Lecture 5

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Cosslett's NPMLE $\qquad y_i = \mathbb{1}(x_i'\beta_0 + \varepsilon_i \geq 0)$

$$\mathcal{L}_N(\beta; F) = \sum_i \{ y_i \log F(x_i'\beta) + (1-y_i)\log(1-F(x_i'\beta)) \}$$

$\beta \in \mathcal{B}$, $F \in \mathcal{F}$ $\qquad \varepsilon_i \perp x_i$

$\beta_0, F_0 \in \mathcal{B} \otimes \mathcal{F}$ $\qquad \| F_1 - F_2 \|^2 = \int (F_1(u) - F_2(u))^2 \cdot w(u)\,du$

$$\| \beta_1 - \beta_2 \| = \sqrt{(\beta_1 - \beta_2)'(\beta_1 - \beta_2)}$$

$\mathcal{B} \otimes \mathcal{F}$ is compact

$$\begin{pmatrix} \hat{\beta} \\ \hat{F} \end{pmatrix} = \underset{\mathcal{B}\otimes\mathcal{F}}{\arg\max}\ \mathcal{L}_N(\beta; F(\cdot))$$

Kiefer-Wolfowitz NPML Results: $\| \hat{\beta} - \beta_0 \| \xrightarrow{p} 0$

$\| \hat{F} - F_0 \| \xrightarrow{p} 0$

$\hat{F}(\cdot)$ a step function, "isotonic regression"

<u>Han Maximum Rank Correlation</u>:

$y_i = \mathbb{1}(x_i'\beta_0 + \varepsilon_i \geq 0) \qquad \varepsilon_i \perp x_i, \ \varepsilon_i \sim \varepsilon_j$

Suppose $y_i \neq y_j$; then $y_i > y_j \Rightarrow x_i'\beta_0 + \varepsilon_i \geq 0 > x_j'\beta_0 + \varepsilon_j$

$$\Longrightarrow (x_i - x_j)'\beta_0 \geq \varepsilon_j - \varepsilon_i$$

$$\text{med}\{ y_i - y_j \mid y_i \neq y_j, x_i, x_j \} = \text{sgn}\{ (x_i - x_j)'\beta_0 \}$$

$$\hat{\beta}_{MRC} = \underset{\text{st } \mathcal{B}}{\arg\max} \binom{N}{2}^{-1} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \{ \mathbb{1}(y_i > y_j) \cdot \mathbb{1}(x_i'\beta > x_j'\beta)$$

$$\hat{\beta}_{MRC} = \underset{\beta \in \mathcal{B}}{\text{argmax}} \binom{N}{2}^{-1} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \{ 1(y_i > y_j) \cdot 1(x_i'\beta > x_j'\beta) + 1(y_i < y_j) \cdot 1(x_i'\beta < x_j'\beta) \}$$

$$= \text{argmin} \binom{N}{2}^{-1} \sum_i \sum_{j>i} \left| \text{sgn}(y_i - y_j) - \text{sgn}((x_i - x_j)'\beta) \right| \cdot 1(y_i \neq y_j)$$

Second-Order U-stat: $U_N = \binom{N}{2}^{-1} \sum_{i<j} p(z_i, z_j)$

$p(z_i, z_j) = p(z_j, z_i)$ $\quad z_i \sim iid$

ex: $s_x^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{X})^2$

$$= \binom{N}{2}^{-1} \sum_i \sum_{j>i} \frac{(X_i - X_j)^2}{2}$$

Projection of U-STATISTIC: $r(z_i) \equiv E[p(z_i, z_j) \mid z_i]$

$$\hat{U} = \theta_0 + \frac{2}{N} \sum_{i=1}^{N} (r(z_i) - \theta_0),$$

$$\theta_0 = E[p(z_i, z_j)] = E(U)$$

If $E[(p(z_i, z_j))^2] < \infty$, then

$$U \xrightarrow{p} \theta_0, \quad \sqrt{N}(U - \hat{U}) \xrightarrow{p} 0$$

q^{th} order U-stat $\quad r(z_i) = E[p(z_i, z_j, \ldots) \mid z_i]$

$$\hat{U} = \theta_0 + \frac{q}{N} \sum_i (r(z_i) - \theta_0)$$

MRC $\quad \hat{\beta} = \underset{\beta \in \mathcal{B}}{\text{argmax}}\ U(\beta)$

$$\max \| U(\beta) - E[U(\beta)] \| \xrightarrow{p} 0$$

$\beta \in B$

with identification, $\hat{\beta} \xrightarrow{P} \beta_0$

Sherman $\sqrt{N}(\hat{\beta}_{MRC} - \tilde{\beta}) \xrightarrow{P} 0$

$\tilde{\beta} = \arg\max \tilde{U}(\beta)$ $\quad \tilde{U}(\beta)$ smooth in β

$\Rightarrow \sqrt{N}(\hat{\beta}_{MRC} - \beta_0) \xrightarrow{d} \mathcal{N}(0, ?)$

Manski's Binary Panel Data Est.

