Lecture 5a

Testing for Jumps in a Discretely Observed Process: High Frequency

Yacine Aït-Sahalia
Princeton University

based on joint work with Jean Jacod

References

- Testing for Jumps in a Discretely Observed Process, with Jean Jacod, forthcoming in the *Annals of Statistics.*

1. Introduction

- Different types of jumps
  - Large jumps, which are rather infrequent, are easy to pick out.
  - But visual inspection of most time series in finance does not provide clear evidence for either the presence or the absence of smaller, more frequent, jumps.

- Models with and without jumps do have quite different properties, both mathematical and financial:
  - Model calibration
  - Volatility estimation
  - Market (in)completeness
  - Option pricing and hedging
  - Risk management
  - Portfolio choice

- Detecting jumps: other approaches
  - Aït-Sahalia (2002)
  - Carr and Wu (2003)
  - BNS (2004), ABD (2004), Huang and Tauchen (2006)
  - Lee and Mykland (2005)
  - Jiang and Oomen (2006)

- This paper: we propose a very simple family of test statistics for jumps which converge as $\Delta_n \to 0$:
  - To 1 if there are jumps
  - To 2 if there are no jumps.

- We provide a distribution theory (hence a test) for the null hypothesis where no jumps are present, but also one for the null where jumps are present.

- This works as soon as the process X is an Itô semimartingale
  - This is a much weaker condition than what is usually assumed (compound Poisson processes, or jump-diffusions)
  - The limit depends neither on the law of the process nor on the coefficients of the (possibly very complicated) SDE
  - So the test does not require any estimation of these coefficients.

2. The Setup

- The structural assumption is that X is an Itô semimartingale on some filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$:

$$\begin{aligned} X_t &= X_0 + A_t + M_t \\ &= X_0 + A_t^C + A_t^J + M_t^C + M_t^J \end{aligned}$$

where A_t is a finite variation and predictable mean component and M_t is a local martingale

- Each decomposable into a continuous and a pure jump part.

- The drift, volatility and jump measure are themselves possibly stochastic and can possibly jump.

- We assume that the continuous part of X is never degenerate, i.e., we have $\int_0^t |\sigma_s| ds > 0$ a.s. for all $t > 0$.

3. The Testing Problem

- X is discretely observed at times $i\Delta_n$ for all $i = 0, 1, \cdots, n$ with $n\Delta_n = T$.

- When the jump measure is finite, there is a positive probability that the path $X(\omega)$ has no jump on $[0, T]$, although the model itself may allow for jumps: this is the peso problem.

3.1. Various Measures of the Variability of X

- Here are processes which measure some kind of variability of X and depend on the whole (unobserved) path of X:

$$A(p)_t = \int_0^t |\sigma_s|^p ds, \qquad B(p)_t = \sum_{s \leq t} |\Delta X_s|^p$$

where $p > 0$ and $\Delta X_s = X_s - X_{s-}$ are the jumps of X.

- The quadratic variation of the process is $[X, X] = A(2) + B(2)$.

- The problem boils down to deciding whether whether $B(p)_T > 0$ for our particular observed path with any given p.

- Let the observed discrete increments of X (not necessarily due to jumps) be

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$$

and for $p > 0$ define the estimator

$$\widehat{B}(p, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p$$

- For $r > 0$, let

$$m_r = \mathbb{E}(|U|^r) = \pi^{-1/2} 2^{r/2}\, \Gamma\left(\frac{r+1}{2}\right)$$

denote the rth absolute moment of a variable $U \sim N(0,1)$.

