Chapter 20
Insurance Versus Incentives

20.1. Insurance with recursive contracts

This chapter studies a planner who designs an efficient contract to supply insurance in the presence of incentive constraints imposed by his limited ability either to enforce contracts or to observe households' actions or incomes. We pursue two themes, one substantive, the other technical. The substantive theme is a tension that exists between offering insurance and providing incentives. A planner can overcome incentive problems by offering "carrots and sticks" that adjust an agent's future consumption and thereby provide less insurance. Balancing incentives against insurance shapes the evolution of distributions of wealth and consumption.

The technical theme is how memory can be encoded recursively and how incentive problems can be managed with contracts that retain memory and make promises. Contracts issue rewards that depend on the history either of publicly observable outcomes or of an agent's announcements about his privately observed outcomes. Histories are large-dimensional objects. But Spear and Srivastava (1987), Thomas and Worrall (1988), Abreu, Pearce, and Stacchetti (1990), and Phelan and Townsend (1991) discovered that the dimension can be contained by using an accounting system cast solely in terms of a "promised value," a one-dimensional object that summarizes relevant aspects of an agent's history. Working with promised values permits us to formulate the contract design problem recursively.

Three basic models are set within a single physical environment but assume different structures of information, enforcement, and storage possibilities. The first adapts a model of Thomas and Worrall (1988) and Kocherlakota (1996b) that focuses on commitment or enforcement problems and has all information being public. The second is a model of Thomas and Worrall (1990) that has an incentive problem coming from private information but that assumes away commitment and enforcement problems. Common to both of these models is that the insurance contract is assumed to be the *only* vehicle for households to transfer wealth across states of the world and over time. The third model, by Allen (1985) Cole and Kocherlakota (2001), extends Thomas and Worrall's (1990) model by introducing private storage that cannot be observed publicly. Ironically, because it lets households self-insure as in chapters 17 and 18, the possibility of private storage reduces *ex ante* welfare by limiting the amount of social insurance that can be attained when incentive constraints are present.

20.2. Basic environment

Imagine a village with a large number of *ex ante* identical households. Each household has preferences over consumption streams that are ordered by

$$E_{-1} \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{20.2.1}$$

where $u(c)$ is an increasing, strictly concave, and twice continuously differentiable function, $\beta \in (0,1)$ is a discount factor, and E_{-1} is the mathematical expectation not conditioning on any information available at time 0 or later. Each household receives a stochastic endowment stream $\{y_t\}_{t=0}^{\infty}$, where for each $t \geq 0$, y_t is independently and identically distributed according to the discrete probability distribution $\text{Prob}(y_t = \overline{y}_s) = \Pi_s$, where $s \in \{1, 2, \ldots, S\} \equiv \mathbf{S}$ and $\overline{y}_{s+1} > \overline{y}_s$. The consumption good is not storable. At time $t \geq 1$, the household has experienced a history of endowments $h_t = (y_t, y_{t-1}, \ldots, y_0)$. The endowment processes are independently and identically distributed both across time and across households.

In this setting, if there were a competitive equilibrium with complete markets as described in chapter 8, at date 0 households would trade history- and date-contingent claims before the realization of endowments. Since all households are *ex ante* identical, each household would end up consuming the per capita endowment in every period, and its lifetime utility would be

$$v_{\text{pool}} = \sum_{t=0}^{\infty} \beta^t \, u\left(\sum_{s=1}^{S} \Pi_s \overline{y}_s \right) = \frac{1}{1-\beta} \, u\left(\sum_{s=1}^{S} \Pi_s \overline{y}_s \right). \tag{20.2.2}$$

Households would thus insure away all of the risk associated with their individual endowment processes. But the incentive constraints that we are about to specify make this allocation unattainable. For each specification of incentive constraints, we shall solve a planning problem for an efficient allocation that respects those constraints.

Following a tradition started by Green (1987), we assume that a "moneylender" or "planner" is the only person in the village who has access to a risk-free loan market outside the village. The moneylender can borrow or lend at a constant risk-free gross interest rate $R = \beta^{-1}$. The households cannot borrow or lend with one another, and can trade only with the moneylender. Furthermore, we assume that the moneylender is committed to honor his promises. We will study three types of incentive constraints.

(a) Both the money lender and the household observe the household's history of endowments at each time t. Although the moneylender *can* commit to honor a contract, households *cannot* commit and at any time are free to walk away from an arrangement with the moneylender and live in perpetual autarky thereafter. They must be induced not to do so by the structure of the contract. This is a model of "one-sided commitment" in which the contract is "self-enforcing" because the household prefers to conform to it.

(b) Households *can* make commitments and enter into enduring and binding contracts with the moneylender, but they have private information about their own incomes. The moneylender can see neither their income nor their consumption. It follows that any exchanges between the moneylender and a household must be based on the household's own reports about income realizations. An incentive-compatible contract must induce a household to report its income truthfully.

(c) The environment is the same as in b except for the additional assumption that households have access to a storage technology that cannot be observed by the moneylender. Households can store nonnegative amounts of goods at a risk-free gross return of R equal to the interest rate that the moneylender faces in the outside credit market. Since the moneylender can both borrow and lend at the interest rate R outside of the village, the private storage technology does not change the economy's aggregate resource constraint, but it does substantially affect the set of incentive-compatible contracts between the moneylender and the households.

When we compute efficient allocations for each of these three environments, we shall find that the dynamics of the implied consumption allocations differ

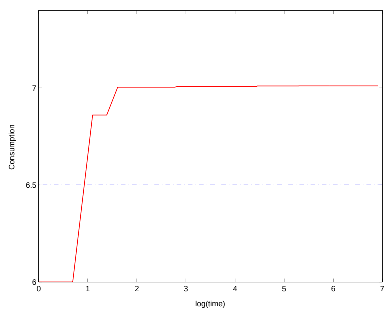

Figure 20.2.1.a: Typical consumption path in environment a.

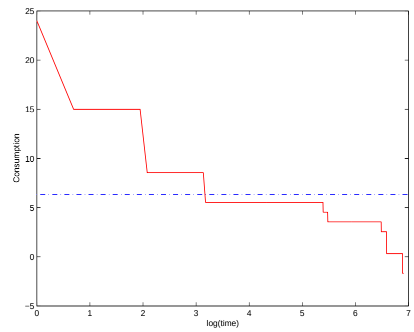

Figure 20.2.1.b: Typical consumption path in environment b.

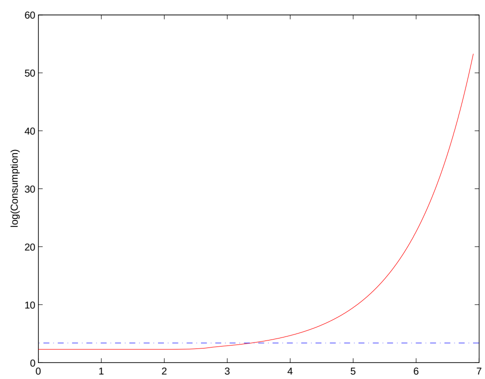

Figure 20.2.2: Typical consumption path in environment c.

dramatically. As a prelude, Figures 20.2.1 and 20.2.2 depict the different consumption streams that are associated with the *same* realization of a random endowment stream for households living in environments a, b, and c, respectively. For all three of these economies, we set $u(c) = -\gamma^{-1}\exp(-\gamma c)$ with

$\gamma = .8$, $\beta = .92$, $[\overline{y}_1, \ldots, \overline{y}_{10}] = [6, \ldots, 15]$, and $\Pi_s = \frac{1-\lambda}{1-\lambda^{10}}\lambda^{s-1}$ with $\lambda = 2/3$. As a benchmark, a horizontal dotted line in each graph depicts the constant consumption level that would be attained in a complete markets equilibrium where there are no incentive problems. In all three environments, prior to date 0, the households have entered into efficient contracts with the moneylender. The dynamics of consumption outcomes evidently differ substantially across the three environments, increasing and then flattening out in environment a, heading "south" in environment b, and heading "north" in environment c. This chapter explains why the sample paths of consumption differ so much across these three settings.

20.3. One-sided no commitment

Our first incentive problem is a lack of commitment. A moneylender is committed to honor his promises, but villagers are free to walk away from their arrangement with the moneylender at any time. The moneylender designs a contract that the villager wants to honor at every moment and contingency. Such a contract is said to be self-enforcing. In chapter 21, we shall study another economy in which there is no moneylender, only another villager, and when no one is able to keep prior commitments. Such a contract design problem with participation constraints on both sides of an exchange represents a problem with two-sided lack of commitment, in contrast to the problem with one-sided lack of commitment treated here.[1]

[1] For an earlier two-period model of a one-sided commitment problem, see Holmström (1983).

20.3.1. Self-enforcing contract

A moneylender can borrow or lend resources from outside the village but the villagers cannot. A *contract* is a sequence of functions $c_t = f_t(h_t)$ for $t \geq 0$, where $h_t = (y_t, \ldots, y_0)$. The sequence of functions $\{f_t\}$ assigns a history-dependent consumption stream $c_t = f_t(h_t)$ to the household. The contract specifies that each period, the villager contributes his time t endowment y_t to the moneylender who then returns c_t to the villager. From this arrangement, the moneylender earns an *ex ante* expected present value

$$P = E_{-1} \sum_{t=0}^{\infty} \beta^t (y_t - c_t). \tag{20.3.1}$$

By plugging the associated consumption process into expression (20.2.1), we find that the contract assigns the villager an expected present value of $v = E_{-1} \sum_{t=0}^{\infty} \beta^t u\left(f_t(h_t)\right)$.

The contract must be self-enforcing. At any point in time, the household is free to walk away from the contract and thereafter consume its endowment stream. Thus, if the household walks away from the contract, it must live in autarky evermore. The *ex ante* value associated with consuming the endowment stream, to be called the autarky value, is

$$v_{\rm aut} = E_{-1} \sum_{t=0}^{\infty} \beta^t u(y_t) = \frac{1}{1-\beta} \sum_{s=1}^{S} \Pi_s u(\overline{y}_s). \tag{20.3.2}$$

At time t, *after* having observed its current-period endowment, the household can guarantee itself a present value of utility of $u(y_t) + \beta v_{\rm aut}$ by consuming its own endowment. The moneylender's contract must offer the household at least this utility at every possible history and every date. Thus, the contract must satisfy

$$u[f_t(h_t)] + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} u[f_{t+j}(h_{t+j})] \geq u(y_t) + \beta v_{\rm aut}, \tag{20.3.3}$$

for all $t \geq 0$ and for all histories h_t. Equation (20.3.3) is called the *participation constraint* for the villager. A contract that satisfies equation (20.3.3) is said to be *sustainable*.

20.3.2. Recursive formulation and solution

A difficulty with constraints like equation (20.3.3) is that there are so many of them: the dimension of the argument h_t grows exponentially with t. Fortunately, there is a recursive way to describe some history-dependent contracts. We can represent the sequence of functions $\{f_t\}$ recursively by finding a state variable x_t such that the contract takes the form

$$c_t = g(x_t, y_t),$$
$$x_{t+1} = \ell(x_t, y_t).$$

Here g and ℓ are time-invariant functions. Notice that by iterating the $\ell(\cdot)$ function t times starting from (x_0, y_0), one obtains

$$x_t = m_t(x_0; y_{t-1}, \dots, y_0), \quad t \geq 1.$$

Thus, x_t summarizes histories of endowments h_{t-1}. In this sense, x_t is a "backward-looking" variable.

A remarkable fact is that the appropriate state variable x_t is a *promised expected discounted future value* $v_t = E_{t-1} \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$. This "forward-looking" variable summarizes a stream of future utilities. We shall formulate the contract recursively by having the moneylender arrive at t, before y_t is realized, with a previously made promised value v_t. He delivers v_t by letting c_t and the continuation value v_{t+1} both respond to y_t.

Thus, we shall treat the promised value v as a *state* variable, then formulate a functional equation for a moneylender. The moneylender gives a prescribed value v by delivering a state-dependent current consumption c and a promised value starting tomorrow, say v', where c and v' each depend on the current endowment y and the preexisting promise v. The moneylender chooses c and v' to let him provide v in a way that maximizes his profits (20.3.1).

