Lecture 5b

Testing for Jumps in a Discretely Observed Process: Low Frequency

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References

- Telling From Discrete Data Whether the Underlying Continuous-Time Model is a Diffusion, *Journal of Finance*, 2002, 57, 2075-2112.

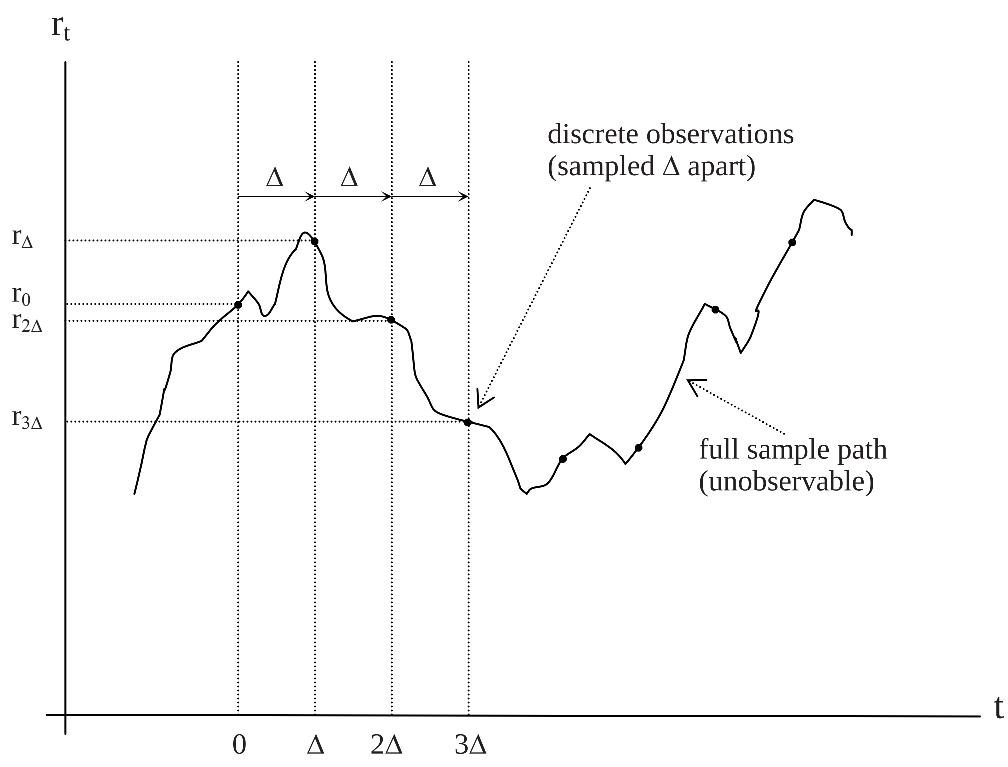
1. Introduction

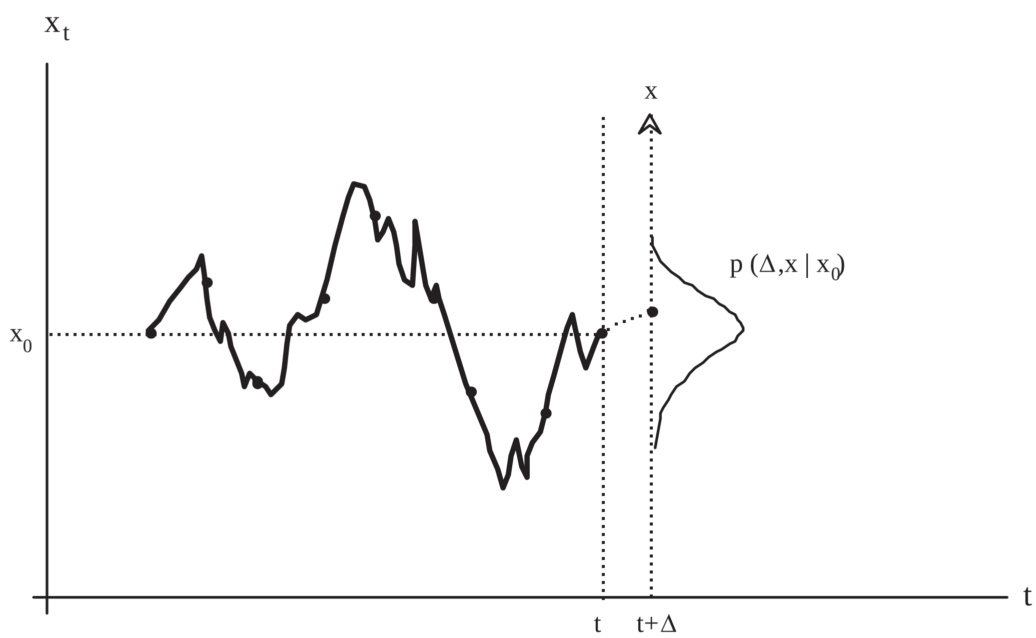
- Assume that X is a Markov process.