- We have the following convergences in probability, locally uniform in t:

$$\begin{cases} p > 2, \text{ all } X & \Rightarrow \quad \widehat{B}(p,\Delta_n)_t \xrightarrow{\mathbb{P}} B(p)_t \\ p = 2, \text{ all } X & \Rightarrow \quad \widehat{B}(p,\Delta_n)_t \xrightarrow{\mathbb{P}} A(2)_t + B(2)_t \\ p < 2, \text{ all } X & \Rightarrow \quad \frac{\Delta_n^{1-p/2}}{m_p}\, \widehat{B}(p,\Delta_n)_t \xrightarrow{\mathbb{P}} A(p)_t \\ \text{all } p, \ X \text{ continuous} & \Rightarrow \quad \frac{\Delta_n^{1-p/2}}{m_p}\, \widehat{B}(p,\Delta_n)_t \xrightarrow{\mathbb{P}} A(p)_t. \end{cases}$$

3.2. The Basic Idea

- $$\begin{cases} p > 2, \text{ all } X & \Rightarrow \quad \widehat{B}(p, \Delta_n)_t \stackrel{\mathbb{P}}{\longrightarrow} B(p)_t \\ \text{all } p, \ X \text{ continuous} & \Rightarrow \quad \frac{\Delta_n^{1-p/2}}{m_p} \widehat{B}(p, \Delta_n)_t \stackrel{\mathbb{P}}{\longrightarrow} A(p)_t. \end{cases}$$

- We see that when $p > 2$ the limit $B(p)_t$ of $\widehat{B}(p, \Delta_n)_t$ does not depend on Δ_n, and $B(p)_t > 0$ is strictly positive if X has jumps between 0 and t.

- On the other hand when X is continuous on $[0, t]$, then the limit is $B(p)_t = 0$ but, after a normalization which does depend on Δ_n, $\widehat{B}(p, \Delta_n)_t$ converges again to a limit $A(p)_t$ not depending on Δ_n.

- These considerations lead us to compare $\widehat{B}(p, \Delta_n)$ on two different Δ_n−scales.

- Specifically, for an integer k, consider:

$$\widehat{S}(p, k, \Delta_n)_t = \frac{\widehat{B}(p, k\Delta_n)_t}{\widehat{B}(p, \Delta_n)_t}.$$

- Theorem: For any $t > 0$ the variables $\widehat{S}(p, k, \Delta_n)_t$ converge in probability to

$$\begin{cases} 1 & \text{if } X \text{ jumps} \\ k^{p/2-1} & \text{if } X \text{ is continuous} \end{cases}$$

4. Testing for Jumps

- The previous theorem provides the first step towards constructing a test for the presence or absence of jumps.

- But to construct actual tests, we need: rates of convergence and asymptotic variances.

- That are applicable under both nulls of jumps and no jumps.

- We also need consistent estimators of the variances.

4.1. CLT for Standardized Statistics

Theorem:

1. Let $p > 3$. The variables $(\widehat{V}_n^j)^{-1/2}\ \left(\widehat{S}(p,k,\Delta_n)_t - 1\right)$ converge stably in law, in restriction to the set Ω_t^j to a variable which, conditionally on $\mathcal{F}$, is centered with variance 1, and which is $N(0,1)$ if in addition the processes σ and X have no common jumps.

2. If X is continuous, then for $p \geq 2$

$$(\widehat{V}_n^c)^{-1/2}\ \left(\widehat{S}(p,k,\Delta_n)_t - k^{p/2-1}\right) \to N(0,1)$$

stably in law, conditionally on $\mathcal{F}$.

4.2. Practical Considerations

- Since we must have $p > 3$, a rather natural choice seems to be $p = 4$.