Each period, the household must be induced to surrender the time t endowment y_t to the moneylender, who possibly gives some of it to other households and invests the rest of it outside the village at a constant one-period gross interest rate of β^{-1}. In exchange, the moneylender delivers a state-contingent consumption stream to the household that keeps it participating in the arrangement every period and after every history. The moneylender wants to do this in the most efficient way, that is, the profit-maximizing way. Let v be the promised expected discounted future utility promised to a villager. Let $P(v)$ be the expected present value of the "profit stream" $\{y_t - c_t\}$ for a moneylender who delivers value v in the optimal way. The optimum value $P(v)$ obeys the functional equation

$$P(v) = \max_{\{c_s, w_s\}} \sum_{s=1}^{S} \Pi_s[(\overline{y}_s - c_s) + \beta P(w_s)] \tag{20.3.4}$$

where the maximization is subject to the constraints

$$\sum_{s=1}^{S} \Pi_s[u(c_s) + \beta w_s] \geq v, \tag{20.3.5}$$

$$u(c_s) + \beta w_s \geq u(\overline{y}_s) + \beta v_{\rm aut}, \quad s = 1, \ldots, S; \tag{20.3.6}$$

$$c_s \in [c_{\min}, c_{\max}], \tag{20.3.7}$$

$$w_s \in [v_{\rm aut}, \bar{v}]. \tag{20.3.8}$$

Here w_s is the promised value with which the consumer enters next period, given that $y = \overline{y}_s$ this period; $[c_{\min}, c_{\max}]$ is a bounded set to which we restrict the choice of c_t each period. We restrict the continuation value w_s to be in the set $[v_{\rm aut}, \bar{v}]$, where $\bar{v}$ is a very large number. Soon we'll compute the highest value that the moneylender would ever want to set w_s. All we require now is that $\bar{v}$ exceed this value. Constraint (20.3.5) is the promise-keeping constraint. It requires that the contract deliver at least promised value v. Constraints (20.3.6), one for each state s, are the participation constraints. Evidently, P must be a decreasing function of v because the higher the consumption stream of the villager, the lower must be the profits of the moneylender.

The constraint set is convex. The one-period return function in equation (20.3.4) is concave. The value function $P(v)$ that solves equation (20.3.4) is concave. In fact, $P(v)$ is strictly concave as will become evident from our characterization of the optimal contract that solves this problem. Form the Lagrangian

$$\begin{aligned} L = & \sum_{s=1}^{S} \Pi_s[(\overline{y}_s - c_s) + \beta P(w_s)] \\ & + \mu \left\{ \sum_{s=1}^{S} \Pi_s[u(c_s) + \beta w_s] - v \right\} \\ & + \sum_{s=1}^{S} \lambda_s \Big\{ u(c_s) + \beta w_s - [u(\overline{y}_s) + \beta v_{\rm aut}] \Big\}. \end{aligned} \tag{20.3.9}$$

For each v and for $s = 1, \ldots, S$, the first-order conditions for maximizing L with respect to c_s, w_s, respectively, are[2]

$$(\lambda_s + \mu \Pi_s) u'(c_s) = \Pi_s, \tag{20.3.10}$$

$$\lambda_s + \mu \Pi_s = -\Pi_s P'(w_s). \tag{20.3.11}$$

By the envelope theorem, if P is differentiable, then $P'(v) = -\mu$. We will proceed under the assumption that P is differentiable but it will become evident that P is indeed differentiable when we learn about the optimal contract that solves this problem.

Equations (20.3.10) and (20.3.11) imply the following relationship between c_s and w_s:

$$u'(c_s) = -P'(w_s)^{-1}. \tag{20.3.12}$$

This condition states that the household's marginal rate of substitution between c_s and w_s, given by $u'(c_s)/\beta$, should equal the moneylender's marginal rate of transformation as given by $-[\beta P'(w_s)]^{-1}$. The concavity of P and u means that equation (20.3.12) traces out a positively sloped curve in the c, w plane, as depicted in Figure 20.3.1. We can interpret this condition as making c_s a function of w_s. To complete the optimal contract, it will be enough to find how w_s depends on the promised value v and the income state $\overline{y}_s$.

Condition (20.3.11) can be written

$$P'(w_s) = P'(v) - \lambda_s / \Pi_s. \tag{20.3.13}$$

How w_s varies with v depends on which of two mutually exclusive and exhaustive sets of states (s, v) falls into after the realization of $\overline{y}_s$: those in which the participation constraint (20.3.6) binds (i.e., states in which $\lambda_s > 0$) or those in which it does not (i.e., states in which $\lambda_s = 0$).

States where $\lambda_s > 0$

When $\lambda_s > 0$, the participation constraint (20.3.6) holds with equality. When $\lambda_s > 0$, (20.3.13) implies that $P'(w_s) < P'(v)$, which in turn implies, by the

[2] Please note that the λ_s's depend on the promised value v. In particular, which λ_s's are positive and which are zero will depend on v, with more of them being zero when the promised value v is higher. See figure 20.3.1.

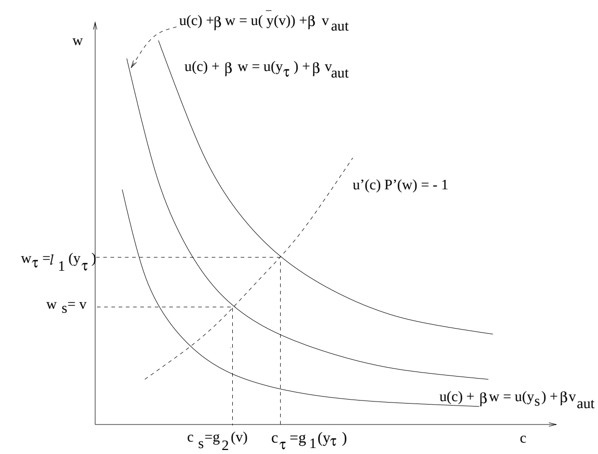

Figure 20.3.1: Determination of consumption and promised utility (c, w). Higher realizations of $\overline{y}_s$ are associated with higher indifference curves $u(c) + \beta w = u(\overline{y}_s) + \beta v_{\text{aut}}$. For a given v, there is a threshold level $\bar{y}(v)$ above which the participation constraint is binding and below which the moneylender awards a constant level of consumption, as a function of v, and maintains the same promised value $w = v$. The cutoff level $\bar{y}(v)$ is determined by the indifference curve going through the intersection of a horizontal line at level v with the "expansion path" $u'(c)P'(w) = -1$.

concavity of P, that $w_s > v$. Further, the participation constraint at equality implies that $c_s < \overline{y}_s$ (because $w_s > v \geq v_{\text{aut}}$). Taken together, these results say that when the participation constraint (20.3.6) binds, the moneylender induces the household to consume less than its endowment today by raising its continuation value.

When $\lambda_s > 0$, c_s and w_s solve the two equations

$$u(c_s) + \beta w_s = u(\overline{y}_s) + \beta v_{\text{aut}}, \tag{20.3.14}$$

$$u'(c_s) = -P'(w_s)^{-1}. \tag{20.3.15}$$

The participation constraint holds with equality. Notice that these equations are independent of v. This property is a key to understanding the form of the optimal contract. It imparts to the contract what Kocherlakota (1996b) calls *amnesia*: when incomes y_t are realized that cause the participation constraint to bind, the contract disposes of all history dependence and makes both consumption and the continuation value depend only on the current income state y_t. We portray amnesia by denoting the solutions of equations (20.3.14) and (20.3.15) by

$$c_s = g_1(\overline{y}_s), \tag{20.3.16a}$$

$$w_s = \ell_1(\overline{y}_s). \tag{20.3.16b}$$

Later, we'll exploit the amnesia property to produce a computational algorithm.

States where $\lambda_s = 0$

When the participation constraint does not bind, $\lambda_s = 0$ and first-order condition (20.3.11) imply that $P'(v) = P'(w_s)$, which implies that $w_s = v$. Therefore, from (20.3.12), we can write $u'(c_s) = -P'(v)^{-1}$, so that consumption in state s depends on promised utility v but not on the endowment in state s. Thus, when the participation constraint does not bind, the moneylender awards

$$c_s = g_2(v), \tag{20.3.17a}$$

$$w_s = v, \tag{20.3.17b}$$

where $g_2(v)$ solves $u'[g_2(v)] = -P'(v)^{-1}$.

The optimal contract

Combining the branches of the policy functions for the cases where the participation constraint does and does not bind, we obtain

$$c = \max\{g_1(y), g_2(v)\}, \tag{20.3.18}$$

$$w = \max\{\ell_1(y), v\}. \tag{20.3.19}$$

The optimal policy is displayed graphically in Figures 20.3.1 and 20.3.2. To interpret the graphs, it is useful to study equations (20.3.6) and (20.3.12) for the case in which $w_s = v$. By setting $w_s = v$, we can solve these equations for a "cutoff value," call it $\bar{y}(v)$, such that the participation constraint binds only when $\overline{y}_s \geq \bar{y}(v)$. To find $\bar{y}(v)$, we first solve equation (20.3.12) for the value c_s associated with v for those states in which the participation constraint is not binding:

$$u'[g_2(v)] = -P'(v)^{-1},$$

and then substitute this value into (20.3.6) at equality to solve for $\bar{y}(v)$:

$$u[\bar{y}(v)] = u[g_2(v)] + \beta(v - v_{\rm aut}). \tag{20.3.20}$$

By the concavity of P, the cutoff value $\bar{y}(v)$ is increasing in v.

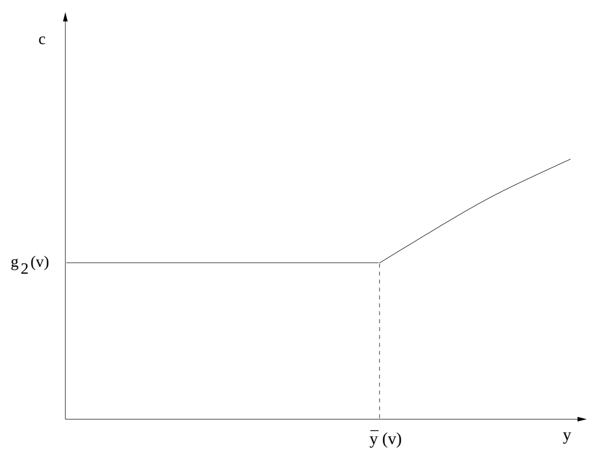

Figure 20.3.2: The shape of consumption as a function of realized endowment, when the promised initial value is v.

Associated with a given level of $v_t \in (v_{\rm aut}, \bar{v})$, there are two numbers $g_2(v_t)$, $\bar{y}(v_t)$ such that if $y_t \leq \bar{y}(v_t)$ the moneylender offers the household $c_t = g_2(v_t)$ and leaves the promised utility unaltered, $v_{t+1} = v_t$. The moneylender is thus insuring against the states $\overline{y}_s \leq \bar{y}(v_t)$ at time t. If $y_t > \bar{y}(v_t)$, the participation constraint is binding, prompting the moneylender to induce the household to surrender some of its current-period endowment in exchange for a raised promised utility $v_{t+1} > v_t$. Promised values never decrease. They stay constant for low-y states $\overline{y}_s < \bar{y}(v_t)$ and increase in high-endowment states that threaten to violate the participation constraint. Consumption stays constant during periods when the participation constraint fails to bind and increases during periods when it threatens to bind. Whenever the participation binds, the household makes a net transfer to the money lender in return for promised continuation utility. A household that realizes the highest endowment y_S is permanently awarded the highest consumption level with an associated promised value $\bar{v}$ that satisfies

$$u[g_2(\bar{v})] + \beta\bar{v} = u(\overline{y}_S) + \beta v_{\text{aut}}.$$

20.3.3. Recursive computation of contract

Suppose that the initial promised value v_0 is v_{aut}. We can compute the optimal contract recursively by using the fact that the villager will ultimately receive a constant welfare level equal to $u(\overline{y}_S) + \beta v_{\text{aut}}$ after ever having experienced the maximum endowment $\overline{y}_S$. We can characterize the optimal policy in terms of numbers $\{\overline{c}_s, \overline{w}_s\}_{s=1}^S \equiv \{g_1(\overline{y}_s), \ell_1(\overline{y}_s)\}_{s=1}^S$ where $g_1(\overline{y}_s)$ and $\ell_1(\overline{s})$ are given by (20.3.16). These numbers can be computed recursively by working backward as follows. Start with $s = S$ and compute $(\overline{c}_S, \overline{w}_S)$ from the nonlinear equations:

$$u(\overline{c}_S) + \beta\overline{w}_S = u(\overline{y}_S) + \beta v_{\text{aut}}, \tag{20.3.21a}$$

$$\overline{w}_S = \frac{u(\overline{c}_S)}{1-\beta}. \tag{20.3.21b}$$

Working backward for $j = S-1, \ldots, 1$, compute $\overline{c}_j, \overline{w}_j$ from the two nonlinear equations

$$u(\overline{c}_j) + \beta\overline{w}_j = u(\overline{y}_j) + \beta v_{\text{aut}}, \tag{20.3.22a}$$

$$\overline{w}_j = [u(\overline{c}_j) + \beta\overline{w}_j] \sum_{k=1}^{j} \Pi_k + \sum_{k=j+1}^{S} \Pi_k [u(\overline{c}_k) + \beta\overline{w}_k]. \tag{20.3.22b}$$

These successive iterations yield the optimal contract characterized by $\{\overline{c}_s, \overline{w}_s\}_{s=1}^S$. *Ex ante*, before the time 0 endowment has been realized, the contract offers the household

$$v_0 = \sum_{k=1}^{S} \Pi_k [u(\overline{c}_k) + \beta\overline{w}_k] = \sum_{k=1}^{S} \Pi_k [u(\overline{y}_k) + \beta v_{\text{aut}}] = v_{\text{aut}}, \tag{20.3.23}$$

where we have used (20.3.22a) to verify that the contract indeed delivers $v_0 = v_{\text{aut}}$.