- Does the process belong to the smaller class of diffusions? That is, does it have continuous sample paths?

- How can we answer that question if we only sample X discretely, and, to compound the problem, not necessarily at high frequency?

Transition function: $p(\Delta, y|x)$ is the conditional density of $X_{t+\Delta} = y$ given $X_t = x$





x_t
x
$p(\Delta, x \mid x_0)$
x_0
t
t
t+Δ

2. Example

- Suppose that we have found empirically that discrete interest rate data approximately follow:

$$r_{t+\Delta}|r_t \sim \mathcal{N}\left(\gamma_0 + \gamma_1 r_t,\ \delta_0^2\right)$$

- We can then construct a continuous-time diffusion:

$$dr_t = \beta\left(\alpha - r_t\right)dt + \sigma dZ_t$$

- For which $p(\Delta, y|x)$ is Gaussian with:

$$E\left[r_{t+\Delta}|r_t\right] = r_t + \left(\alpha - r_t\right)e^{-\beta\Delta}$$
$$V\left[r_{t+\Delta}|r_t\right] = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta\Delta}\right)$$

- Now set:

$$\beta = -Ln\,(1 - \gamma_1)/\Delta$$
$$\alpha = \gamma_0/(1 - \gamma_1)$$
$$\sigma^2 = -2\delta_0^2 Ln\,(1 - \gamma_1)\,/\,\Big((1 - \gamma_1)^2\Delta\Big)$$

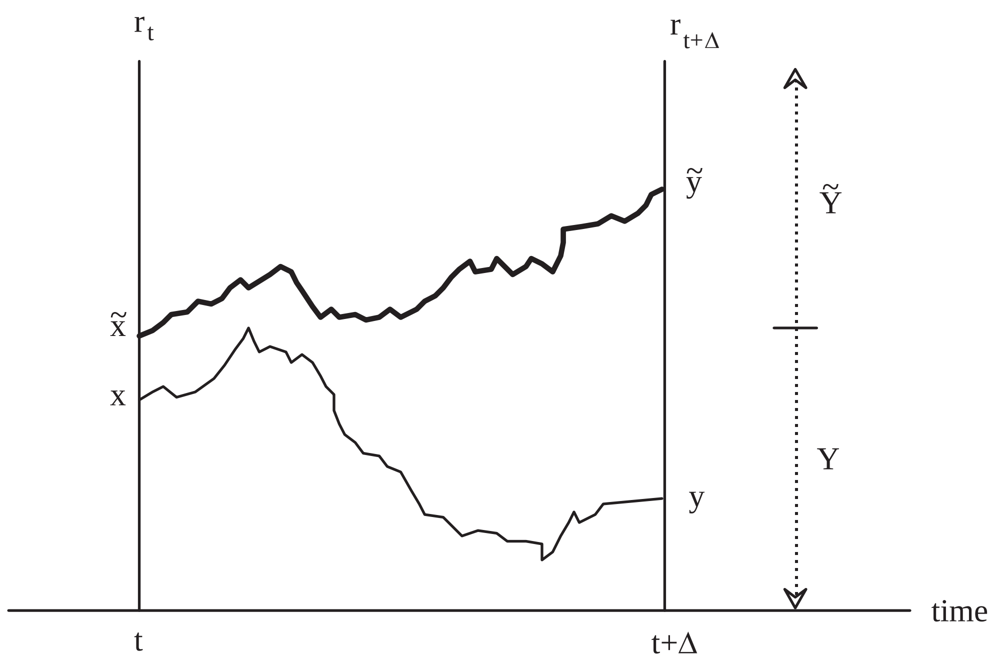
- The continuous-time diffusion $dr_t = \beta(\alpha - r_t)dt + \sigma dW_t$ is fully determined

- Unfortunately, such calculations are impossible to conduct in most cases!

