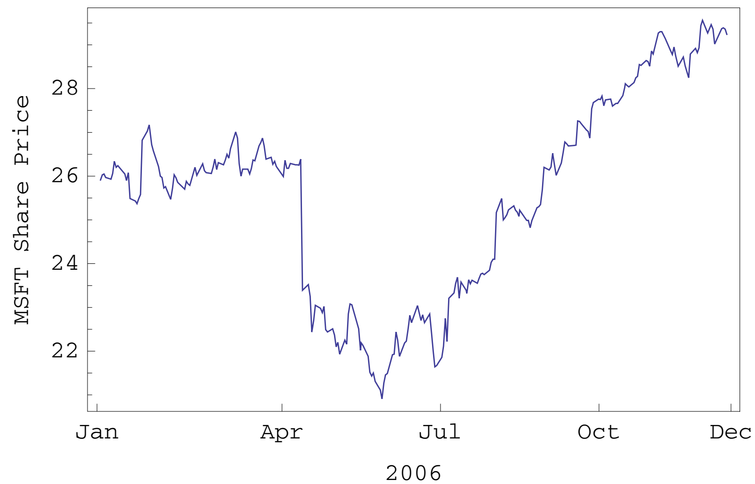
Simulations: $\beta = 0.25$ and $\beta = 0$