Some additional manipulations will enable us to express $\{\overline{c}_j\}_{j=1}^S$ solely in terms of the utility function and the endowment process. First, solve for $\overline{w}_j$ from (20.3.22b),

$$\overline{w}_j = \frac{u(\overline{c}_j)\sum_{k=1}^{j}\Pi_k + \sum_{k=j+1}^{S}\Pi_k[u(\overline{y}_k)+\beta v_{\rm aut}]}{1-\beta\sum_{k=1}^{j}\Pi_k}, \tag{20.3.24}$$

where we have invoked (20.3.22a) when replacing $[u(\overline{c}_k)+\beta\overline{w}_k]$ by $[u(\overline{y}_k)+\beta v_{\rm aut}]$. Next, substitute (20.3.24) into (20.3.22a) and solve for $u(\overline{c}_j)$,

$$\begin{aligned}
u(\overline{c}_j) &= \left[1-\beta\sum_{k=1}^{j}\Pi_k\right]\left[u(\overline{y}_j)+\beta v_{\rm aut}\right] - \beta\sum_{k=j+1}^{S}\Pi_k\left[u(\overline{y}_k)+\beta v_{\rm aut}\right] \\
&= u(\overline{y}_j)+\beta v_{\rm aut} - \beta u(\overline{y}_j)\sum_{k=1}^{j}\Pi_k - \beta^2 v_{\rm aut} - \beta\sum_{k=j+1}^{S}\Pi_k u(\overline{y}_k) \\
&= u(\overline{y}_j)+\beta v_{\rm aut} - \beta u(\overline{y}_j)\sum_{k=1}^{j}\Pi_k - \beta^2 v_{\rm aut} - \beta\left[(1-\beta)v_{\rm aut} - \sum_{k=1}^{j}\Pi_k u(\overline{y}_k)\right] \\
&= u(\overline{y}_j) - \beta\sum_{k=1}^{j}\Pi_k\left[u(\overline{y}_j)-u(\overline{y}_k)\right].
\end{aligned} \tag{20.3.25}$$

According to (20.3.25), $u(\overline{c}_1)=u(\overline{y}_1)$ and $u(\overline{c}_j)<u(\overline{y}_j)$ for $j\geq 2$. That is, a household that realizes a record high endowment of $\overline{y}_j$ must surrender some of that endowment to the moneylender unless the endowment is the lowest possible value $\overline{y}_1$. Households are willing to surrender parts of their endowments in exchange for promises of insurance (i.e., future state-contingent transfers) that are encoded in the associated continuation values, $\{\overline{w}_j\}_{j=1}^S$. For those unlucky households that have so far realized only endowments equal to $\overline{y}_1$, the profit-maximizing contract prescribes that the households retain their endowment, $\overline{c}_1 = \overline{y}_1$ and by (20.3.22a), the associated continuation value is $\overline{w}_1 = v_{\rm aut}$. That is, to induce those low-endowment households to adhere to the contract, the moneylender has only to offer a contract that assures them an autarky continuation value in the next period.

Contracts when $v_0 > \overline{w}_1 = v_{\rm aut}$

We have shown how to compute the optimal contract when $v_0 = \overline{w}_1 = v_{\rm aut}$ by computing quantities $(\overline{c}_s, \overline{w}_s)$ for $s=1,\ldots,S$. Now suppose that we want to construct a contract that assigns initial value $v_0 \in [\overline{w}_{k-1}, \overline{w}_k)$ for $1 < k \leq S$. Given v_0, we can deduce k, then solve for $\tilde{c}$ satisfying

$$v_0 = \left(\sum_{j=1}^{k-1} \Pi_j\right) [u(\tilde{c}) + \beta v_0] + \sum_{j=k}^{S} \Pi_j [u(\overline{c}_j) + \beta \overline{w}_j]. \tag{20.3.26}$$

The optimal contract promises $(\tilde{c}, v_0)$ so long as the maximum y_t to date is less than or equal to $\overline{y}_{k-1}$. When the maximum y_t experienced to date equals $\overline{y}_j$ for $j \geq k$, the contract offers $(\overline{c}_j, \overline{w}_j)$.

It is plausible that a higher initial expected promised value $v_0 > v_{\text{aut}}$ can be delivered in the most cost-effective way by choosing a higher consumption level $\tilde{c}$ for households that experience low endowment realizations, $\tilde{c} > \overline{c}_j$ for $j = 1, \ldots, k-1$. The reason is that those unlucky households have high marginal utilities of consumption. Therefore, transferring resources to them minimizes the resources that are needed to increase the *ex ante* promised expected utility. As for those lucky households that have received relatively high endowment realizations, the optimal contract prescribes an unchanged allocation characterized by $\{\overline{c}_j, \overline{w}_j\}_{j=k}^{S}$.

If we want to construct a contract that assigns initial value $v_0 \geq \overline{w}_S$, the efficient solution is simply to find the constant consumption level $\tilde{c}$ that delivers lifetime utility v_0:

$$v_0 = \sum_{j=1}^{S} \Pi_j [u(\tilde{c}) + \beta v_0] \qquad \Longrightarrow \qquad v_0 = \frac{u(\tilde{c})}{1-\beta}.$$

This contract trivially satisfies all participation constraints, and a constant consumption level maximizes the expected profit of delivering v_0.

Summary of optimal contract

Define

$$s(t) = \{j : \overline{y}_j = \max\{y_0, y_1, \ldots, y_t\}\}.$$

That is, $\overline{y}_{s(t)}$ is the maximum endowment that the household has experienced up until period t.

The optimal contract has the following features. To deliver promised value $v_0 \in [v_{\text{aut}}, \overline{w}_S]$ to the household, the contract offers stochastic consumption and continuation values, $\{c_t, v_{t+1}\}_{t=0}^{\infty}$, that satisfy

$$c_t = \max\{\tilde{c},\, \overline{c}_{s(t)}\}, \tag{20.3.27a}$$

$$v_{t+1} = \max\{v_0,\, \overline{w}_{s(t)}\}, \tag{20.3.27b}$$

where $\tilde{c}$ is given by (20.3.26).

20.3.4. Profits

We can use (20.3.4) to compute expected profits from offering continuation value $\overline{w}_j$, $j = 1, \ldots, S$. Starting with $P(\overline{w}_S)$, we work backward to compute $P(\overline{w}_k)$, $k = S-1, S-2, \ldots, 1$:

$$P(\overline{w}_S) = \sum_{j=1}^{S} \Pi_j \left(\frac{\overline{y}_j - \overline{c}_S}{1-\beta} \right), \tag{20.3.28a}$$

$$\begin{aligned} P(\overline{w}_k) = &\sum_{j=1}^{k} \Pi_j(\overline{y}_j - \overline{c}_k) + \sum_{j=k+1}^{S} \Pi_j(\overline{y}_j - \overline{c}_j) \\ &+ \beta \left[\sum_{j=1}^{k} \Pi_j P(\overline{w}_k) + \sum_{j=k+1}^{S} \Pi_j P(\overline{w}_j) \right]. \end{aligned} \tag{20.3.28b}$$

Strictly positive profits for $v_0 = v_{\text{aut}}$

We will now demonstrate that a contract that offers an initial promised value of v_{aut} is associated with strictly positive expected profits. In order to show that $P(v_{\text{aut}}) > 0$, let us first examine the expected profit implications of the following limited obligation. Suppose that a household has just experienced $\overline{y}_j$ for the first time and that the limited obligation amounts to delivering $\overline{c}_j$ to the household in that period and in all future periods until the household realizes an endowment higher than $\overline{y}_j$. At the time of such a higher endowment realization in the future, the limited obligation ceases without any further transfers. Would such a limited obligation be associated with positive or negative expected profits? In the case of $\overline{y}_j = \overline{y}_1$, this would entail a deterministic profit equal to zero, since we have shown above that $\overline{c}_1 = \overline{y}_1$. But what is true for other endowment realizations?

To study the expected profit implications of such a limited obligation for any given $\overline{y}_j$, we first compute an upper bound for the obligation's consumption level $\overline{c}_j$ by using (20.3.25):

$$u(\overline{c}_j) = \left[1 - \beta \sum_{k=1}^{j} \Pi_k\right] u(\overline{y}_j) + \beta \sum_{k=1}^{j} \Pi_k u(\overline{y}_k)$$
$$\leq u\left(\left[1 - \beta \sum_{k=1}^{j} \Pi_k\right] \overline{y}_j + \beta \sum_{k=1}^{j} \Pi_k \overline{y}_k\right),$$

where the weak inequality is implied by the strict concavity of the utility function, and evidently the expression holds with strict inequality for $j > 1$. Therefore, an upper bound for $\overline{c}_j$ is

$$\overline{c}_j \leq \left[1 - \beta \sum_{k=1}^{j} \Pi_k\right] \overline{y}_j + \beta \sum_{k=1}^{j} \Pi_k \overline{y}_k. \tag{20.3.29}$$

We can sort out the financial consequences of the limited obligation by looking separately at the first period and then at all future periods. In the first period, the moneylender obtains a nonnegative profit,

$$\begin{aligned} \overline{y}_j - \overline{c}_j &\geq \overline{y}_j - \left(\left[1 - \beta \sum_{k=1}^{j} \Pi_k\right] \overline{y}_j + \beta \sum_{k=1}^{j} \Pi_k \overline{y}_k\right) \\ &= \beta \sum_{k=1}^{j} \Pi_k \left[\overline{y}_j - \overline{y}_k\right], \end{aligned} \tag{20.3.30}$$

where we have invoked the upper bound on $\overline{c}_j$ in (20.3.29). After that first period, the moneylender must continue to deliver $\overline{c}_j$ for as long as the household does not realize an endowment greater than $\overline{y}_j$. So the probability that the household remains within the limited obligation for another t number of periods is $(\sum_{i=1}^{j} \Pi_i)^t$. Conditional on remaining within the limited obligation, the household's average endowment realization is $(\sum_{k=1}^{j} \Pi_k \overline{y}_k)/(\sum_{k=1}^{j} \Pi_k)$. Consequently, the expected discounted profit stream associated with all future periods of the limited obligation, expressed in first-period values, is

$$\begin{aligned} \sum_{t=1}^{\infty} \beta^t \left[\sum_{i=1}^{j} \Pi_i\right]^t \left[\frac{\sum_{k=1}^{j} \Pi_k \overline{y}_k}{\sum_{k=1}^{j} \Pi_k} - \overline{c}_j\right] &= \frac{\left[\beta \sum_{i=1}^{j} \Pi_i\right]}{1 - \beta \sum_{i=1}^{j} \Pi_i} \left[\frac{\sum_{k=1}^{j} \Pi_k \overline{y}_k}{\sum_{k=1}^{j} \Pi_k} - \overline{c}_j\right] \\ &\geq -\beta \sum_{k=1}^{j} \Pi_k \left[\overline{y}_j - \overline{y}_k\right], \end{aligned} \tag{20.3.31}$$

where the inequality is obtained after invoking the upper bound on $\overline{c}_j$ in (20.3.29). Since the sum of (20.3.30) and (20.3.31) is nonnegative, we conclude that the limited obligation at least breaks even in expectation. In fact, for $\overline{y}_j > \overline{y}_1$ we have that (20.3.30) and (20.3.31) hold with strict inequalities, and thus, each such limited obligation is associated with strictly positive profits.

Since the optimal contract with an initial promised value of $v_{\rm aut}$ can be viewed as a particular constellation of all of the described limited obligations, it follows immediately that $P(v_{\rm aut}) > 0$.

Contracts with $P(v_0) = 0$

In exercise *20.2*, you will be asked to compute v_0 such that $P(v_0) = 0$. Here is a good way to do this. After computing the optimal contract for $v_0 = v_{\rm aut}$, suppose that we can find some k satisfying $1 < k \leq S$ such that for $j \geq k, P(\overline{w}_j) \leq 0$ and for $j < k$, $P(\overline{w}_k) > 0$. Use a zero-profit condition to find an initial $\tilde{c}$ level:

$$0 = \sum_{j=1}^{k-1} \Pi_j(\overline{y}_j - \tilde{c}) + \sum_{j=k}^{S} \Pi_j \left[\overline{y}_j - \overline{c}_j + \beta P(\overline{w}_j)\right].$$

Given $\tilde{c}$, we can solve (20.3.26) for v_0.

However, such a k will fail to exist if $P(\overline{w}_S) > 0$. In that case, the efficient allocation associated with $P(v_0) = 0$ is a trivial one. The moneylender would simply set consumption equal to the average endowment value. This contract breaks even on average, and the household's utility is equal to the first-best unconstrained outcome, $v_0 = v_{\rm pool}$, as given in (20.2.2).