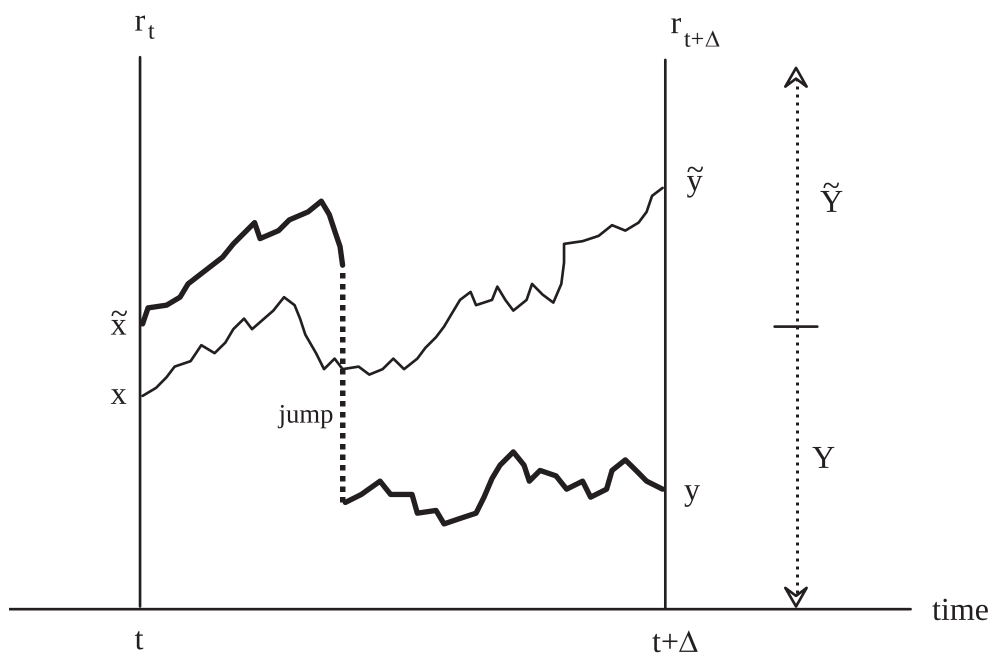
3. Geometric Implication of the Continuity of Sample Paths

- Consider two processes $\{r_t, t \geq 0\}$ and $\{\tilde{r}_t, t \geq 0\}$ on $\mathbb{R}$ with the same distribution, starting at $r_t = x$ and $\tilde{r}_t = \tilde{x}$ with $x < \tilde{x}$.

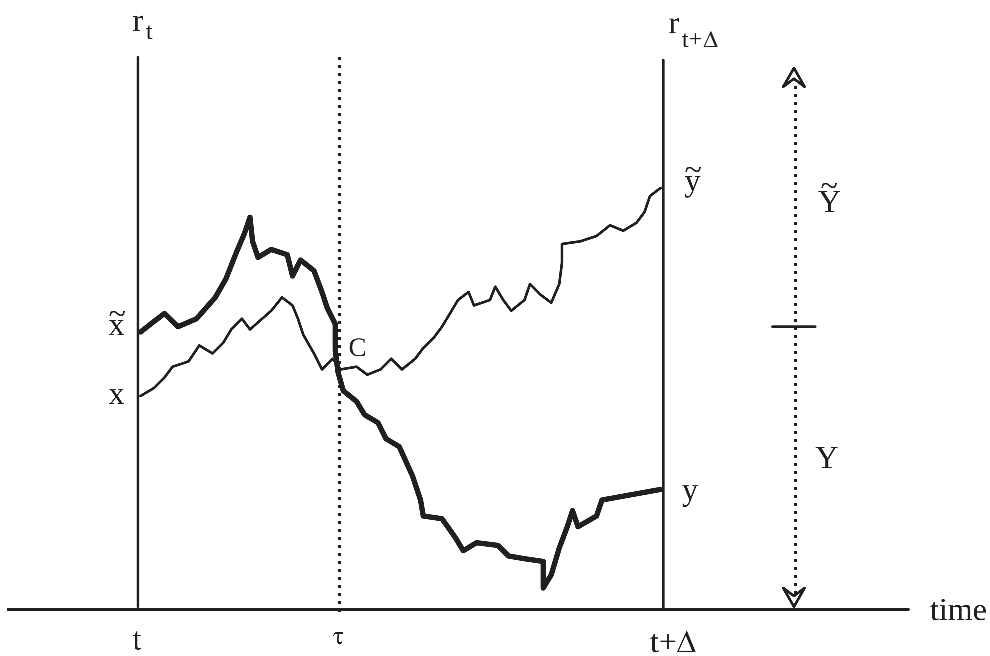
- If the process has continuous sample paths, then at any future date $t + \Delta$, the process r cannot be above $\tilde{r}$ without their sample paths having crossed at least once.