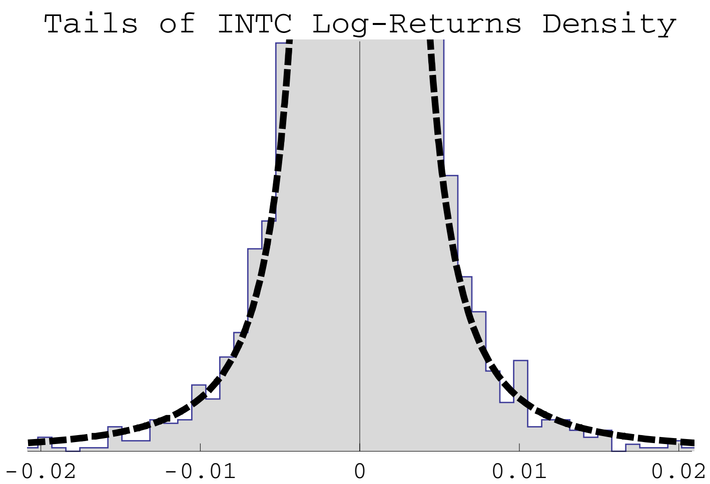


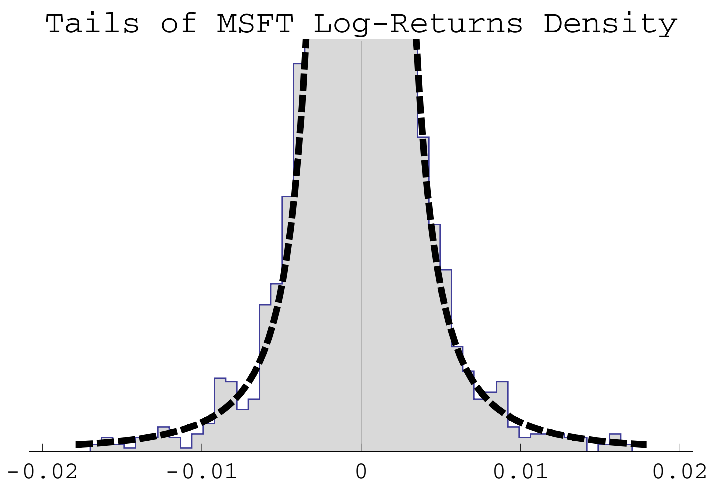


7. Empirical Results: Intel & Microsoft 2006

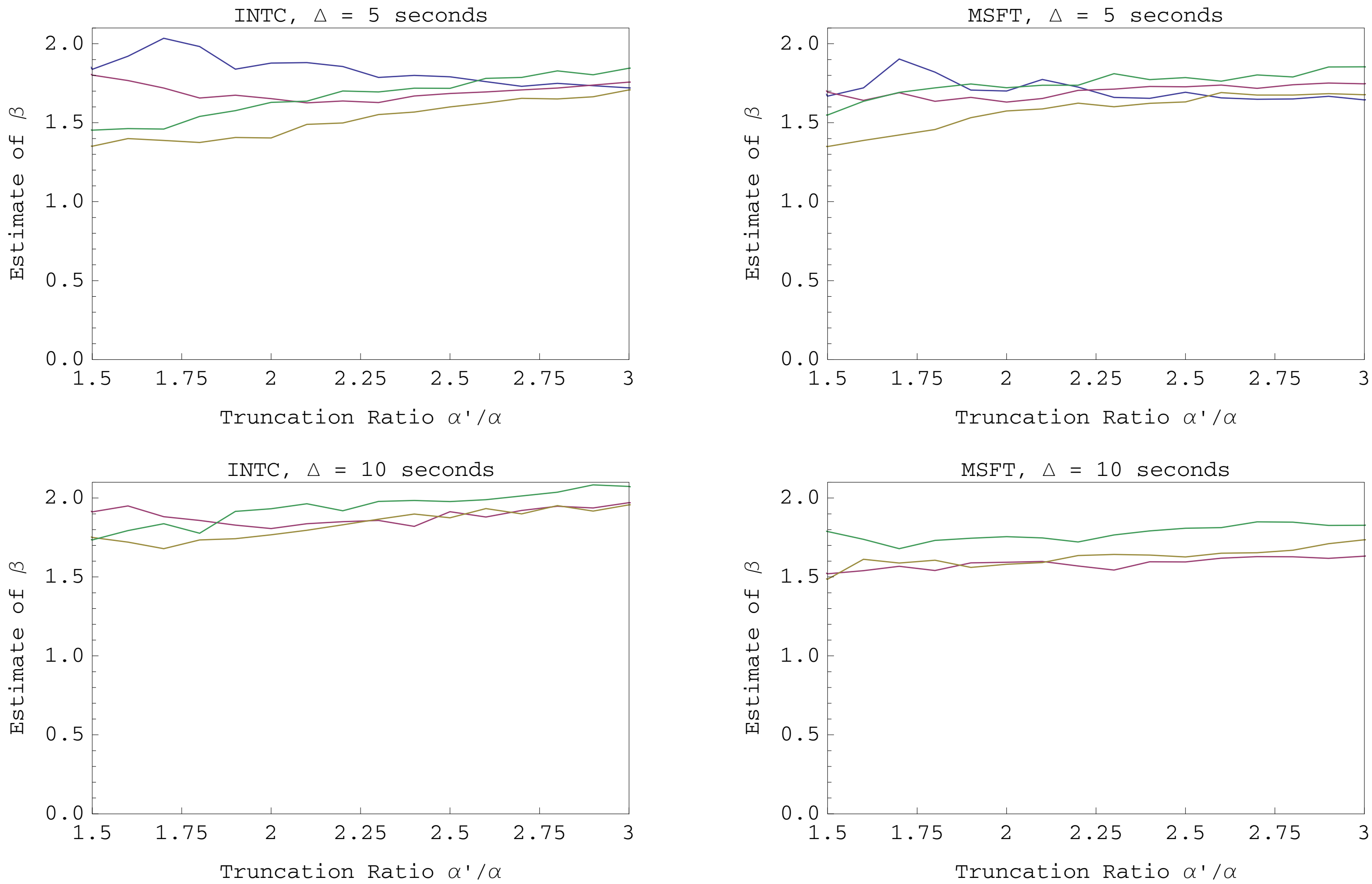





Tails of INTC Log-Returns Density
-0.02
-0.01
0
0.01
0.02


Tails of MSFT Log-Returns Density
-0.02
-0.01
0
0.01
0.02

Estimates of the Degree of Jump Activity, 2006



	INTC				MSFT			
Δ_n	5 sec		15 sec		5 sec		15 sec	
α	7	AVG	7	AVG	7	AVG	7	AVG
α'/α	2	AVG	2	AVG	2	AVG	2	AVG
$\hat{\beta}_n$	1.43	1.56	1.76	1.72	1.69	1.62	1.60	1.61
$\tilde{\beta}_n$		1.52		1.69		1.60		1.59
	(0.04)	(0.003)	(0.05)	(0.006)	(0.05)	(0.004)	(0.05)	(0.005)

8. Conclusions

- Jumps are prevalent in these data

- Especially if one accounts for small, infinite activity, jumps.