20.3.5. $P(v)$ *is strictly concave and continuously differentiable*

Consider a promised value $v_0 \in [\overline{w}_{k-1}, \overline{w}_k)$ for $1 < k \leq S$. We can then use equation (20.3.26) to compute the amount of consumption $\tilde{c}(v_0)$ awarded to a household with promised value v_0, as long as the household is not experiencing an endowment greater than $\overline{y}_{k-1}$:

$$u[\tilde{c}(v_0)] = \frac{\left[1 - \beta \sum_{j=1}^{k-1} \Pi_j\right] v_0 - \sum_{j=k}^{S} \Pi_j \left[u(\overline{c}_j) + \beta \overline{w}_j\right]}{\sum_{j=1}^{k-1} \Pi_j} \equiv \Phi_k(v_0), \quad (20.3.32)$$

that is,

$$\tilde{c}(v_0) = u^{-1}\left[\Phi_k(v_0)\right]. \quad (20.3.33)$$

Since the utility function is strictly concave, it follows that $\tilde{c}(v_0)$ is strictly convex in the promised value v_0:

$$\tilde{c}'(v_0) = \frac{\left[1 - \beta \sum_{j=1}^{k-1} \Pi_j\right]}{\sum_{j=1}^{k-1} \Pi_j} u^{-1\prime}\left[\Phi_k(v_0)\right] > 0, \quad (20.3.34a)$$

$$\tilde{c}''(v_0) = \frac{\left[1 - \beta \sum_{j=1}^{k-1} \Pi_j\right]^2}{\left[\sum_{j=1}^{k-1} \Pi_j\right]^2} u^{-1\prime\prime}\left[\Phi_k(v_0)\right] > 0. \quad (20.3.34b)$$

Next, we evaluate the expression for expected profits in (20.3.4) at the optimal contract,

$$P(v_0) = \sum_{j=1}^{k-1} \Pi_j \left[\overline{y}_j - \tilde{c}(v_0) + \beta P(v_0)\right] + \sum_{j=k}^{S} \Pi_j \left[\overline{y}_j - \overline{c}_j + \beta P(\overline{w}_j)\right],$$

which can be rewritten as

$$P(v_0) = \frac{\sum_{j=1}^{k-1} \Pi_j \left[\overline{y}_j - \tilde{c}(v_0)\right] + \sum_{j=k}^{S} \Pi_j \left[\overline{y}_j - \overline{c}_j + \beta P(\overline{w}_j)\right]}{1 - \beta \sum_{j=1}^{k-1} \Pi_j}.$$

We can now verify that $P(v_0)$ is strictly concave for $v_0 \in [\overline{w}_{k-1}, \overline{w}_k)$,

$$P'(v_0) = -\frac{\sum_{j=1}^{k-1} \Pi_j}{1 - \beta \sum_{j=1}^{k-1} \Pi_j} \tilde{c}'(v_0) = -u^{-1\prime}\left[\Phi_k(v_0)\right] < 0, \quad (20.3.35a)$$

$$P''(v_0) = -\frac{\sum_{j=1}^{k-1}\Pi_j}{1-\beta\sum_{j=1}^{k-1}\Pi_j}\,\tilde{c}''(v_0)$$
$$= -\frac{\left[1-\beta\sum_{j=1}^{k-1}\Pi_j\right]}{\sum_{j=1}^{k-1}\Pi_j}\,u^{-1''}\left[\Phi_k(v_0)\right] < 0, \qquad (20.3.35b)$$

where we have invoked expressions (20.3.34).

To shed light on the properties of the value function $P(v_0)$ around the promised value $\overline{w}_k$, we can establish that

$$\lim_{v_0 \uparrow \overline{w}_k} \Phi_k(v_0) = \Phi_k(\overline{w}_k) \; = \; \Phi_{k+1}(\overline{w}_k), \qquad (20.3.36)$$

where the first equality is a trivial limit of expression (20.3.32) while the second equality can be shown to hold because a rearrangement of that equality becomes merely a restatement of a version of expression (20.3.22b). On the basis of (20.3.36) and (20.3.33), we can conclude that the consumption level $\tilde{c}(v_0)$ is continuous in the promised value which in turn implies continuity of the value function $P(v_0)$. Moreover, expressions (20.3.36) and (20.3.35a) ensure that the value function $P(v_0)$ is continuously differentiable in the promised value.

20.3.6. Many households

Consider a large village in which a moneylender faces a continuum of such households. At the beginning of time $t = 0$, before the realization of y_0, the moneylender offers each household $v_{\rm aut}$ (or maybe just a small amount more). As time unfolds, the moneylender executes the contract for each household. A society of such households would experience a “fanning out” of the distributions of consumption and continuation values across households for a while, to be followed by an eventual “fanning in” as the cross-sectional distribution of consumption asymptotically becomes concentrated at the single point $g_2(\bar{v})$ computed earlier (i.e., the minimum c such that the participation constraint will never again be binding). Notice that early on the moneylender would on average, across villagers, be collecting money from the villagers, depositing it in the bank, and receiving the gross interest rate β^{-1} on the bank balance. Later he could be using the interest on his account outside the village to finance payments to the villagers. Eventually, the villagers are completely insured, i.e., they experience no fluctuations in their consumptions.

For a contract that offers initial promised value $v_0 \in [v_{\text{aut}}, \overline{w}_S]$, constructed as above, we can compute the dynamics of the cross-section distribution of consumption by appealing to a law of large numbers of the kind used in chapter 18. At time 0, after the time 0 endowments have been realized, the cross-section distribution of consumption is evidently

$$\text{Prob}\{c_0 = \tilde{c}\} = \left(\sum_{s=1}^{k-1} \Pi_s\right) \tag{20.3.37a}$$

$$\text{Prob}\{c_0 \leq \overline{c}_j\} = \left(\sum_{s=1}^{j} \Pi_s\right), \ j \geq k. \tag{20.3.37b}$$

After t periods,

$$\text{Prob}\{c_t = \tilde{c}\} = \left(\sum_{s=1}^{k-1} \Pi_s\right)^{t+1} \tag{20.3.38a}$$

$$\text{Prob}\{c_t \leq \overline{c}_j\} = \left(\sum_{s=1}^{j} \Pi_s\right)^{t+1}, \ j \geq k. \tag{20.3.38b}$$

From the cumulative distribution functions (20.3.37) and (20.3.38), it is easy to compute the corresponding densities

$$f_{j,t} = \text{Prob}(c_t = \overline{c}_j) \tag{20.3.39}$$

where here we set $\overline{c}_j = \tilde{c}$ for all $j < k$. These densities allow us to compute the evolution over time of the moneylender's bank balance. Starting with initial balance $\beta^{-1}B_{-1} = 0$ at time 0, the moneylender's balance at the bank evolves according to

$$B_t = \beta^{-1}B_{t-1} + \left(\sum_{j=1}^{S} \Pi_j \overline{y}_j - \sum_{j=1}^{S} f_{j,t}\overline{c}_j\right) \tag{20.3.40}$$

for $t \geq 0$, where B_t denotes the end-of-period balance in period t. Let $\beta^{-1} = 1 + r$. After the cross-section distribution of consumption has converged to a distribution concentrated on $\overline{c}_S$, the moneylender's bank balance will obey the difference equation

$$B_t = (1 + r)B_{t-1} + E(y) - \overline{c}_S, \tag{20.3.41}$$

where $E(y)$ is the mean of y.

A convenient formula links $P(v_0)$ to the tail behavior of B_t, in particular, to the behavior of B_t after the consumption distribution has converged to $\overline{c}_S$. Here we are once again appealing to a law of large numbers so that the expected profits $P(v_0)$ becomes a nonstochastic present value of profits associated with making a promise v_0 to a large number of households. Since the moneylender lets all surpluses and deficits accumulate in the bank account, it follows that $P(v_0)$ is equal to the present value of the sum of any future balances B_t and the continuation value of the remaining profit stream. After all households' promised values have converged to $\overline{w}_S$, the continuation value of the remaining profit stream is evidently equal to $\beta P(\overline{w}_S)$. Thus, for t such that the distribution of c has converged to $\overline{c}_s$, we deduce that

$$P(v_0) = \frac{B_t + \beta P(\overline{w}_S)}{(1+r)^t}. \tag{20.3.42}$$

Since the term $\beta P(\overline{w}_S)/(1+r)^t$ in expression (20.3.42) will vanish in the limit, the expression implies that the bank balances B_t will eventually change at the gross rate of interest. If the initial v_0 is set so that $P(v_0) > 0$ ($P(v_0) < 0$), then the balances will eventually go to plus infinity (minus infinity) at an exponential rate. The asymptotic balances would be constant only if the initial v_0 is set so that $P(v_0) = 0$. This has the following implications. First, recall from our calculations above that there can exist an initial promised value $v_0 \in [v_{\rm aut}, \overline{w}_S]$ such that $P(v_0) = 0$ only if it is true that $P(\overline{w}_S) \leq 0$, which by (20.3.28a) implies that $E(y) \leq \overline{c}_S$. After imposing $P(v_0) = 0$ and using the expression for $P(\overline{w}_S)$ in (20.3.28a), equation (20.3.42) becomes $B_t = -\beta \frac{E(y) - \overline{c}_S}{1-\beta}$, or

$$B_t = \frac{\overline{c}_S - E(y)}{r} \geq 0,$$

where we have used the definition $\beta^{-1} = 1+r$. Thus, if the initial promised value v_0 is such that $P(v_0) = 0$, then the balances will converge when all households' promised values converge to $\overline{w}_S$. The interest earnings on those stationary balances will equal the one-period deficit associated with delivering $\overline{c}_S$ to every household while collecting endowments per capita equal to $E(y) \leq \overline{c}_S$.

After enough time has passed, all of the villagers will be perfectly insured because according to (20.3.38), $\lim_{t \to +\infty} {\rm Prob}(c_t = \overline{c}_S) = 1$. How much time it takes to converge depends on the distribution Π. Eventually, everyone will have received the highest endowment realization sometime in the past, after which his continuation value remains fixed. Thus, this is a model of temporary imperfect insurance, as indicated by the eventual "fanning in" of the distribution of continuation values.

20.3.7. An example

Figures 20.3.3 and 20.3.4 summarize aspects of the optimal contract for a version of our economy in which each household has an i.i.d. endowment process that is distributed as

$$\text{Prob}(y_t = \overline{y}_s) = \frac{1-\lambda}{1-\lambda^S}\lambda^{s-1}$$

where $\lambda \in (0,1)$ and $\overline{y}_s = s+5$ is the sth possible endowment value, $s = 1,\ldots,S$. The typical household's one-period utility function is $u(c) = (1-\gamma)^{-1}c^{1-\gamma}$, where γ is the household's coefficient of relative risk aversion. We have assumed the parameter values $(\beta, S, \gamma, \lambda) = (.5, 20, 2, .95)$. The initial promised value v_0 is set so that $P(v_0) = 0$.

The moneylender's bank balance in Figure 20.3.3, panel d, starts at zero. The moneylender makes money at first, which he deposits in the bank. But as time passes, the moneylender's bank balance converges to the point that he is earning just enough interest on his balance to finance the extra payments he must make to pay $\overline{c}_S$ to each household each period. These interest earnings make up for the deficiency of his per capita period income $E(y)$, which is less than his per period per capita expenditures $\overline{c}_S$.

20.4. A Lagrangian method

Marcet and Marimon (1992, 1999) have proposed an approach that applies to most of the contract design problems of this chapter. They form a Lagrangian and use the Lagrange multipliers on incentive constraints to keep track of promises. Their approach extends the work of Kydland and Prescott (1980) and is related to Hansen, Epple, and Roberds' (1985) formulation for linear quadratic environments.[3] We can illustrate the method in the context of the preceding model.

[3] Marcet and Marimon's method is a variant of the method used to compute Stackelberg or Ramsey plans in chapter 19. See chapter 19 for a more extensive review of the history of

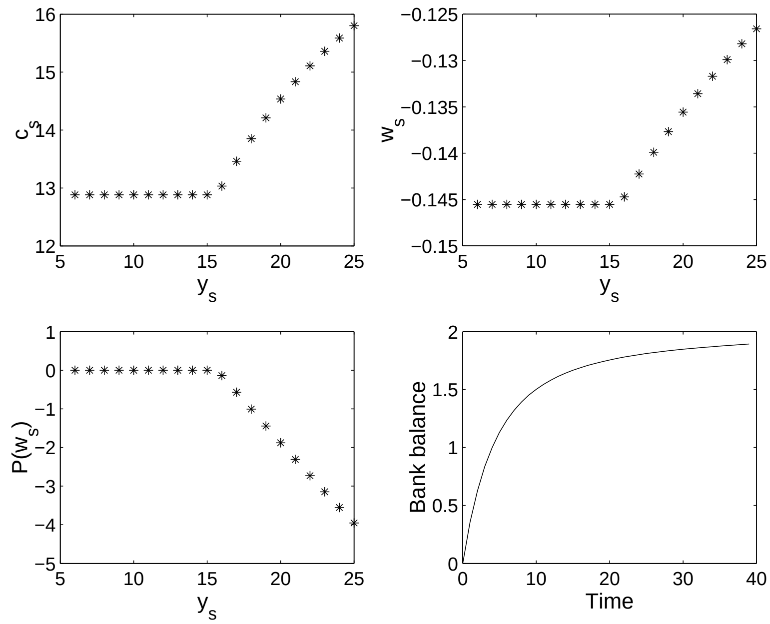

Figure 20.3.3: Optimal contract when $P(v_0) = 0$. Panel a: $\overline{c}_s$ as function of maximum $\overline{y}_s$ experienced to date. Panel b: $\overline{w}_s$ as function of maximum $\overline{y}_s$ experienced. Panel c: $P(\overline{w}_s)$ as function of maximum $\overline{y}_s$ experienced. Panel d: The moneylender's bank balance.