r t
r t+Δ
x̃
x
ỹ
y
Ỹ
Y
time
t
t+Δ


r t
r t+Δ
ỹ
Ỹ
x̃
x
jump
Y
y
time
t
t+Δ

- Immediately after they cross, they are indistinguishable by the Markov property and we can interchange them.



- We will see that this simple observation implies that the process is a diffusion if and only if its transition function satisfies:

$$\frac{\partial^2}{\partial x \partial y} \ln \left(p(\Delta, y|x) \right) > 0$$

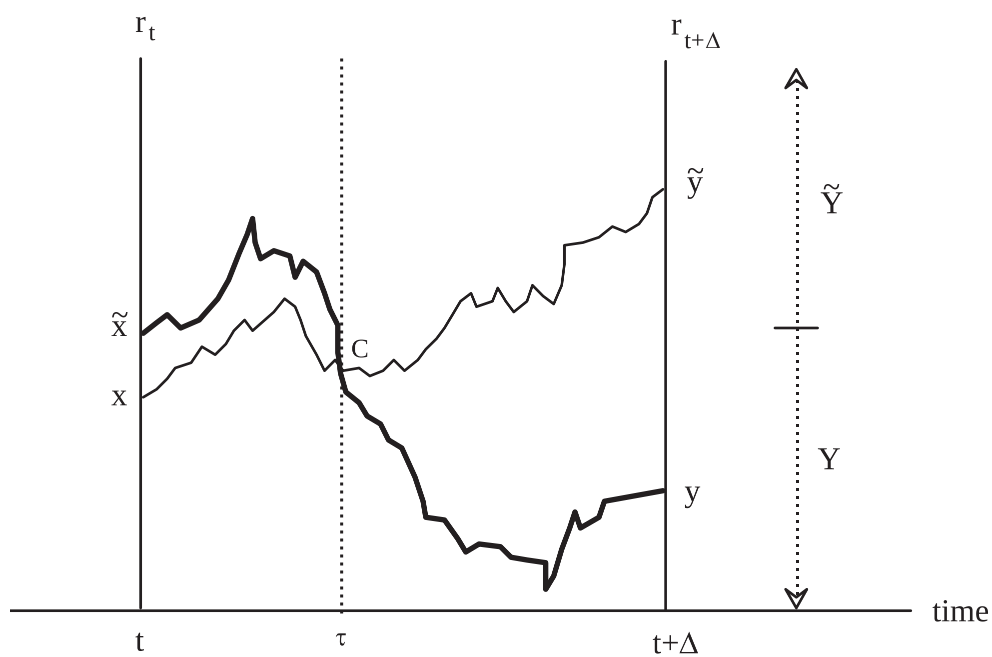
- For all (x, y) and $\Delta > 0$. Not just for small Δ.

- Sketch of proof of the necessary part

- Consider two sets Y and $\tilde{Y}$, with $Y < \tilde{Y}$ (meaning each element of Y and $\tilde{Y}$ satisfy the inequality).

- Coincidence probability (Karlin and McGregor)

$$\begin{aligned}
&\mathsf{Pr}\left(r_{t+\Delta} \in Y,\ \tilde{r}_{t+\Delta} \in \tilde{Y},\ \{\exists s \in [t, t+\Delta] \,/ r_s = \tilde{r}_s\}\,|r_t = x,\ \tilde{r}_t = \tilde{x}\right) \\
&= \mathsf{Pr}\left(\tilde{r}_{t+\Delta} \in Y,\ r_{t+\Delta} \in \tilde{Y},\ \{\exists s \in [t, t+\Delta] \,/ r_s = \tilde{r}_s\}\,|r_t = x,\ \tilde{r}_t = \tilde{x}\right) \\
&= \mathsf{Pr}\left(\tilde{r}_{t+\Delta} \in Y,\ r_{t+\Delta} \in \tilde{Y}|r_t = x,\ \tilde{r}_t = \tilde{x}\right)
\end{aligned}$$