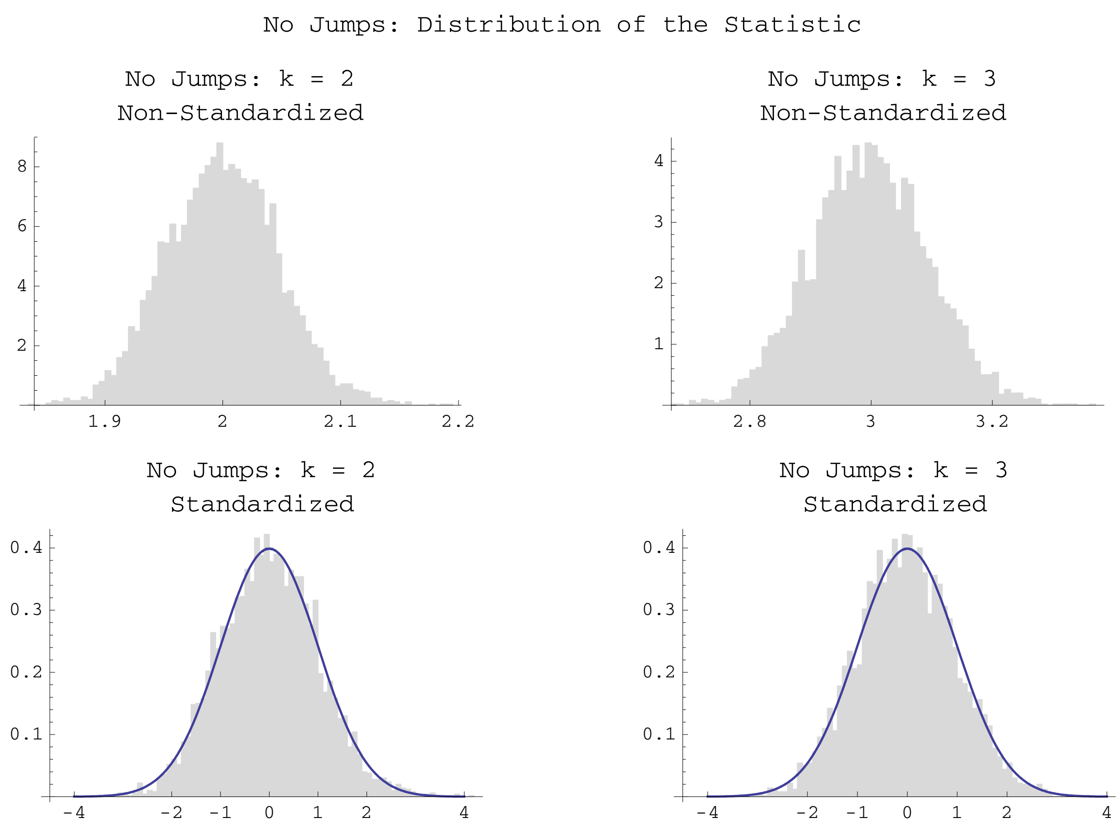
- We see that the variances are increasing with k, so it is probably wise to take $k = 2$ (although when $k > 2$ we have to separate the two points 1 and k, which are further apart than 1 and 2).