Marcet and Marimon's approach would be to formulate the problem directly in the space of stochastic processes (i.e., random sequences) and to form a Lagrangian for the moneylender. The contract specifies a stochastic process for consumption obeying the following constraints:

$$u(c_t) + E_t \sum_{j=1}^{\infty} \beta^j u(c_{t+j}) \geq u(y_t) + \beta v_{\rm aut} \ , \forall t \geq 0, \tag{20.4.1a}$$

$$E_{-1} \sum_{t=0}^{\infty} \beta^t u(c_t) \geq v, \tag{20.4.1b}$$

the ideas underlying Marcet and Marimon's approach, in particular, some work from Great Britain in the 1980s by Miller, Salmon, Pearlman, Currie, and Levine.

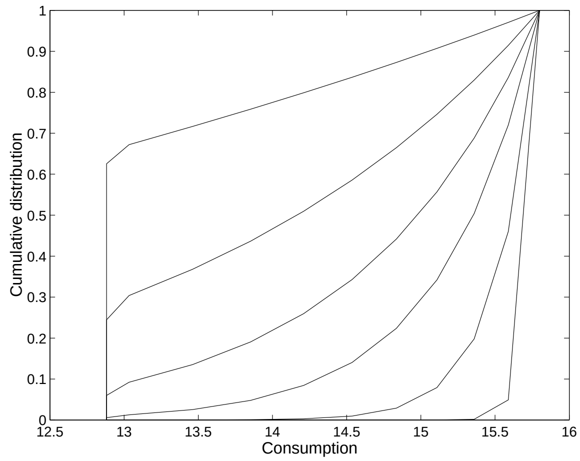

Figure 20.3.4: Cumulative distribution functions $F_t(c_t)$ for consumption for $t = 0, 2, 5, 10, 25, 100$ when $P(v_0) = 0$ (later dates have c.d.f.s shifted to right).

where $E_{-1}(\cdot)$ denotes the conditional expectation before y_0 has been realized. Here v is the initial promised value to be delivered to the villager starting in period 0. Equation $(20.4.1a)$ gives the participation constraints.

The moneylender's Lagrangian is

$$\begin{aligned} J = E_{-1} \sum_{t=0}^{\infty} \beta^t \Big\{ (y_t - c_t) + \alpha_t \Big[E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) - [u(y_t) + \beta v_{\rm aut}] \Big] \Big\} \\ + \phi \Big[E_{-1} \sum_{t=0}^{\infty} \beta^t u(c_t) - v \Big], \end{aligned} \tag{20.4.2}$$

where $\{\alpha_t\}_{t=0}^{\infty}$ is a stochastic process of nonnegative Lagrange multipliers on the participation constraint of the villager and ϕ is the strictly positive multiplier on the initial promise-keeping constraint that states that the moneylender must deliver v. It is useful to transform the Lagrangian by making use of the following equality, which is a version of the "partial summation formula of Abel" (see Apostol, 1975, p. 194):

$$\sum_{t=0}^{\infty} \beta^t \alpha_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) = \sum_{t=0}^{\infty} \beta^t \mu_t u(c_t), \tag{20.4.3}$$

where

$$\mu_t = \mu_{t-1} + \alpha_t, \qquad \text{with} \quad \mu_{-1} = 0. \tag{20.4.4}$$

Formula (20.4.3) can be verified directly. If we substitute formula (20.4.3) into formula (20.4.2) and use the law of iterated expectations to justify $E_{-1}E_t(\cdot) = E_{-1}(\cdot)$, we obtain

$$J = E_{-1}\sum_{t=0}^{\infty}\beta^t \left\{(y_t - c_t) + (\mu_t + \phi)u(c_t) \right. \\ \left. -(\mu_t - \mu_{t-1})\left[u(y_t) + \beta v_{\rm aut}\right]\right\} - \phi v. \tag{20.4.5}$$

For a given value v, we seek a saddle point: a maximum with respect to $\{c_t\}$, a minimum with respect to $\{\mu_t\}$ and ϕ. The first-order condition with respect to c_t is

$$u'(c_t) = \frac{1}{\mu_t + \phi}, \tag{20.4.6a}$$

which is a version of equation (20.3.12). Thus, $-(\mu_t+\phi)$ equals $P'(w)$ from the previous section, so that the multipliers encode the information contained in the derivative of the moneylender's value function. We also have the complementary slackness conditions

$$u(c_t) + E_t\sum_{j=1}^{\infty}\beta^j u(c_{t+j}) - \left[u(y_t) + \beta v_{\rm aut}\right] \geq 0, \qquad = 0 \text{ if } \alpha_t > 0; \tag{20.4.6b}$$

$$E_{-1}\sum_{t=0}^{\infty}\beta^t u(c_t) - v = 0. \tag{20.4.6c}$$

Equation (20.4.6) together with the transition law (20.4.4) characterizes the solution of the moneylender's maximization problem.

To explore the time profile of the optimal consumption process, we now consider some period $t \geq 0$ when (y_t, μ_{t-1}, ϕ) are known. First, we tentatively try the solution $\alpha_t = 0$ (i.e., the participation constraint is not binding). Equation (20.4.4) instructs us then to set $\mu_t = \mu_{t-1}$, which by first-order condition (20.4.6a) implies that $c_t = c_{t-1}$. If this outcome satisfies participation constraint (20.4.6b), we have our solution for period t. If not, it signifies that the participation constraint binds. In other words, the solution has $\alpha_t > 0$ and $c_t > c_{t-1}$. Thus, equations (20.4.4) and (20.4.6a) immediately show us that c_t

is a nondecreasing random sequence, that c_t stays constant when the participation constraint is not binding, and that it rises when the participation constraint binds.

The numerical computation of a solution to equation (20.4.5) is complicated by the fact that slackness conditions (20.4.6b) and (20.4.6c) involve conditional expectations of future endogenous variables $\{c_{t+j}\}$. Marcet and Marimon (1992) handle this complication by resorting to the parameterized expectation approach; that is, they replace the conditional expectation by a parameterized function of the state variables.[4] Marcet and Marimon (1992, 1999) describe a variety of other examples using the Lagrangian method. See Kehoe and Perri (2002) for an application to an international trade model.

20.5. Insurance with asymmetric information

The moneylender-villager environment of section 20.3 has a commitment problem because agents are free to choose autarky each period; but there is no information problem. We now study a contract design problem where the incentive problem comes not from a commitment problem, but instead from asymmetric information. As before, the moneylender or planner can borrow or lend outside the village at the constant risk-free gross interest rate of β^{-1}, and each household's income y_t is independently and identically distributed across time and across households. However, we now assume that both the planner and households can enter into enduring and binding contracts. At the beginning of time, let v^o be the expected lifetime utility that the planner promises to deliver to a household. The initial promise v^o could presumably not be less than $v_{\rm aut}$, since a household would not accept a contract that gives a lower utility than he could attain at time 0 by choosing autarky. We defer discussing how v^o is determined until the end of the section. The other new assumption here is that households have private information about their own income, and that the planner can see neither their income nor their consumption. It follows that any transfers between the planner and a household must be based on the household's

[4] For details on the implementation of the parameterized expectation approach in a simple growth model, see den Haan and Marcet (1990).

own reports about income realizations. An incentive-compatible contract makes households choose to report their incomes truthfully.

Our analysis follows the work by Thomas and Worrall (1990), who make a few additional assumptions about the preferences in expression (20.2.1): $u : (a, \infty) \to \mathbf{R}$ is twice continuously differentiable with $\sup u(c) < \infty$, $\inf u(c) = -\infty$, $\lim_{c \to a} u'(c) = \infty$. Thomas and Worrall also use the following special assumption:

Condition A: $-u''/u'$ is nonincreasing.

This is a sufficient condition to make the value function concave, as we will discuss. The roles of the other restrictions on preferences will also be revealed.

An efficient insurance contract solves a dynamic programming problem.[5] A planner maximizes expected discounted profits, $P(v)$, where v is the household's promised utility from last period. The planner's current payment to the household, denoted b (repayments from the household register as negative numbers), is a function of the state variable v and the household's reported current income y. Let b_s and w_s be the payment and continuation utility awarded to the household if it reports income $\overline{y}_s$. The optimum value function $P(v)$ obeys the functional equation

$$P(v) = \max_{\{b_s, w_s\}} \sum_{s=1}^{S} \Pi_s [-b_s + \beta P(w_s)] \tag{20.5.1}$$

where the maximization is subject to the constraints

$$\sum_{s=1}^{S} \Pi_s \left[u(\overline{y}_s + b_s) + \beta w_s\right] = v \tag{20.5.2}$$

$$C_{s,k} \equiv u(\overline{y}_s + b_s) + \beta w_s - \Big[u(\overline{y}_s + b_k) + \beta w_k\Big] \geq 0, \;\; s, k \in \mathbf{S} \times \mathbf{S} \tag{20.5.3}$$

$$b_s \in [a - \overline{y}_s, \, \infty] \, , \;\; s \in \mathbf{S} \tag{20.5.4}$$

$$w_s \in [-\infty, \, v_{\max}] \, , \;\; s \in \mathbf{S} \tag{20.5.5}$$

where $v_{\max} = \sup u(c)/(1 - \beta)$. Equation (20.5.2) is the "promise-keeping" constraint guaranteeing that the promised utility v is delivered. Note that

[5] It is important that the endowment is independently distributed over time. See Fernandes and Phelan (2000) for a related analysis that shows complications that arise when the iid assumption is relaxed

our earlier weak inequality in (20.3.5) is replaced by an equality. The planner cannot award a higher utility than v because that could violate an incentive-compatibility constraint for telling the truth in earlier periods. The set of constraints (20.5.3) ensures that the households have no incentive to lie about their endowment realization in each state $s \in \mathbf{S}$. Here s indexes the actual income state, and k indexes the reported income state. We express the incentive compatibility constraints when the endowment is in state s as $C_{s,k} \geq 0$ for $k \in \mathbf{S}$. Note also that there are no "participation constraints" like expression (20.3.6) from our earlier model, an absence that reflects the assumption that both parties are committed to the contract.

It is instructive to establish bounds on the value function $P(v)$. Consider first a contract that pays a constant amount $\bar{b} = \bar{b}(v)$ in all periods, where $\bar{b}(v)$ satisfies $\sum_{s=1}^{S} \Pi_s u(\overline{y}_s + \bar{b})/(1-\beta) = v$. It is trivially incentive compatible and delivers the promised utility v. Therefore, the discounted profits from this contract, $-\bar{b}/(1-\beta)$, provide a lower bound on $P(v)$. In addition, $P(v)$ cannot exceed the value of the unconstrained first-best contract that pays $\bar{c} - \overline{y}_s$ in all periods, where $\bar{c}$ satisfies $\sum_{s=1}^{S} \Pi_s u(\bar{c})/(1-\beta) = v$. Thus, the value function is bounded by

$$-\bar{b}(v)/(1-\beta) \;\leq\; P(v) \;\leq\; \sum_{s=1}^{S} \Pi_s[\overline{y}_s - \bar{c}(v)]/(1-\beta)\,. \tag{20.5.6}$$

The bounds are depicted in Figure 20.5.1, which also illustrates a few other properties of $P(v)$. Since $\lim_{c\to a} u'(c) = \infty$, it becomes very cheap for the planner to increase the promised utility when the current promise is very low, that is, $\lim_{v\to-\infty} P'(v) = 0$. The situation is different when the household's promised utility is close to the upper bound $v_{\max}$ where the household has a low marginal utility of additional consumption, which implies that both $\lim_{v\to v_{\max}} P'(v) = -\infty$ and $\lim_{v\to v_{\max}} P(v) = -\infty$.

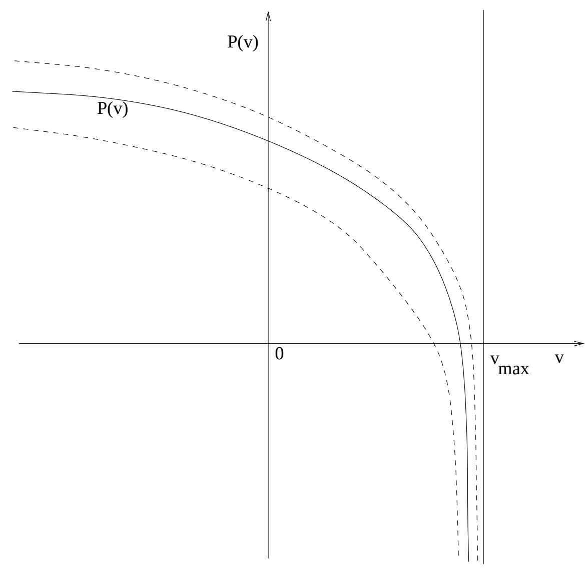

Figure 20.5.1: Value function $P(v)$ and the two dashed curves depict the bounds on the value function. The vertical solid line indicates $v_{\max} = \sup u(c)/(1-\beta)$.