r t
r t+Δ
x̃
x
C
ỹ
y
Ỹ
Y
t
τ
t+Δ
time

- By independence of r and $\tilde{r}$,

$$\Pr\left(r_{t+\Delta} \in Y,\ \tilde{r}_{t+\Delta} \in \tilde{Y} | r_t = x,\ \tilde{r}_t = \tilde{x}\right) = P\left(\Delta, Y|x\right) P\left(\Delta, \tilde{Y}|\tilde{x}\right)$$
$$\Pr\left(\tilde{r}_{t+\Delta} \in Y,\ r_{t+\Delta} \in \tilde{Y} | r_t = x,\ \tilde{r}_t = \tilde{x}\right) = P\left(\Delta, Y|\tilde{x}\right) P\left(\Delta, \tilde{Y}|x\right)$$

where

$$P\left(\Delta, Y|x\right) \equiv \int_{y\in Y} p\left(\Delta, y|x\right) dx$$

- The probability that $r_{t+\Delta} \in Y$ and $\tilde{r}_{t+\Delta} \in \tilde{Y}$ without their sample paths having ever crossed between t and $t+\Delta$ is therefore

$$P\left(\Delta, Y|x\right) P\left(\Delta, \tilde{Y}|\tilde{x}\right) - P\left(\Delta, Y|\tilde{x}\right) P\left(\Delta, \tilde{Y}|x\right)$$

- Since its a probability, it must be positive.

- Therefore:

$$p\,(\Delta, y|x)\,p\,(\Delta, \tilde{y}|\tilde{x}) \;>\; p\,(\Delta, y|\tilde{x})\,p\,(\Delta, \tilde{y}|x)$$

  for all $x < \tilde{x}$ and $y < \tilde{y}$ in $\mathbb{R}$.

- This is equivalent by taking limits as $\tilde{y} \to y$ and $\tilde{x} \to x$ to

$$\frac{\partial^2}{\partial x \partial y} \ln\left(p(\Delta, y|x)\right) > 0.$$

- The sufficiency part is more involved (see paper).

4. Example: SDEs driven by Brownian vs. Cauchy

- Let's distinguish:

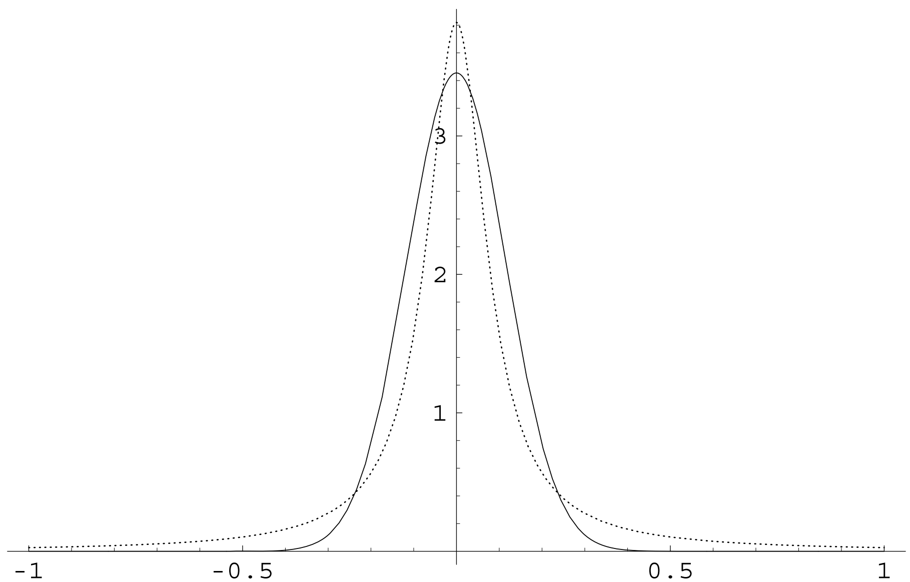
$$dr_t = \beta\left(\alpha - r_t\right) dt + \sigma dW_t$$

from:

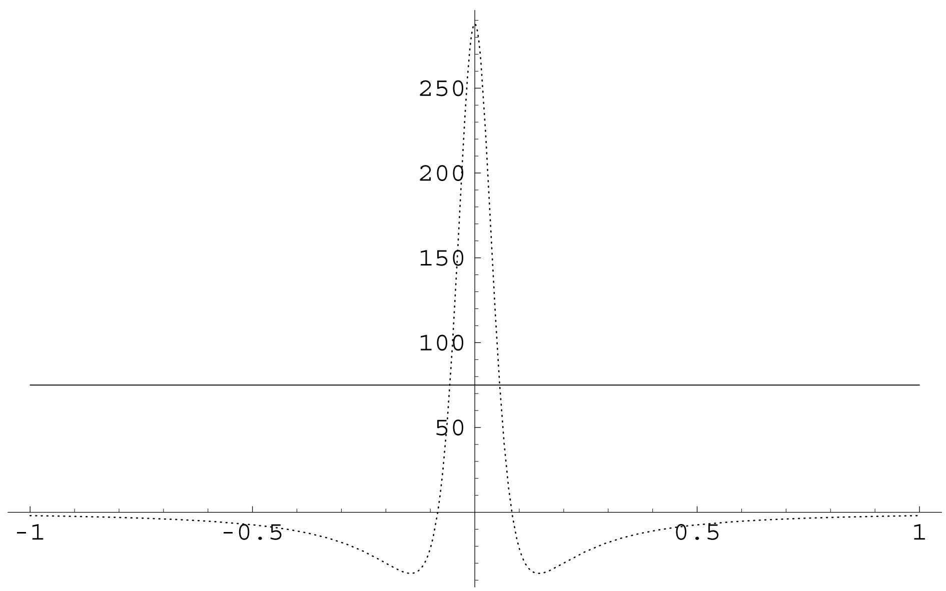
$$dr_t = \beta\left(\alpha - r_t\right) dt + \sigma dC_t$$

where W is a Brownian motion and C a Cauchy process.

The two transition functions

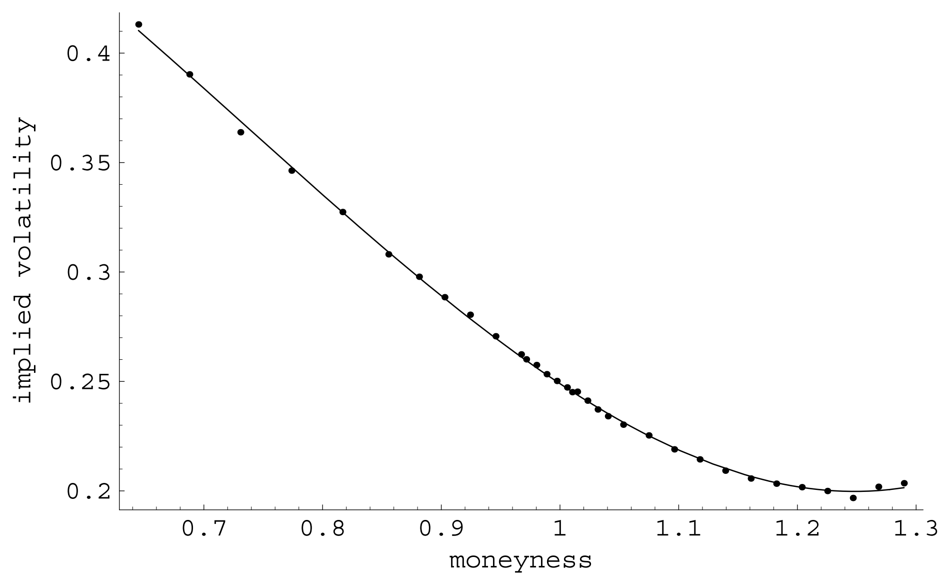


$\frac{\partial^2}{\partial x \partial y} \ln p$ for the two transition functions

