Lecture 3

Introduction to Jump Processes

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1. Lévy Processes

- A Lévy process is a cadlag process X on $\mathbb{R}^d$ with $X_0 = 0$ and the following three properties:
  - Independent increments: for any $t_0 < t_1 < t_2 < ...$, $X_{t_1} - X_{t_0}$, $X_{t_2} - X_{t_1}$, etc. are independent.
  - Stationary increments: the law of $X_{t+s} - X_t$ does not depend on t
  - Stochastic continuity: $\forall t > 0$, $\forall \varepsilon > 0$, $\lim_{s \to 0} \mathbb{P}(|X_{t+s} - X_t| > \varepsilon) = 0$

- The last condition does not make the paths of X continuous. But it excludes jumps at deterministic times. It also does not mean that a path on a finite interval $[0, T]$ contains a finite number of jumps. All it says is that at any given time t, the probability of seeing a jump is 0.

1.1. Infinite divisibility

- If we sample a Lévy process at time intervals Δ, we have $X_{n\Delta} = \sum_{i=1}^{n} Y_{i\Delta}$ where $Y_{i\Delta} = X_{i\Delta} - X_{(i-1)\Delta}$ so by choosing combinations of n and Δ such as $n\Delta = t$ we see that X_t can be written in an infinite number of ways, or divided, into a sum of iid rvs whose distribution is that of $X_{t/n} - X_0 = X_{t/n}$.

- Since the increments are iid, the distribution of their sum (that of X_t) is the convolution of the individual increments distribution (that of $X_{t/n}$).

- A cdf F on $\mathbb{R}^d$ is said to be infinitely divisible if for any $n \geq 2$ there exist n iid rvs $Y_1, ..., Y_n$ such that $Y_1 + ... + Y_n$ has distribution F.

- The distribution F of X_t must be infinitely divisible. Examples are the Gaussian, Gamma, $\alpha-$stable, Poisson. In the Gaussian case, we see that if $X \sim N(m, v)$ then X can be written as the sum of n iid rvs $Y_i \sim N(m/n, v/n)$.

- Conversely, if F is ∞d, then one can construct a Lévy process such that the distribution function of X_1 is F.

- The laws of X_t for all t's are completely characterized by the law of X at any single time, for example that of X_1.

- Consequence of ∞d for the characteristic function: there exists a function $\Psi : \mathbb{R}^d \to \mathbb{R}$, called the characteristic exponent of X such that

$$E\left[e^{iu \cdot X_t}\right] = \exp\left(t\Psi(u)\right), \; u \in \mathbb{R}^d, \; t > 0$$

1.2. Compound Poisson Process

- A CPP with jump intensity λ and jump distribution F is a stochastic process

$$X_t = \sum_{i=1}^{N_t} J_i$$

where N_t is a Poisson process with intensity λ and the jump sizes J_i's are iid rvs with distribution F, all mutually independent.

- X_t jumps whenever N_t does, and by an amount J_i.

- X_t has a finite amount of jumps on any finite time interval.

- N_t is the special case of a CPP on $\mathbb{R}$ where $J_i \equiv 1$ (F is a Dirac mass at 1).

- X_t is a CPP iff it is a Lévy process and its sample paths are piecewise constant.

- The characteristic function of X_t is

$$
\begin{aligned}
E\left[e^{iu\cdot X_t}\right] &= \exp\left(t\Psi(u)\right) \\
&= \exp\left(t\lambda \int_{\mathbb{R}^d}\left(e^{iu\cdot x}-1\right)F(dx)\right) \\
&= \exp\left(t \int_{\mathbb{R}^d}\left(e^{iu\cdot x}-1\right)\nu(dx)\right)
\end{aligned}
$$

  where the measure $\nu(dx) = \lambda F(dx)$ is the Lévy jump measure of X_t.

  - ν is not a probability measure, because $\nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(dx) = \lambda \neq 1$ in general.

2. Lévy processes and Poisson random measures

- To every cadlag process X on $\mathbb{R}^d$ one can associate a random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ describing the jumps of X : for any measurable Borel set $A \subset \mathbb{R}_+ \times \mathbb{R}^d$

$$M(A) = \# \{(t, \Delta X_t) \in A\}$$

When $A = [s,t] \times B$, $M(A)$ counts the number of jumps of X that took place between times s and t, and whose sizes where contained in the set $B \subset \mathbb{R}^d$. B needs to be bounded away from 0.