20.5.1. Efficiency implies $b_{s-1} \geq b_s, w_{s-1} \leq w_s$

An incentive-compatible contract must satisfy $b_{s-1} \geq b_s$ (insurance) and $w_{s-1} \leq w_s$ (*partial* insurance). This can be established by adding the "downward constraint" $C_{s,s-1} \geq 0$ and the "upward constraint" $C_{s-1,s} \geq 0$ to get

$$u(\overline{y}_s + b_s) \; - \; u(\overline{y}_{s-1} + b_s) \; \geq \; u(\overline{y}_s + b_{s-1}) \; - \; u(\overline{y}_{s-1} + b_{s-1}) \, ,$$

where the concavity of $u(c)$ implies $b_s \leq b_{s-1}$. It then follows directly from $C_{s,s-1} \geq 0$ that $w_s \geq w_{s-1}$. Thus, for any v, a household reporting a lower income receives a higher transfer from the planner in exchange for a lower future utility.

20.5.2. Local upward and downward constraints are enough

Constraint set (20.5.3) can be simplified. We can show that if the local downward constraints $C_{s,s-1} \geq 0$ and upward constraints $C_{s,s+1} \geq 0$ hold for each $s \in \mathbf{S}$, then the global constraints $C_{s,k} \geq 0$ hold for each $s, k \in \mathbf{S}$. The argument goes as follows: Suppose we know that the downward constraint $C_{s,k} \geq 0$ holds for some $s > k$,

$$u(\overline{y}_s + b_s) + \beta w_s \;\geq\; u(\overline{y}_s + b_k) + \beta w_k \,. \tag{20.5.7}$$

From above we know that $b_s \leq b_k$, so the concavity of $u(c)$ implies

$$u(\overline{y}_{s+1} + b_s) \,-\, u(\overline{y}_s + b_s) \;\geq\; u(\overline{y}_{s+1} + b_k) \,-\, u(\overline{y}_s + b_k)\,. \tag{20.5.8}$$

By adding expressions (20.5.7) and (20.5.8) and using the local downward constraint $C_{s+1,s} \geq 0$, we arrive at

$$u(\overline{y}_{s+1} + b_{s+1}) + \beta w_{s+1} \;\geq\; u(\overline{y}_{s+1} + b_k) + \beta w_k,$$

that is, we have shown that the downward constraint $C_{s+1,k} \geq 0$ holds. In this recursive fashion we can verify that all global downward constraints are satisfied when the local downward constraints hold. A symmetric reasoning applies to the upward constraints. Starting from any upward constraint $C_{k,s} \geq 0$ with $k < s$, we can show that the local upward constraint $C_{k-1,k} \geq 0$ implies that the upward constraint $C_{k-1,s} \geq 0$ must also hold, and so forth.

20.5.3. Concavity of P

Thus far, we have not appealed to the concavity of the value function, but henceforth we shall have to. Thomas and Worrall showed that under condition A, P is concave.

Proposition: The value function $P(v)$ is concave.

We recommend just skimming the following proof on first reading:

Proof: Let $T(P)$ be the operator associated with the right side of equation (20.5.1). We could compute the optimum value function by iterating to convergence on T. We want to show that T maps strictly concave P to strictly concave function $T(P)$. Thomas and Worrall use the following argument:

Let $P_{k-1}(v)$ be the $k-1$ iterate on T. Assume that $P_{k-1}(v)$ is strictly concave. We want to show that $P_k(v)$ is strictly concave. Consider any v^o and v' with associated contracts $(b_s^o, w_s^o)_{s\in S}, (b_s', w_s')_{s\in S}$. Let $w_s^* = \delta w_s^o + (1-\delta)w_s'$ and define b_s^* by $u(b_s^* + \overline{y}_s) = \delta u(b_s^o + \overline{y}_s) + (1-\delta)u(b_s' + \overline{y}_s)$ where $\delta \in (0,1)$. Therefore, $(b_s^*, w_s^*)_{s\in S}$ gives the borrower a utility that is the weighted average of the two utilities, and gives the lender no less than the average utility $\delta P_k(v^o) + (1-\delta)P_k(v')$. Then $C_{s,s-1}^* = \delta C_{s,s-1}^o + (1-\delta)C_{s,s-1}' + [\delta u(b_{s-1}^o + \overline{y}_s) + (1-\delta)u(b_{s-1}' + \overline{y}_s) - u(b_{s-1}^* + \overline{y}_s)]$. Because the downward constraints $C_{s,s-1}^o$ and $C_{s,s-1}'$ are satisfied, and because the third term is nonnegative under condition A, the downward incentive constraints $C_{s,s-1}^* \geq 0$ are satisfied. However, $(b_s^*, w_s^*)_{s\in S}$ may violate the upward incentive constraints. But Thomas and Worrall use the following argument to construct a new contract from $(b_s^*, w_s^*)_{s\in S}$ that is incentive compatible and that offers both the lender and the borrower no less utility. Thus, keep w_1 fixed and reduce w_2 until $C_{2,1} = 0$ or $w_2 = w_1$. Then reduce w_3 in the same way, and so on. Add the constant necessary to leave $\sum_s \Pi_s w_s$ constant. This step will not make the lender worse off, by the concavity of $P_{k-1}(v)$. Now if $w_2 = w_1$, which implies $b_2^* > b_1^*$, reduce b_2 until $C_{2,1} = 0$, and proceed in the same way for b_3, and so on. Since $b_s + \overline{y}_s > b_{s-1} + \overline{y}_{s-1}$, adding a constant to each b_s to leave $\sum_s \Pi_s b_s$ constant cannot make the borrowers worse off. So in this new contract, $C_{s,s-1} = 0$ and $b_{s-1} \geq b_s$. Thus, the upward constraints also hold. Strict concavity of $P_k(v)$ then follows because it is not possible to have both $b_s^o = b_s'$ and $w_s^o = w_s'$ for all $s \in S$ and $v^o \neq v'$, so the contract (b_s^*, w_s^*) yields the lender strictly more than $\delta P_k(v^o) + (1-\delta)P_k(v')$. To complete the induction argument, note that starting from $P_0(v) = 0$, $P_1(v)$ is strictly concave. Therefore, $\lim_{k=\infty} P_k(v)$ is concave. ∎

We now turn to some properties of the optimal allocation that require strict concavity of the value function. Thomas and Worrall derive these results for the finite horizon problem with value function $P_k(v)$, which is strictly concave by the preceding proposition. In order for us to stay with the infinite horizon value function $P(v)$, we make the following assumption about $\lim_{k=\infty} P_k(v)$:[6]

ASSUMPTION: The value function $P(v)$ is strictly concave.

[6] To get the main result reported below that all households become impoverished in the limit, Thomas and Worrall provide a proof that requires only concavity of $P(v)$ as established in the preceding proposition.

20.5.4. Local downward constraints always bind

At the optimal solution, the local downward incentive constraints always bind, while the local upward constraints never do. That is, a household is always indifferent between reporting the truth and reporting that its income is actually a little lower than it is; but it never wants to report that its income is higher. To see that the downward constraints must bind, suppose to the contrary that $C_{k,k-1} > 0$ for some $k \in \mathbf{S}$. Since $b_k \leq b_{k-1}$, it must then be the case that $w_k > w_{k-1}$. Consider changing $\{b_s, w_s; s \in \mathbf{S}\}$ as follows. Keep w_1 fixed, and if necessary reduce w_2 until $C_{2,1} = 0$. Next reduce w_3 until $C_{3,2} = 0$, and so on, until $C_{s,s-1} = 0$ for all $s \in \mathbf{S}$. (Note that any reductions cumulate when moving up the sequence of constraints.) Thereafter, add the necessary constant to each w_s to leave the expected value of all future promises unchanged, $\sum_{s=1}^{S} \Pi_s w_s$. The new contract offers the household the same utility and is incentive compatible because $b_s \leq b_{s-1}$ and $C_{s,s-1} = 0$ together imply that the local upward constraint $C_{s-1,s} \geq 0$ does not bind. At the same time, since the mean of promised values is unchanged and the differences $(w_s - w_{s-1})$ have either been left the same or reduced, the strict concavity of the value function $P(v)$ implies that the planner's profits have increased. That is, we have engineered a mean-preserving *decrease* in the spread in the continuation values w. Because $P(v)$ is strictly concave, $\sum_{s \in \mathrm{S}} \Pi_s P(w_s)$ rises and therefore $P(v)$ rises. Thus, the original contract with a nonbinding local downward constraint could not have been an optimal solution.

20.5.5. Coinsurance

The optimal contract is characterized by *coinsurance*, meaning that the household's utility and the planner's profits both increase with a higher income realization:

$$u(\overline{y}_s + b_s) + \beta w_s \ > \ u(\overline{y}_{s-1} + b_{s-1}) + \beta w_{s-1} \tag{20.5.9}$$

$$-b_s + \beta P(w_s) \ \geq \ -b_{s-1} + \beta P(w_{s-1}) \, . \tag{20.5.10}$$

The higher utility of the household in expression (20.5.9) follows trivially from the downward incentive-compatibility constraint $C_{s,s-1} = 0$. Concerning the planner's profits in expression (20.5.10), suppose to the contrary that $-b_s + \beta P(w_s) < -b_{s-1} + \beta P(w_{s-1})$. Then replacing (b_s, w_s) in the contract by

(b_{s-1}, w_{s-1}) raises the planner's profits but leaves the household's utility unchanged because $C_{s,s-1} = 0$, and the change is also incentive compatible. Thus, an optimal contract must be such that the planner's profits weakly increase in the household's income realization.

20.5.6. $P'(v)$ is a martingale

If we let λ and μ_s, $s = 2, \ldots, S$, be Lagrange multipliers associated with the constraints (20.5.2) and $C_{s,s-1} \geq 0$, $s = 2, \ldots, S$, respectively, the first-order necessary conditions with respect to b_s and w_s, $s \in \mathbf{S}$, are

$$\Pi_s \Big[1 - \lambda\, u'(\overline{y}_s + b_s) \Big] \;=\; \mu_s\, u'(\overline{y}_s + b_s) \;-\; \mu_{s+1}\, u'(\overline{y}_{s+1} + b_s), \tag{20.5.11}$$

$$\Pi_s \Big[P'(w_s) \,+\, \lambda \Big] \;=\; \mu_{s+1} \,-\, \mu_s\,, \tag{20.5.12}$$

for $s \in \mathbf{S}$, where $\mu_1 = \mu_{S+1} = 0$. (There are no constraints corresponding to μ_1 and μ_{S+1}.) From the envelope condition,

$$P'(v) \;=\; -\lambda\,. \tag{20.5.13}$$

Summing equation (20.5.12) over $s \in \mathbf{S}$ and using $\sum_{s=1}^{S} (\mu_{s+1} - \mu_s) = \mu_{S+1} - \mu_1 = 0$ and equation (20.5.13) yields

$$\sum_{s=1}^{S} \Pi_s\, P'(w_s) \;=\; P'(v)\,. \tag{20.5.14}$$

This equation states that P' is a martingale.

20.5.7. *Comparison to model with commitment problem*

In the model with a commitment problem studied in section 20.3, the efficient allocation had to satisfy equation (20.3.12), i.e., $u'(\overline{y}_s + b_s) = -P'(w_s)^{-1}$. As we explained then, this condition sets the household's marginal rate of substitution equal to the planner's marginal rate of transformation with respect to transfers in the current period and continuation values in the next period. This condition fails to hold in the present framework with incentive-compatibility constraints associated with telling the truth. The efficient trade-off between current consumption and a continuation value for a household with income realization $\overline{y}_s$ can not be determined without taking into account the incentives that other households have to report $\overline{y}_s$ untruthfully in order to obtain the corresponding bundle of current and future transfers from the planner. It is instructive to note that equation (20.3.12) *would* continue to hold in the present framework if the incentive-compatibility constraints for truth telling were not binding. That is, set the multipliers μ_s, $s = 2, \ldots, S$, equal to zero and substitute first-order condition (20.5.12) into (20.5.11) to obtain $u'(\overline{y}_s + b_s) = -P'(w_s)^{-1}$.