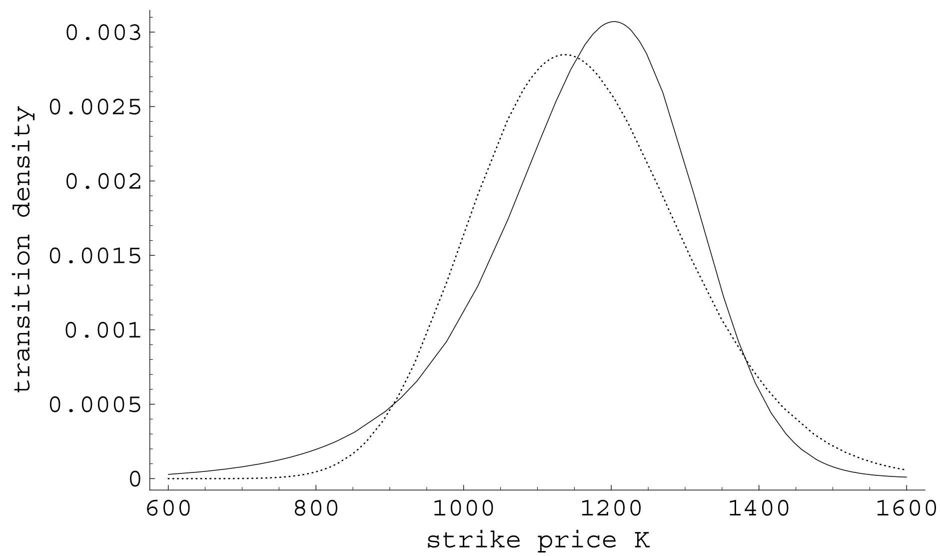


5. Empirical Results: SPX Options

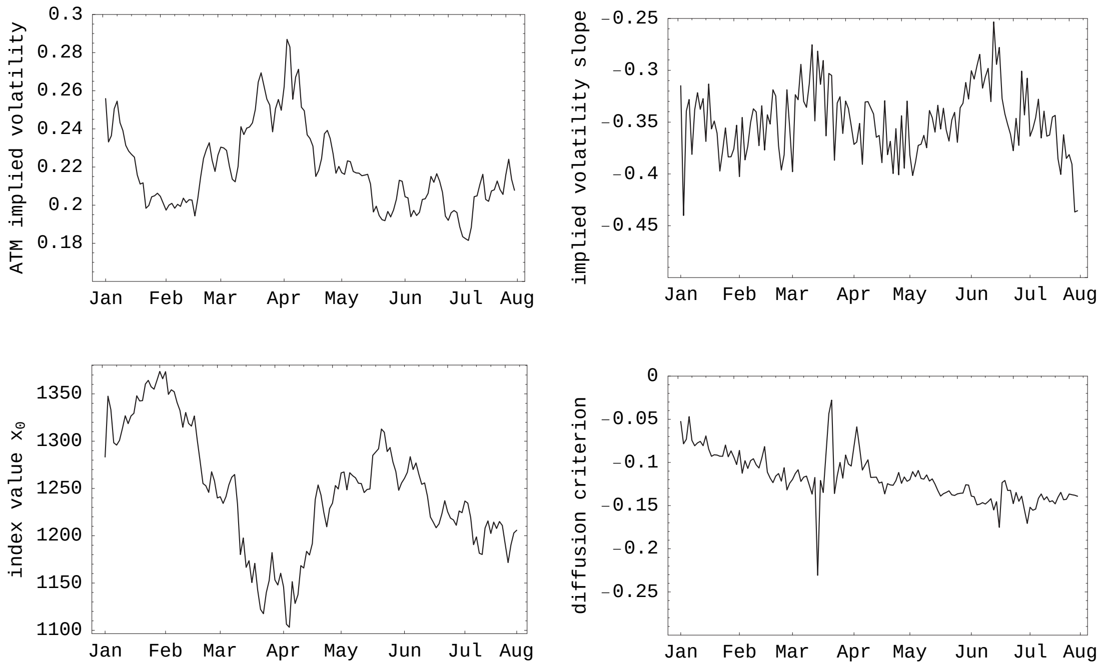
SPX Implied Volatility Smile



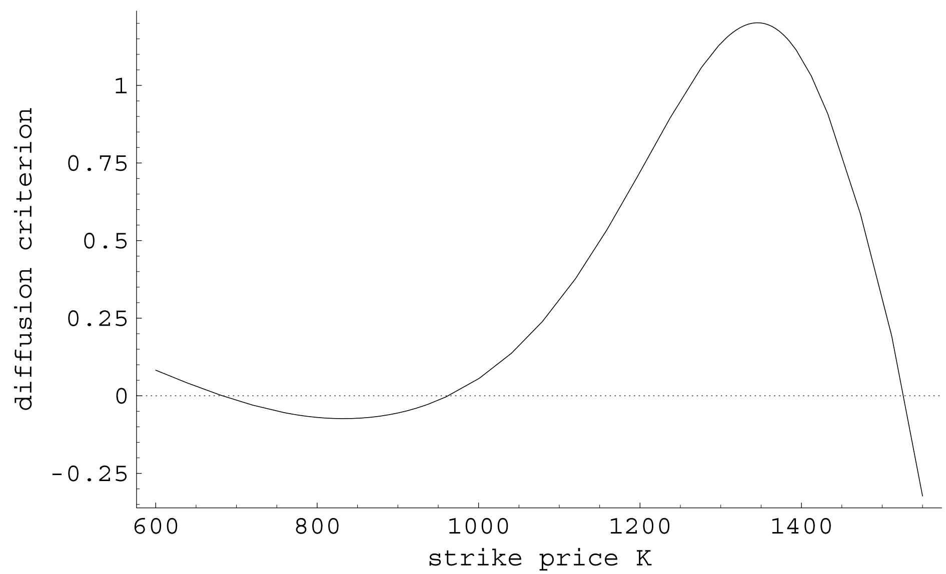
SPX State-Price Density

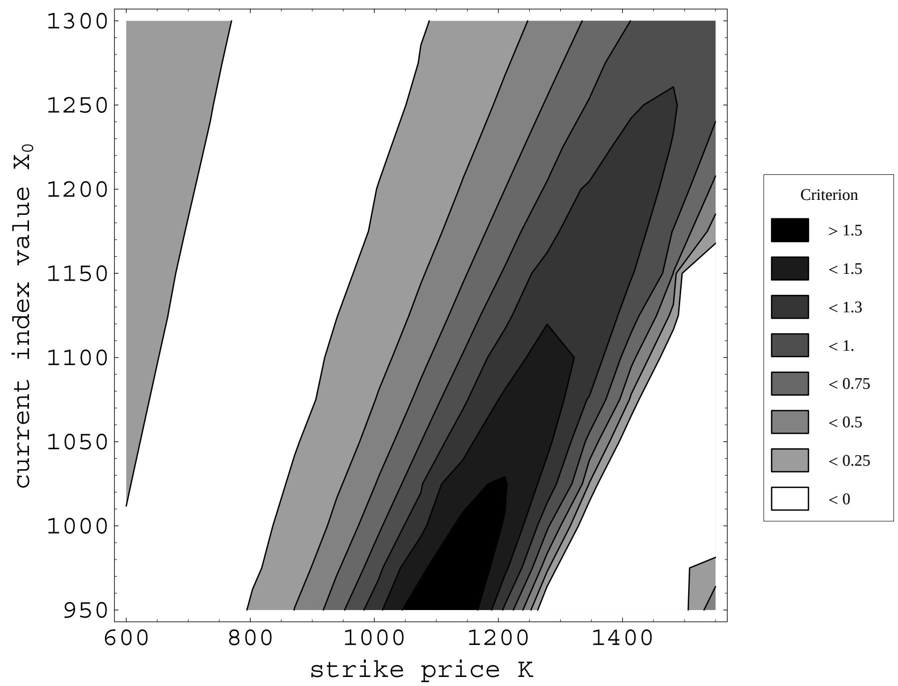


Time Series


0.3
0.28
0.26
0.24
0.22
0.2
0.18
ATM implied volatility
Jan Feb Mar Apr May Jun Jul Aug
-0.25
-0.3
-0.35
-0.4
-0.45
implied volatility slope
Jan Feb Mar Apr May Jun Jul Aug
1350
1300
1250
1200
1150
1100
index value x0
Jan Feb Mar Apr May Jun Jul Aug
0
-0.05
-0.1
-0.15
-0.2
-0.25
diffusion criterion
Jan Feb Mar Apr May Jun Jul Aug

$\frac{\partial^2}{\partial x \partial y} \ln p$ for the data SPX




1300
1250
1200
1150
1100
1050
1000
950
current index value X_0
600
800
1000
1200
1400
strike price K
Criterion
> 1.5
< 1.5
< 1.3
< 1.
< 0.75
< 0.5
< 0.25
< 0

6. Conclusions

- Option prices say jumps are present.

- No need for jumps to be observed, the mere possibility that they happen is sufficient.

- No need for high frequency data.