$y_{it} = \mathbb{1}(x_{it}'\beta_0 + \varepsilon_{it} + \alpha_i \geq 0) \qquad t = 1, 2$

$y_{i1} \neq y_{i2} \qquad \mathrm{sgn}(y_{i2} - y_{i1}) = 1 \Rightarrow$

$(x_{i2} - x_{i1})'\beta + (\alpha_i - \alpha_i) \geq (\varepsilon_{i1} - \varepsilon_{i2})$

$\mathrm{med}\{\mathrm{sgn}(y_{i2} - y_{i1}) \mid y_{i2} \neq y_{i1}, x_{i2}, x_{i1}\}$

$= \mathrm{sgn}\{(x_{i2} - x_{i1})'\beta_0\}$

Assuming $(\varepsilon_{i2} \sim \varepsilon_{i1}) \mid x_{i1}, x_{i2}$

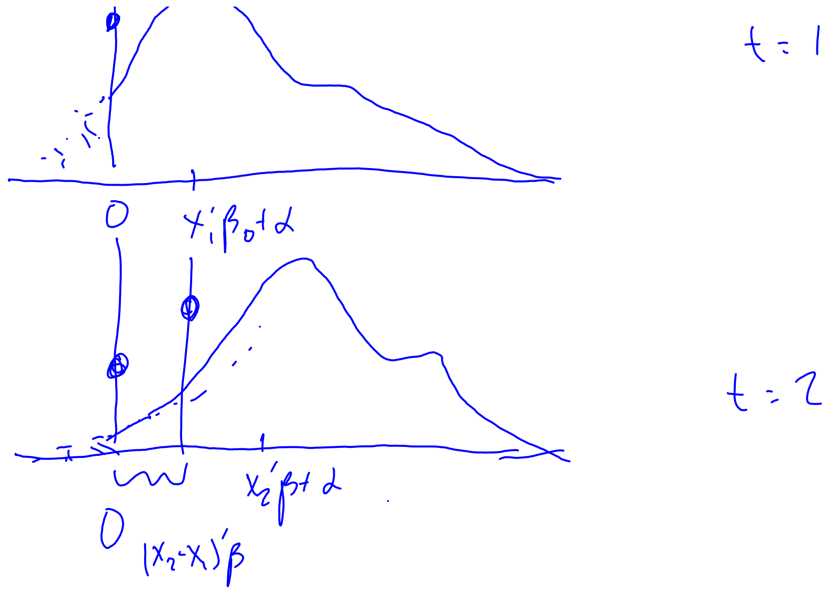
$\hat{\beta} \xrightarrow{P} \beta_0$

$$\hat{\beta} = \arg\max \sum_i \mathbb{1}(y_{i2} > y_{i1}) \cdot \mathbb{1}(x_{i2}'\beta > x_{i1}'\beta) + \mathbb{1}(y_{i2} < y_{i1}) \cdot \mathbb{1}(x_{i2}'\beta < x_{i1}'\beta)$$

Not $\sqrt{N}$ - consistent

Honoré's Censored Fixed Effect Est. ("Tobit")

$y_{it} = \max\{0, x_{it}'\beta_0 + \alpha_i + \varepsilon_{it}\} \qquad t = 1, 2$



$$e_{it}(\beta) = \max\{ y_{it}, -(x_{is} - x_{it})'\beta \} \quad s \neq t$$

$$- x_{it}'\beta \;.$$

$e_{i2}(\beta_0) - e_{i1}(\beta_0)$ odd function of $\varepsilon_{i2} - \varepsilon_{i1}$

$$E[\psi(e_{i2}(\beta) - e_{i1}(\beta))(x_{i2} - x_{i1})] = 0$$

if $\psi(u) = -\psi(-u)$

if $\varepsilon_{i2} | x_i \sim \varepsilon_{i1} | x_i$

$\psi(u) = u$ or $\psi(u) = \text{sgn}(u)$

$$\hat{\beta} = \arg\min \sum_{i=1}^{N} \underline{\Psi}(e_{i2}(\beta) - e_{i1}(\beta))$$

$\underline{\Psi}(u) = \Psi(-u)$, convex

$$\sqrt{N}(\hat{\beta} - \beta_0) \xrightarrow{d} \mathcal{N}(0, ?)$$

Honoré-Powell $\quad \hat{\beta} = \arg\min \sum_i \sum_{j>i} \underline{\Psi}(e_{ij}(\beta) - e_{ji}(\beta))$

Index Restrictions: $\quad \varepsilon_i | x_i \sim \varepsilon_i | x\beta_0$

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$\Rightarrow y_i^* = x_i'\beta_0 + \varepsilon_i \,|\, x_i \sim y_i^* \,|\, x_i'\beta_0$