20.5.8. *Spreading continuation values*

An efficient contract requires that the promised future utility falls (rises) when the household reports the lowest (highest) income realization, that is, that $w_1 < v < w_S$. To show that $w_S > v$, suppose to the contrary that $w_S \leq v$. That this assumption leads to a contradiction is established by the following line of argument. Since $w_S \geq w_s$ for all $s \in \mathbf{S}$ and $P(v)$ is strictly concave, equation (20.5.14) implies that $w_s = v$ for all $s \in \mathbf{S}$. Substitution of equation (20.5.13) into equation (20.5.12) then yields a zero on the left side of equation (20.5.12). Moreover, the right side of equation (20.5.12) is equal to μ_2 when $s = 1$ and $-\mu_S$ when $s = S$, so we can successively unravel from the constraint set (20.5.12) that $\mu_s = 0$ for all $s \in \mathbf{S}$. Turning to equation (20.5.11), it follows that the marginal utility of consumption is equalized across income realizations, $u'(\overline{y}_s + b_s) = \lambda^{-1}$ for all $s \in \mathbf{S}$. Such consumption smoothing requires $b_{s-1} > b_s$, but from incentive compatibility, $w_{s-1} = w_s$ implies $b_{s-1} = b_s$, a contradiction. We conclude that an efficient contract must have $w_S > v$. A symmetric argument establishes $w_1 < v$.

The planner must spread out promises to future utility because otherwise it would be impossible to provide any insurance in the form of contingent payments today. Equation (20.5.14) describes how the planner balances the delivery of utility today versus tomorrow. To understand this expression, consider having the planner increase the household's promised utility v by one unit. One way of doing so is to increase every w_s by an increment $1/\beta$ while keeping every b_s constant. Such a change preserves incentive compatibility at an expected discounted cost to the planner of $\sum_{s=1}^{S} \Pi_s P'(w_s)$. By the envelope theorem, locally this is as good a way to increase v as any other, and its cost is therefore equal to $P'(v)$; that is, we obtain expression (20.5.14). In other words, given a planner's obligation to deliver utility v to the agent, it is cost-efficient to balance today's contingent deliveries of goods, $\{b_s\}$, and the bundle of future utilities, $\{w_s\}$, so that the expected marginal cost of next period's promises, $\sum_{s=1}^{S} \Pi_s P'(w_s)$, becomes equal to the marginal cost of the current obligation, $P'(v)$. No intertemporal price affects this trade-off, since any interest earnings on postponed payments are just sufficient to compensate the agent for his own subjective rate of discounting, $(1+r) = \beta^{-1}$.

20.5.9. Martingale convergence and poverty

The martingale property (20.5.14) for $P'(v)$ has an intriguing implication for the long-run tendency of a household's promised future utility. Recall that $\lim_{v \to -\infty} P'(v) = 0$ and $\lim_{v \to v_{\max}} P'(v) = -\infty$, so $P'(v)$ in expression (20.5.14) is a nonpositive martingale. By a theorem of Doob (1953, p. 324), $P'(v)$ then converges almost surely. We can show that $P'(v)$ must converge to 0, so that v converges to $-\infty$. Suppose to the contrary that $P'(v)$ converges to a nonzero limit, which implies that v converges to a finite limit. However, this assumption contradicts our earlier result that future w_s always spread out to provide incentives. The contradiction is avoided only for v converging to $-\infty$; therefore, the limit of $P'(v)$ must be zero.

The result that all households become impoverished in the limit can be understood in terms of the concavity of $P(v)$. First, if there were no asymmetric information, the least expensive way of delivering lifetime utility v would be to assign the household a constant consumption stream, given by the upper bound on the value function in expression (20.5.6). The concavity of $P(v)$

and standard intertemporal considerations favor a time-invariant consumption stream. But the presence of asymmetric information makes it necessary for the planner to vary promises of future utility to induce truth telling, which is costly due to the concavity of $P(v)$. For example, Thomas and Worrall pointed out that if $S = 2$, the cost of spreading w_1 and w_2 an equally small amount ϵ on either side of their average value $\bar{w}$ is approximately $-0.5\epsilon^2 P''(\bar{w})$.[7] In general, we cannot say how this cost differs for any two values of $\bar{w}$, but it follows from the properties of $P(v)$ at its endpoints that $\lim_{v\to-\infty} P''(v) = 0$, and $\lim_{v\to v_{\max}} P''(v) = -\infty$. Thus, the cost of spreading promised values goes to zero at one endpoint and to infinity at the other endpoint. Therefore, the concavity of $P(v)$ and incentive compatibility considerations impart a downward drift to future utilities and, consequently, consumption. That is, with private information the ideal time-invariant consumption level without private information is abandoned in favor of random consumption paths that are expected to be tilted toward the present.

One possibility is that the initial utility level v^o is determined in competition between insurance providers. If there are no costs associated with administering contracts, v^o would then be implicitly determined by the zero-profit condition, $P(v^o) = 0$. Such a contract must be enforceable because, as we have seen, the household will almost surely eventually wish that it could revert to autarky. However, since the contract is the solution to a dynamic programming problem where the continuation of the contract is always efficient at every date, the insurer and the household will never mutually agree to renegotiate the contract.

7 The expected discounted profits of providing promised values $w_1 = \bar{w} - \epsilon$ and $w_2 = \bar{w} + \epsilon$ with equal probabilities, can be approximated with a Taylor series expansion around $\bar{w}$, $\sum_{s=1}^{2} \frac{1}{2} P(w_s) \approx \sum_{s=1}^{2} \frac{1}{2} \left[P(\bar{w}) + (w_s - \bar{w}) P'(\bar{w}) + \frac{(w_s - \bar{w})^2}{2} P''(\bar{w}) \right] = P(\bar{w}) + \frac{\epsilon^2}{2} P''(\bar{w})$.

20.5.10. Extension to general equilibrium

Atkeson and Lucas (1992) provide examples of closed economies where the constrained efficient allocation also has each household's expected utility converging to the minimum level with probability 1. Here the planner chooses the incentive-compatible allocation for all agents subject to a constraint that the total consumption handed out in each period to the population of households cannot exceed some constant endowment level. Households are assumed to experience unobserved idiosyncratic taste shocks ϵ that are i.i.d. over time and households. The taste shock enters multiplicatively into preferences that take either the logarithmic form $u(c, \epsilon) = \epsilon \log(c)$, the constant relative risk aversion (CRRA) form $u(c, \epsilon) = \epsilon c^\gamma / \gamma,\ \gamma < 1,\ \gamma \neq 0$, or the constant absolute risk aversion (CARA) form $u(c, \epsilon) = -\epsilon \exp(-\gamma c),\ \gamma > 0$. The assumption that the utility function belongs to one of these families greatly simplifies the analytics of the evolution of the wealth distribution. Atkeson and Lucas show that an equilibrium of this model yields an efficient allocation that assigns an ever-increasing fraction of resources to an ever-diminishing fraction of the economy's population.

20.5.11. Comparison with self-insurance

We have just seen how in the Thomas and Worrall model, the planner responds to the incentive problem created by the consumer's private information by putting a downward tilt into temporal consumption profiles. It is useful to recall how in the savings problem of chapters 17 and 18, the martingale convergence theorem was used to show that the consumption profile acquired an upward tilt coming from the motive of the consumer to self-insure.

20.6. Insurance with unobservable storage

In the spirit of an analysis of Franklin Allen (1985), we now augment the model of the previous section by assuming that households have access to a technology that enables them to store nonnegative amounts of goods at a risk-free gross return of $R > 0$. The planner cannot observe private storage. The planner can borrow and lend outside the village at a risk-free gross interest rate that also equals R, so that private and public storage yield identical rates of return. The planner retains an advantage over households of being the only one able to *borrow* outside of the village.

The outcome of our analysis will be to show that allowing households to store amounts that are not observable to the planner so impedes the planner's ability to manipulate the household's continuation valuations that no social insurance can be supplied. Instead, the planner helps households overcome the nonnegativity constraint on households' storage by in effect allowing them to engage also in private *borrowing* at the risk-free rate R, subject to natural borrowing limits. Thus, outcomes share many features of the allocations studied in chapters 17 and 18.

Our analysis partly follows Cole and Kocherlakota (2001), who assume that a household's utility function $u(\cdot)$ is strictly concave and twice continuously differentiable over $(0, \infty)$ with $\lim_{c \to 0} u'(c) = \infty$. The domain of u is the entire real line with $u(c) = -\infty$ for $c < 0$.[8] They also assume that u satisfies condition A above. This preference specification allows Cole and Kocherlakota to characterize an efficient allocation in a finite horizon model. Their extension to an infinite horizon involves a few other assumptions, including upper and lower bounds on the utility function.

We retain our earlier assumption that the planner has access to a risk-free loan market outside of the village. Cole and Kocherlakota (2001) postulate a closed economy where the planner is constrained to choose nonnegative amounts of storage. Hence, our concept of feasibility differs from theirs.

[8] Allowing for negative consumption while setting utility equal to $-\infty$ is a convenient device for avoiding having to deal with transfers that exceed the household's resources.

20.6.1. Feasibility

Anticipating that our characterization of efficient outcomes will be in terms of sequences of quantities, we let the history of a household's reported income enter as an argument in the function specifying the planner's transfer scheme. In period t, a household with an earlier history h_{t-1} and a currently reported income of y_t receives a transfer $b_t(\{h_{t-1}, y_t\})$ that can be either positive or negative. If all households report their incomes truthfully, the planner's time t budget constraint is

$$K_t + \sum_{h_t} \pi(h_t) b_t(h_t) \leq RK_{t-1}, \tag{20.6.1}$$

where K_t is the planner's end-of-period savings (or, if negative, borrowing) and $\pi(h_t)$ is the unconditional probability that a household experiences history h_t, which in the planner's budget constraint equals the fraction of households that experience history h_t. Given a finite horizon with a final period T, solvency of the planner requires that $K_T \geq 0$.

We use a household's history h_t to index consumption and private storage at time t; $c_t(h_t) \geq 0$ and $k_t(h_t) \geq 0$. The household's resource constraint at history h_t at time t is

$$c_t(h_t) + k_t(h_t) \leq y_t(h_t) + Rk_{t-1}(h_{t-1}) + b_t(h_t), \tag{20.6.2}$$

where the function for current income $y_t(h_t)$ returns the tth element of the household's history h_t. We assume that the household has always reported its income truthfully, so that the transfer in period t is given by $b_t(h_t)$.

Given initial conditions $K_0 = k_0 = 0$, an allocation $(c, k, b, K) \equiv \{c_t(h_t), k_t(h_t), b_t(h_t), K_t\}$ is physically feasible if inequalities (20.6.1), (20.6.2) and $k_t(h_t) \geq 0$ are satisfied for all periods t and all histories h_t, and $K_T \geq 0$.

20.6.2. Incentive compatibility

Since income realizations and private storage are both unobservable, households are free to deviate from an allocation (c, k, b, K) in two ways. First, households can lie about their income and thereby receive the transfer payments associated with the reported but untrue income history. Second, households can choose different levels of storage. Let Ω^T be the set of reporting and storage strategies $(\hat{y}, \hat{k}) \equiv \{\hat{y}_t(h_t), \hat{k}_t(h_t); \text{for all } t, h_t\}$, where h_t denotes the household's true history.

Let $\hat{h}_t$ denote the history of reported incomes, $\hat{h}_t(h_t) = \{\hat{y}_1(h_1), \hat{y}_2(h_2), \ldots, \hat{y}_t(h_t)\}$. With some abuse of notation, we let y denote the truth-telling strategy for which $\hat{y}_t(\{h_{t-1}, y_t\}) = y_t$ for all (t, h_{t-1}), and hence for which $\hat{h}_t(h_t) = h_t$.

Given a transfer scheme b, the expected utility of following reporting and storage strategy $(\hat{y}, \hat{k})$ is

$$\Gamma(\hat{y}, \hat{k}; b) \equiv \sum_{t=1}^{T} \beta^{t-1} \sum_{h_t} \pi(h_t) \cdot u\Big(y_t(h_t) + R\hat{k}_{t-1}(h_{t-1}) + b_t(\hat{h}_t(h_t)) - \hat{k}_t(h_t)\Big), \tag{20.6.3}$$

given $k_0 = 0$. An allocation is incentive compatible if

$$\Gamma(y, k; b) = \max_{(\hat{y}, \hat{k}) \in \Omega^T} \Gamma(\hat{y}, \hat{k}; b). \tag{20.6.4}$$

An allocation that is both incentive compatible and feasible is called an *incentive feasible* allocation. The following proposition asserts that any incentive feasible allocation with private storage can be attained with an alternative incentive feasible allocation without private storage.

Proposition 1: Given any incentive feasible allocation (c, k, b, K), there exists another incentive feasible allocation $(c, 0, b^o, K^o)$.

Proof: We claim that $(c, 0, b^o, K^o)$ is incentive feasible where

$$b_t^o(h_t) \equiv b_t(h_t) - k_t(h_t) + Rk_{t-1}(h_{t-1}), \tag{20.6.5}$$

$$K_t^o \equiv \sum_{h_t} \pi(h_t) k_t(h_t) + K_t. \tag{20.6.6}$$

Feasibility follows from the assumed feasibility of (c, k, b, K). Note also that $\Gamma(y, 0; b^o) = \Gamma(y, k; b)$. The proof of incentive compatibility is by contradiction.