5. Simulation Results

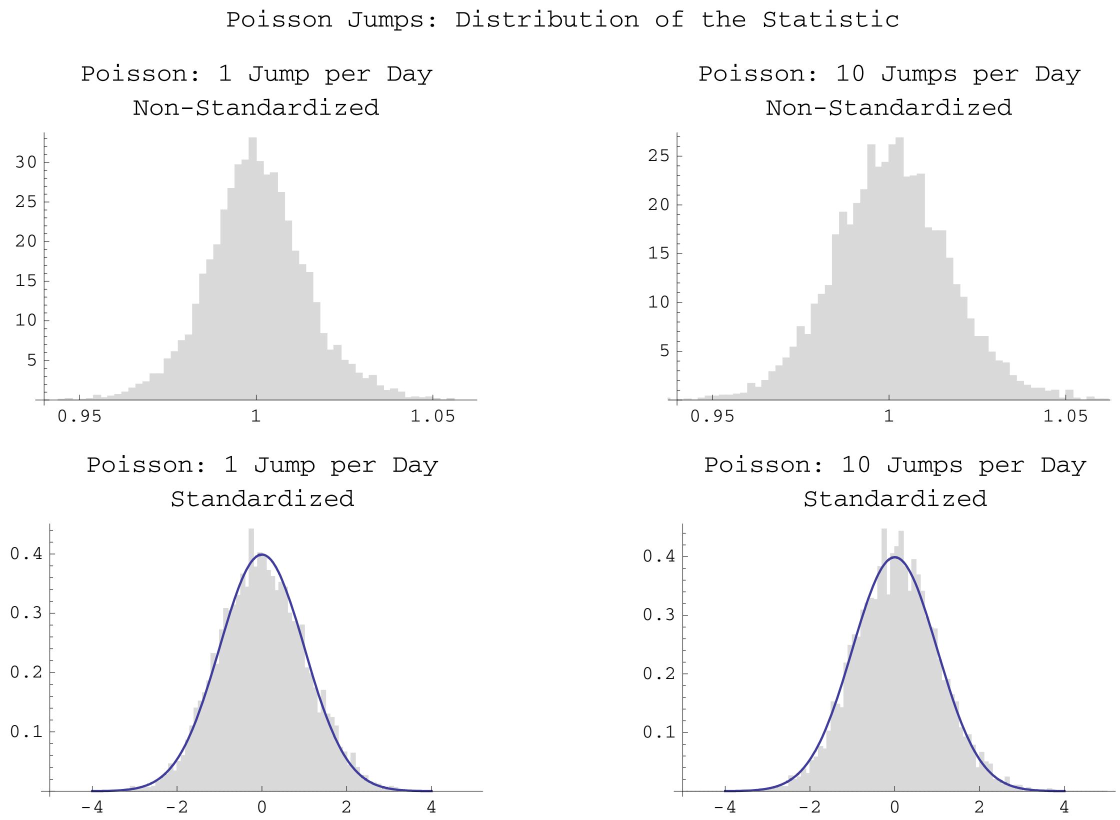
- We calibrate the values to be realistic for a liquid stock trading on the NYSE.

- We use an observation length of $T = 1$ day, consisting of 6.5 hours of trading, that is $23,400$ seconds.

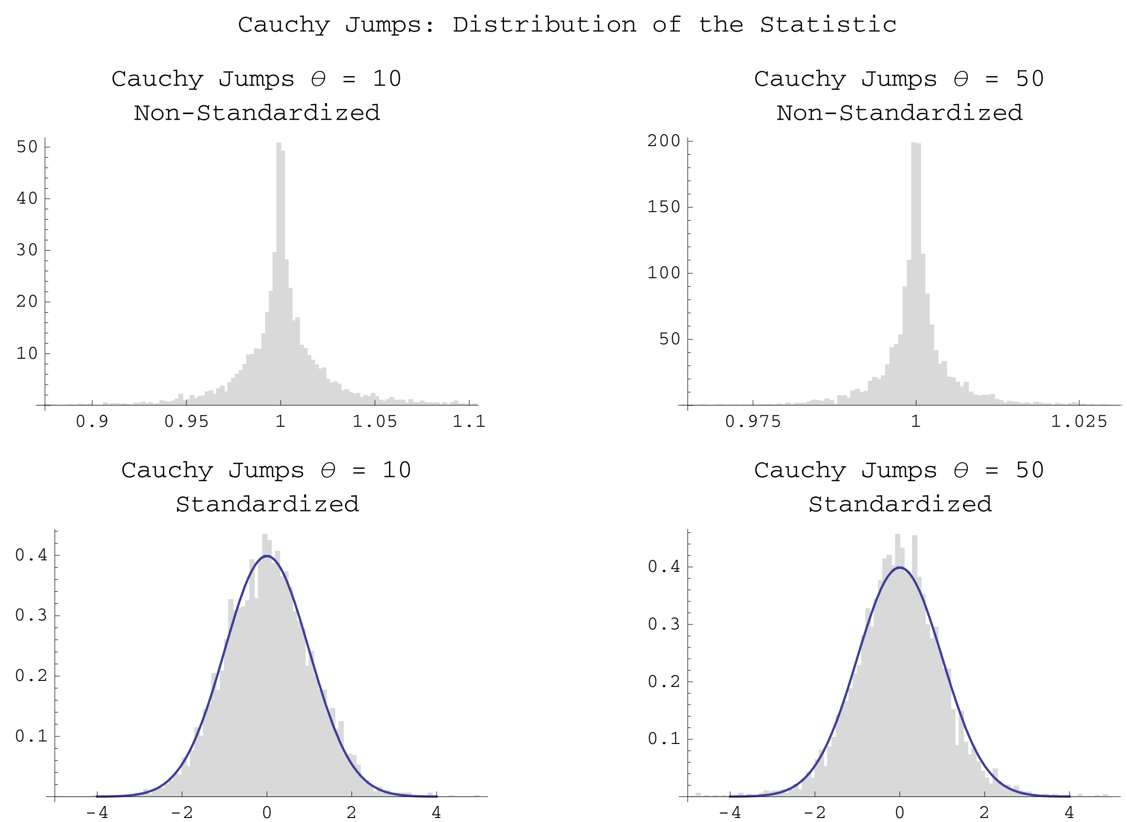
Simulations: Null of No Jumps, $k = 2$ and 3