- One alternative way to write it is to define

$$T_B^1 = \inf \{t > 0 : \Delta X_t \in B\}$$
$$\vdots$$
$$T_B^{n+1} = \inf \{t > T_B^n : \Delta X_t \in B\}$$

and then define for $A = [0, t] \times B$

$$\begin{aligned} M_t(B) &= M(\omega, A) \\ &= M(\omega, [0, t] \times B) \\ &= \textstyle\sum_{0<s\leq t} 1_{\{\Delta X_s \in B\}} \\ &= \textstyle\sum_{n=1}^{\infty} 1_{\{T_B^n \leq t\}} \end{aligned}$$

- Since X is a Lévy process, $t \mapsto M_t(B)$ is a Poisson process because by stationarity of the increments, for $t > s$, $M_t(B) - M_s(B)$ is the number of jumps that X has between s in t in B, and so has the same distribution as $M_{t-s}(B)$; by independence of the increments of X, those of $M_t(B)$ are also independent. Therefore $t \mapsto M_t(B)$ is a counting process with stationary and independent increments = a Poisson process.

- So whenever X is a Lévy process, $t \mapsto M_t(B)$ is a Poisson process, with intensity parameter given by

$$\nu(B) = E\left[M_1(B)\right].$$

- This means that $\nu(B)$ is the expected number of jumps of size that falls in B, per unit of time.

- $\nu(B) < \infty$ as long as $0 \notin B$.

- This is in fact the definition of the Lévy measure ν : $\nu(B)$ is the expected number, per unit of time, of jumps of size that belongs to B. For a Lévy process X on $\mathbb{R}^d$, define

$$\nu(B) = E\left[\#\left\{t \in [0,1] : \Delta X_t \in B,\ \Delta X_t \neq 0\right\}\right] = E\left[M_1(B)\right]$$

where $\Delta X_t = X_t - X_{t-}$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

3. Infinite Activity Lévy Processes

- A CPP X can be written as

$$\begin{aligned} X_t &= \textstyle\sum_{0 \leq s \leq t} \Delta X_s \\ &= \int_0^t \int_{\mathbb{R}^d} x \; M(dt \times dx) \end{aligned}$$

where M is the Poisson random measure with intensity $\mu(dt \times dx) = dt\ \nu(dx) = dt\ \lambda F(dx)$ where F is the distribution function of jump sizes.

- Not every jump process can be written as $\sum_{0 \leq s \leq t} \Delta X_s$ because the sum may diverge. A CPP has a finite number of jumps, so this is not an issue when X is a CPP.

- Every Lévy process with piecewise constant paths is a CPP. Add a Brownian motion (independent of the CPP) plus drift to it, and get

$$\begin{aligned} X_t &= bt + cW_t + X_t^{CPP} \\ &= bt + cW_t + \sum_{0 \leq s \leq t} \Delta X_s \\ &= bt + cW_t + \int_0^t \int_{\mathbb{R}^d} x \; M(dt \times dx) \end{aligned}$$

  where M has intensity $dt \; \lambda F(dx)$.

- Can every Lévy process be decomposed in this way?
  - It is still possible to define $\nu(B)$ as above
  - $\nu(B) < \infty$ as long as $0 \notin B$, otherwise the paths of X would have an infinite number of jumps of finite size on any $[0, T]$ and not be cadlag.

- This makes v a Radon measure on $\mathbb{R}^d \backslash \{0\}$.

- But v is not necessarily a finite measure, because it can diverge ($\to \infty$) near 0, in which case X may have an ∞ number of small jumps in $[0, T]$.

- In this case, the sum $\sum_{0 \leq s \leq t} \Delta X_s$ becomes an infinite series and its convergence will place some constraints on how fast v can diverge near 0.

- The Lévy-Itô decomposition: any Lévy process X is such that

  - There exists a Radon measure ν on $\mathbb{R}^d \backslash \{0\}$ satisfying $\int_{|x| \leq 1} |x|^2 \nu(dx) < \infty$ and $\int_{|x| \geq 1} \nu(dx) < \infty$, written in short as

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge 1 \right) \nu(dx) < \infty.$$

– The jump measure of X, M, is a Poisson jump measure with intensity $\mu = dt\ \nu(dx)$.