$\Rightarrow y_i = t(y_i^*) \,|\, x_i \sim y_i \,|\, x_i'\beta_0$

Weak Index $E[\varepsilon_i | x_i] = E[\varepsilon_i | x_i'\beta_0]$

Binary Response: $1(y_i^* \geq 0) = y_i$

$g(x_i) = E[y_i | x_i] = Pr\{y_i = 1 | x_i\}$
$= G(x_i'\beta_0)$

Stoker's Avg Derivatives: $x_i \sim f(x)dx$

$$\frac{\partial g(x_i)}{\partial x} = G'(x_i'\beta_0)\beta_0 \propto \beta_0$$

$$E\left[\frac{\partial g(x_i)}{\partial x}\right] = E[G'(x_i'\beta_0)]\beta_0 \equiv K_0 \cdot \beta_0$$

Estimate $f(x)$ parametrically, ie $f(x) = f(x; \gamma_0)$

$$E\left[\frac{\partial g(x_i)}{\partial x}\right] = \int \frac{\partial g(x)}{\partial x} f(x)\,dx$$

$$= g(x)f(x)\Big|_{\Omega} - \int g(x)\frac{\partial f(x)}{\partial x}dx$$

$(=0)$

$$= -\int g(x)\frac{\partial f(x)}{\partial x}dx$$

$$= -\int g(x)\frac{\partial \log f(x)}{\partial x} f(x)\,dx$$

$$= -E\left[g(x_i)\frac{\partial \log f(x)}{\partial x}\right]$$

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$$= -E\left[y_i \frac{\partial \log f(x)}{\partial x}\right]$$

$$\hat{\delta} = \frac{-1}{N}\sum_i y_i \frac{\partial \log \hat{f}(x_i; \hat{\tau})}{\partial x}$$

$$\sqrt{N}(\hat{\delta} - \delta_0) \xrightarrow{d} \mathcal{N}(0, ?), \quad \delta_0 = E[G'(x_i;\beta_0)]\beta_0$$

PSS Weighted avg Derivatives:

$$E[y_i | x_i] = G(x_i; \beta_0) \equiv g(x_i)$$

$$E\left[w(x_i)\frac{\partial g(x_i)}{\partial x}\right] = E[w(x_i)G'(x_i;\beta_0)]\cdot\beta_0$$

Take $w(x_i) = f(x_i)$

$$\delta_w \equiv E\left[f(x_i)\frac{\partial g(x_i)}{\partial x}\right] = -2\,E\left[y_i \frac{\partial f(x)}{\partial x}\right]$$

if $[f(x)]^2 g(x)$ vanishes on boundary of integral

$$\hat{\delta}_w = \frac{-2}{N}\sum_{i=1}^{N} y_i \frac{\partial \hat{f}_{(i)}(x_i)}{\partial x}$$

$$\hat{f}_{(i)}(x) = \frac{1}{n-1}\sum_{j\neq i} K\left(\frac{x - x_j}{h}\right)\frac{1}{h^p} \qquad p = \dim(x_i)$$

$$\hat{\delta}_w = \binom{N}{2}^{-1}\sum_i \sum_{j>i} p_N(z_i, z_j) \qquad h = h_N$$

$$= \binom{N}{2}^{-1}\sum_i (y_i - y_j)\frac{\partial K}{\partial u}\left(\frac{x_i - x_j}{h}\right)\left(\frac{1}{h^{p+1}}\right)$$

"Smoothed" U-stat $\quad U_N = \binom{N}{2}^{-1} \sum_i \sum_{j>i} p_N(z_i, z_j)$

$$r_N(z_i) = E\left[p_N(z_i, z_j) \mid z_i\right] \quad E(r_N(z_i)) = \theta_N$$

$$\tilde{U}_N = \theta_N + \frac{2}{N} \sum_{i=1}^{N} (r_N(z_i) - \theta_N)$$

If $\quad E\left[\|p_N(z_i, z_j)\|^2\right] = o(N)$

then $\quad U_N - \theta_N \xrightarrow{p} 0$

$$\sqrt{N}(U_N - \tilde{U}_N) \xrightarrow{p} 0$$

For $\hat{\delta}_W$, $\quad E\left[\|p_N(z_i, z_j)\|^2\right] = o(N)$

if $\quad Nh^{p+2} \xrightarrow{\infty}$ as $N \to \infty$

want $\quad \sqrt{N}(\theta_N - \theta_0) \to 0, \quad \theta_0 = \lim_{N \to \infty} \theta_N$

using Higher-order kernels,

$$\theta_N - \theta_0 = O(h^q) \qquad \text{set } \sqrt{N} h^q \to 0$$

$$N h^{p+2} \to \infty$$

$$\sqrt{N}(\hat{\delta}_W - \delta_W) \xrightarrow{d} \mathcal{N}(0, ?).$$