Simulations: Poisson Jumps

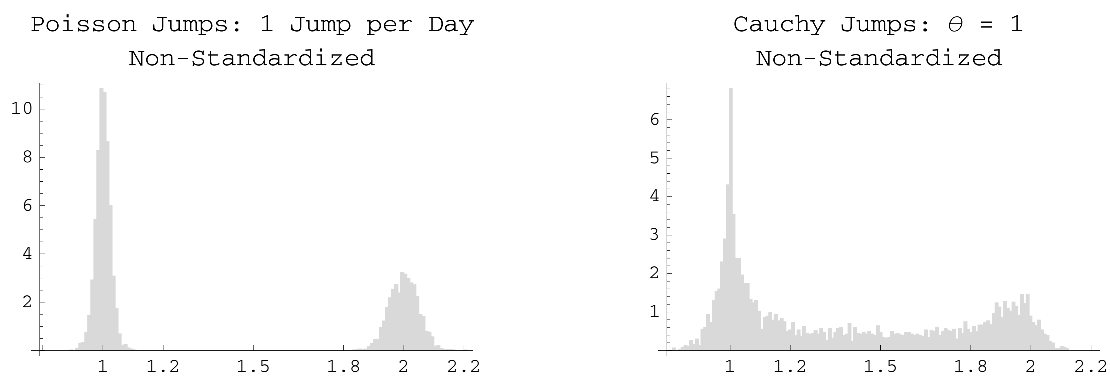


Simulations: Cauchy Jumps



Simulations: Tiny or No Jumps

Tiny Jumps or No Jumps: Distribution of the Statistic



6. Real Data Analysis

- In real data, observations of the process X are blurred by market microstructure noise, which messes things up at very high frequency.

- Assume that each observation is affected by an additive noise, that is instead of $X_{i\Delta_n}$ we really observe $Y_{i\Delta_n} = X_{i\Delta_n} + \varepsilon_i$, and the ε_i are i.i.d. with $E(\varepsilon_i^2)$ and $E(\varepsilon_i^4)$ finite.

- We show that, in the presence of noise, the limit of our test statistics $\widehat{S}(4, k, \Delta_n)_t$ becomes as $\Delta_n \to 0$:

$$\widehat{S}(4, k, \Delta_n)_t \ \xrightarrow{\mathbb{P}} \ \frac{1}{k}$$

Real Data Analysis: 30 DJIA Stocks, All 2005 Trading Days

Empirical Distribution of the Test Statistic: DJIA30 All 2005 Trading Days