– There exists a drift vector b, diffusion matrix c and BM W_t such that

$$X_t = bt + cW_t + Y_t^1 + \lim_{\varepsilon \longrightarrow 0} \tilde{Y}_t^\varepsilon$$

where

$$\begin{aligned} Y_t^1 &= \int_0^t \int_{|x| \geq 1} x\ M(dt \times dx) \\ &= \sum\nolimits_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| \geq 1\}} \\ \tilde{Y}_t^\varepsilon &= \int_0^t \int_{\varepsilon \leq |x| \leq 1} x\ \{M(dt \times dx) - \nu(dx)dt\} \\ &= \int_0^t \int_{\varepsilon \leq |x| \leq 1} x\ \tilde{M}(dt \times dx) \end{aligned}$$

– W_t, Y_t^1 and $\tilde{Y}_t^\varepsilon$ are independent.

– Y_t^1 and $\tilde{Y}_t^\varepsilon$ are pure jump processes.

– Because $\int_{|x|\geq 1} \nu(dx) < \infty$, there is (a.s.) a finite number of jumps of size ≥ 1 and so Y_t^1 is a CPP and we can write $\sum_{0\leq s\leq t} \Delta X_s 1_{\{|\Delta X_s|\geq 1\}}$.

– There is nothing special about the threshold 1. This holds for any finite value:

$$Y_t^\varepsilon = \int_0^t \int_{\varepsilon\leq|x|\leq 1} x \; M(dt \times dx)$$
$$= \sum\nolimits_{0\leq s\leq t:\; \varepsilon\leq|\Delta X_s|\leq 1} \Delta X_s$$

is a properly defined CPP for ε fixed. But the sum may not converge as $\varepsilon \longrightarrow 0$ because there can be an ∞, non-summable, number of jumps of size less than ε.

– To obtain convergence of the sum of small jumps, we need to compensate: hence the use of $\tilde{M}$ in $\tilde{Y}_t^\varepsilon$.

– Y_t^ε is the sum of an ∞ number of independent CPPs. $\tilde{Y}_t^\varepsilon$ is the sum of an ∞ number of independent, compensated, CPPs.

– $\tilde{Y}_t^\varepsilon$ is a martingale.

– By compensating Y_t^ε to get $\tilde{Y}_t^\varepsilon$, we are subtracting a deterministic term from Y_t^ε (recall that M is random, but ν is not), so the difference can be viewed as an adjustment to the drift.

– The compensation centers the CPPs, and we can conceivably get CLTs for $\tilde{Y}_t^\varepsilon$.

– (b, c, ν) is called the characteristic triplet of X.

4. The Lévy-Khintchine formula

- Given that $X_t = bt + cW_t + Y_t^1 + \lim_{\varepsilon \to 0} \tilde{Y}_t^\varepsilon$ with independent terms, the characteristic function of X is the product of the characteristic functions of the four terms.

- The characteristic function of a Lévy process with characteristic triplet (b, c, ν) is $E\left[e^{iu \cdot X_t}\right] = \exp\left(t\Psi(u)\right)$ where the characteristic exponent is

$$\Psi(u) = iu\cdot b - \frac{1}{2}u'cu + \int_{\mathbb{R}^d}\left(e^{iu\cdot x} - 1 - iu\cdot x\mathbf{1}_{|x|\le 1}\right)\nu(dx)$$

$$= \underbrace{iu\cdot b}_{\text{from } bt} - \underbrace{\frac{1}{2}u'cu}_{\text{from } cW_t} + \underbrace{\int_{|x|\ge 1}\left(e^{iu\cdot x} - 1\right)\nu(dx)}_{\text{from } Y_t^1\text{: char fct of a CPP}}$$

$$+ \underbrace{\int_{|x|\le 1}\left(e^{iu\cdot x} - 1 - iu\cdot x\right)\nu(dx)}_{\text{from } \lim_{\varepsilon\to 0}\tilde{Y}_t^{\varepsilon}}$$

4.1. Moments of Lévy Processes

- A Lévy process X with characteristics (b, c, ν) has $E[|X_t|^n] < \infty$ for some t (or equivalently for all t) iff

$$\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$$

  - Obviously, the divergence of the integral $E[|X_t|^n]$ for $n \geq 1$ can only happen for large values of x, hence it's the tail behavior of $\nu(dx)$ near ∞ that matters.

  - In particular, if the jumps of X are bounded

$$\sup_t |X_t| \leq C < \infty \text{ a.s.}$$

    then $E[|X_t|^n] < \infty$ for all $n \geq 1$. This is not the case in typical examples.

- Because of ∞d, if the moments are finite for small t they are finite for all t.

- Cumulants can be computed from the log-characteristic function by differentiation:

$$c_n = \frac{1}{i^n}\frac{\partial^n}{\partial u^n}(t\Psi(u))_{|u=0} = n\text{th cumulant}$$

where

$$\begin{aligned} c_1 &= m_1 = E[X] \\ c_2 &= \mu_2 = m_2 - m_1^2 = \text{Var}(X) \\ c_3 &= \mu_3 = m_3 - 3m_2m_1 + 2m_1^3 \\ c_4 &= \mu_4 - 3\mu_2 \end{aligned}$$

where $m =$ non-centered moments of X, $\mu =$ centered.

- Here

$$\Psi(u) = iu \cdot b - \frac{1}{2}u'cu + \int_{\mathbb{R}^d} \left(e^{iu\cdot x} - 1 - iu \cdot x 1_{|x|\le 1}\right) \nu(dx)$$

and so

$$c_1 = E[X_t] = \frac{1}{i} t\Psi'(0) = t\left(b + \int_{|x|\ge 1} x\nu(dx)\right)$$

$$c_2 = \text{Var}(X_t) = -t\Psi''(0) = t\left(c + \int_{-\infty}^{+\infty} x^2\nu(dx)\right)$$

$$c_n = t\int_{-\infty}^{+\infty} x^n \nu(dx) \quad \text{for all } n \ge 3.$$

  - All the cumulants increase linearly with the time interval t.
  - All ∞d distributions are leptokurtic since $c_4 = t\int_{-\infty}^{+\infty} x^4\nu(dx) > 0$.

5. Stable Processes

- Self–similar processes
  - BM is self-similar: $\forall a > 0$, $\frac{W_{at}}{a^{1/2}} \overset{d}{=} W_t$.
  - BM with drift, $B_t = bt + W_t$, is not self-similar: $\forall a > 0$, $\frac{B_{at}}{a^{1/2}} \overset{d}{=} B_t + a^{1/2}bt$.

- What are the other self-similar Lévy processes?

$$\exists \phi : \; \forall a > 0, \quad \frac{X_{at}}{\phi(a)} \overset{d}{=} X_t.$$

  - These processes are stable under addition: if $X^{(1)}, \ldots, X^{(n)}$ are n

independent copies of X then there exists a positive number κ_n and a vector k such that $X^{(1)} + ... + X^{(n)} \stackrel{d}{=} \kappa_n X + k$.

– It can be shown that necessarily $\phi(a) = a^{1/\beta}$ for some fixed $0 < \beta \leq 2$, called the stable index of X.

– The only stable distribution with $\beta = 2$ is the Gaussian. This corresponds to BM.

– A real-valued Lévy process with characteristic triplet (b, c, ν) is $\beta-$stable with $0 < \beta < 2$ iff $b = 0$, and

$$\nu(x) = \frac{A}{x^{1+\beta}} \mathbf{1}_{x>0} + \frac{B}{|x|^{1+\beta}} \mathbf{1}_{x<0}$$

for some nonnegative constants A and B.

– When the distribution is symmetric ($A = B$), the characteristic exponent is simply $\Psi(u) = C|u|^\beta$ where C is a constant.

– From the form of ν we see that $E[X^2] = \infty$ for all $\beta < 2$ and that $E[X] = \infty$ if $\beta \leq 1$.

- The densities corresponding to stable processes are not known in closed form except in the three special cases where:

$\beta = 2$: Gaussian $N(\mu, \sigma^2)$

$\beta = 1$: Cauchy: $\frac{\kappa}{\pi((x-\mu)^2+\kappa^2)}$ symmetric around μ

$\beta = \frac{1}{2}$: Lévy: $\left(\frac{\kappa}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp(\frac{-\kappa}{2(x-\mu)}) 1_{x>\mu}$ defined only on $(\mu, +\infty)$

- Tempered stable process: corresponds to

$$\nu(x) = \frac{A}{x^{1+\beta}} e^{-ax} 1_{x>0} + \frac{B}{|x|^{1+\beta}} e^{-b|x|} 1_{x<0}$$

with $A \geq 0$, $B \geq 0$, $a \geq 0$, $b \geq 0$. The objective is to limit the occurence of the very large jumps relative to a stable process. Its characteristic function is explicit.