Suppose that $(c, 0, b^o, K^o)$ is not incentive compatible, i.e., that there exists a reporting and storage strategy $(\hat{y}, \hat{k}) \in \Omega^T$ such that

$$\Gamma(\hat{y}, \hat{k}; b^o) > \Gamma(y, 0; b^o) = \Gamma(y, k; b). \tag{20.6.7}$$

After invoking expression (20.6.5) for transfer payment $b_t^o(\hat{h}_t(h_t))$, the left side of inequality (20.6.7) becomes

$$\begin{aligned}\Gamma(\hat{y}, \hat{k}; b^o) = &\sum_{t=1}^{T} \beta^{t-1} \sum_{h_t} \pi(h_t)\, u\Big(y_t(h_t) + R\hat{k}_{t-1}(h_{t-1}) - \hat{k}_t(h_t) \\ &+ \Big[b_t(\hat{h}_t(h_t)) - k_t(\hat{h}_t(h_t)) + Rk_{t-1}(\hat{h}_{t-1}(h_{t-1}))\Big]\Big) \\ = &\ \Gamma(\hat{y}, k^*; b),\end{aligned}$$

where we have defined $k_t^*(h_t) \equiv \hat{k}_t(h_t) + k_t(\hat{h}_t(h_t))$. Thus, inequality (20.6.7) implies that

$$\Gamma(\hat{y}, k^*; b) > \Gamma(y, k; b),$$

which contradicts the assumed incentive compatibility of (c, k, b, K). ∎

20.6.3. Efficient allocation

An incentive feasible allocation that maximizes *ex ante* utility is called an efficient allocation. It solves the following problem:

$$\text{(P1)} \qquad \max_{\{c,k,b,K\}} \sum_{t=1}^{T} \beta^{t-1} \sum_{h_t} \pi(h_t) u(c_t(h_t))$$

subject to

$$\begin{aligned}&\Gamma(y, k; b) = \max_{(\hat{y}, \hat{k}) \in \Omega^T} \Gamma(\hat{y}, \hat{k}; b) \\ &c_t(h_t) + k_t(h_t) = y_t(h_t) + Rk_{t-1}(h_{t-1}) + b_t(h_t), \qquad \forall t, h_t \\ &K_t + \sum_{h_t} \pi(h_t) b_t(h_t) \le RK_{t-1}, \qquad \forall t \\ &k_t(h_t) \ge 0, \qquad \forall t, h_t \\ &K_T \ge 0, \\ &K_0 = k_0 = 0.\end{aligned}$$

The incentive compatibility constraint with unobservable private storage makes problem (P1) exceedingly difficult to solve. To find the efficient allocation we will adopt a guess-and-verify approach. We will guess that the consumption allocation that solves (P1) coincides with the optimal consumption allocation in another economic environment. For example, we might guess that the consumption allocation that solves (P1) is the same as in a complete markets economy with complete enforcement. A better guess might be the autarkic consumption allocation where each household stores goods only for its own use, behaving according to a version of the chapter 17 model with a no-borrowing constraint. Our analysis of the model without private storage in the previous section makes the first guess doubtful. In fact, both guesses are wrong. What turns out to be true is the following.

PROPOSITION 2: An incentive feasible allocation (c, k, b, K) is efficient if and only if $c = c^*$, where c^* is the consumption allocation that solves

$$\text{(P2)} \qquad \max_{\{c\}} \sum_{t=1}^{T} \beta^{t-1} \sum_{h_t} \pi(h_t) u(c_t(h_t))$$

subject to

$$\sum_{t=1}^{T} R^{1-t} \left[y_t(h_T) - c_t(h_t(h_T)) \right] \geq 0, \qquad \forall h_T.$$

The proposition says that the consumption allocation that solves (P1) is the same as that in an economy where each household can borrow or lend outside the village at the risk-free gross interest rate R subject to a solvency requirement.[9] Below we will provide a proof for the case of two periods ($T = 2$). We refer readers to Cole and Kocherlakota (2001) for a general proof.

Central to the proof are the first-order conditions of problem (P2), namely,

$$u'(c_t(h_t)) = \beta R \sum_{s=1}^{S} \Pi_s u'\left(c_{t+1}(\{h_t, \overline{y}_s\})\right), \quad \forall t, h_t \tag{20.6.8}$$

$$\sum_{t=1}^{T} R^{1-t} \left[y_t(h_T) - c_t(h_t(h_T)) \right] = 0, \quad \forall h_T. \tag{20.6.9}$$

[9] The solvency requirement is equivalent to the *natural debt limit* discussed in chapters 17 and 18.

Given the continuous, strictly concave objective function and the compact, convex constraint set in problem (P2), the solution c^* is unique and the first-order conditions are both necessary and sufficient.

In the efficient allocation, the planner chooses transfers that in effect relax the nonnegativity constraint on a household's storage is not binding, i.e., consumption smoothing condition (20.6.8) is satisfied. However, the optimal transfer scheme offers no insurance across households because the present value of transfers is zero for any history h_T, i.e., the net-present value condition (20.6.9) is satisfied.

20.6.4. The case of two periods ($T = 2$)

In a finite horizon model, an immediate implication of the incentive constraints is that transfers in the final period T must be independent of households' reported values of y_T. In the case of two periods, we can therefore encode permissible transfer schemes as

$$
\begin{aligned}
b_1(\overline{y}_s) &= b_s, \quad \forall s \in \mathbf{S}, \\
b_2(\{\overline{y}_s, \overline{y}_j\}) &= e_s, \quad \forall s, j \in \mathbf{S},
\end{aligned}
$$

where b_s and e_s denote the transfer in the first and second period, respectively, when the household reports income $\overline{y}_s$ in the first period and income $\overline{y}_j$ in the second period.

Following Cole and Kocherlakota (2001), we will first characterize the solution to the modified planner's problem (P3) stated below. It has the same objective function as (P1) but a larger constraint set. In particular, we enlarge the constraint set by considering a smaller set of reporting strategies for the households, Ω_R^2. A household strategy $(\hat{y}, \hat{k})$ is an element of Ω_R^2 if

$$
\begin{aligned}
&\hat{y}_1(\overline{y}_s) \in \{\overline{y}_{s-1}, \overline{y}_s\}, \qquad \text{for } s = 2, 3, \ldots, S \\
&\hat{y}_1(\overline{y}_1) = \overline{y}_1.
\end{aligned}
$$

That is, a household can either tell the truth or lie downward by one notch in the grid of possible income realizations. There is no restriction on possible storage strategies.

Given $T = 2$, we state problem (P3) as follows. Choose $\{b_s, e_s\}_{s=1}^{S}$ to maximize

$$\text{(P3)} \qquad \sum_{s=1}^{S} \Pi_s \left[u(\overline{y}_s + b_s) + \beta \sum_{j=1}^{S} \Pi_j u(\overline{y}_j + e_s) \right]$$

subject to

$$\begin{aligned}
&\Gamma(y, 0; b) = \max_{(\hat{y},\hat{k}) \in \Omega_R^2} \Gamma(\hat{y}, \hat{k}; b) \\
&c_t(h_t) = y_t(h_t) + b_t(h_t), \qquad \forall t, h_t \\
&k_t(h_t) = 0, \qquad \forall t, h_t \\
&K_t + \sum_{h_t} \pi(h_t) b_t(h_t) \leq R K_{t-1}, \qquad \forall t \\
&K_2 \geq 0, \\
&\text{given } K_0 = k_0 = 0.
\end{aligned}$$

Beyond the restricted strategy space Ω_R^2, problem (P3) differs from (P1) in considering only allocations that have zero private storage. But by Proposition 1, we know that this is an innocuous restriction that does not affect the maximized value of the objective.

Here it is useful to explain why we are first studying the contrived problem (P3) rather than turning immediately to the real problem (P1). Certainly problem (P3) is easier to solve because we are exogenously restricting the households' reporting strategies to either telling the truth or making one specific lie. But how can knowledge of the solution to problem (P3) help us understand problem (P1)? Well, suppose it happens that problem (P3) has a unique solution equal to the optimal consumption allocation c^* from Proposition 2 (which will in fact turn out to be true). In that case, it follows that c^* is also the solution to problem (P1) because of the following argument. First, it is straightforward to verify that c^* is incentive compatible with respect to the unrestricted set Ω^2 of reporting strategies. Second, given that no better allocation than c^* can be supported with the restricted set Ω_R^2 of reporting strategies (telling the truth or making one specific lie), it is impossible that we can attain better outcomes by merely introducing additional ways of lying.

Let us therefore first study problem (P3). In particular, using a proof by contradiction, we now show that any allocation $(c, 0, b, K)$ that solves problem (P3) must satisfy three conditions:[10]

(i) The aggregate resource constraint (20.6.1) holds with equality in both periods and $K_2 = 0$;

(ii) $u'(c_1(\overline{y}_s)) = \beta R \sum_{j=1}^{S} \Pi_j u'\left(c_2(\{\overline{y}_s, \overline{y}_j\})\right), \quad \forall s;$

(iii) $b_s + R^{-1} e_s = 0, \quad \forall s.$

Condition (i) is easy to establish given the restricted strategy space Ω_R^2. Suppose that condition (i) is violated and hence, some aggregate resources have not been transferred to the households. In that case, the planner should store all unused resources until period 2 and give them to any household who reported the highest income in period 1. Given strategy space Ω_R^2, households are only allowed to lie downward so the proposed allocation cannot violate the incentive constraints for truthful reporting. Also, transferring more consumption in the last period will not lead to any private storage. We conclude that condition (i) must hold for any solution to problem (P3).

Next, suppose that condition (ii) is violated, i.e., for some $i \in \mathbf{S}$,

$$u'(c_1(\overline{y}_i)) > \beta R \sum_{s=1}^{S} \Pi_s u'\left(c_2(\{\overline{y}_i, \overline{y}_s\})\right). \tag{20.6.10}$$

(The reverse inequality is obviously inconsistent with the incentive constraints since households are free to store goods between periods.) We can then construct an alternative incentive feasible allocation that yields higher *ex ante* utility as follows. Set $K_1^o = K_1 - \epsilon \Pi_i$, $b_i^o = b_i + \epsilon$, $e_i^o = e_i - \delta$, and choose (ϵ, δ) such that

$$\begin{aligned} u(\overline{y}_i + b_i + \epsilon) &+ \beta \sum_{s=1}^{S} \Pi_s u\left(\overline{y}_s + e_i - \delta\right) \\ &= u(\overline{y}_i + b_i) + \beta \sum_{s=1}^{S} \Pi_s u\left(\overline{y}_s + e_i\right), \end{aligned} \tag{20.6.11}$$

[10] The proof by contradiction goes as follows. Suppose that an allocation $(c, 0, b, K)$ solves problem (P3) but violates one of our conditions. Then we can show either that $(c, 0, b, K)$ cannot be incentive feasible with respect to (P3) or that there exists another incentive feasible allocation $(c^o, 0, b^o, K^o)$ that yields an even higher *ex ante* utility than $(c, 0, b, K)$.

$$u'(\overline{y}_i + b_i + \epsilon) \geq \beta R \sum_{s=1}^{S} \Pi_s u'\left(\overline{y}_s + e_i - \delta\right). \tag{20.6.12}$$

By the envelope condition, (20.6.10) implies that $\delta > R\epsilon$, so this alternative allocation frees up resources that can be used to improve *ex ante* utility. But we have to check that the incentive constraints are respected. Concerning households experiencing $\overline{y}_i$, the proposed allocation is clearly incentive compatible, since their payoffs from reporting truthfully or lying are unchanged, and condition (20.6.12) ensures that they are not deviating from zero private storage. It remains to be checked that households with the next higher income shock $\overline{y}_{i+1}$ would not like to lie downward. This is also true, since a household with a higher income $\overline{y}_{i+1}$ would not like to accept the proposed loan against the future at the implied interest rate, $\delta/\epsilon > R$, at which the lower-income household is indifferent to the transaction. The following lemma shows this formally.

LEMMA: Let ϵ, $\delta > 0$ satisfy $\delta > R\epsilon$, and define

$$Z(m) \equiv \max_{k\geq 0}\Big[u(m-k) + \beta E_y u(y + Rk)\Big]$$

$$W(m) \equiv \max_{k\geq 0}\Big[u(m-k+\epsilon) + \beta E_y u(y + Rk - \delta)\Big],$$

where u is a strictly concave function and the expectation E_y is taken with respect to a random second-period income y. If $Z(m_a) = W(m_a)$ and $m_b > m_a$, then $Z(m_b) > W(m_b)$.

PROOF: Let the unique, weakly increasing sequence of maximizers of the savings problems Z and W be denoted $k_Z(m)$ and $k_W(m)$, respectively, which are guaranteed to exist by the strict concavity of u. The proof of the lemma proceeds by contradiction. Suppose that $Z(m_b) \leq W(m_b)$. Then by the mean value theorem, there exists $m_c \in (m_a, m_b)$ such that $Z'(m_c) \leq W'(m_c)$. This implies that

$$u'(m_c - k_Z(m_c)) \leq u'(m_c - k_W(m_c) + \epsilon).$$

The concavity of u implies that

$$0 \leq k_Z(m_c) \leq k_W(m_c) - \epsilon.$$

The weak monotonicity of k_W implies that $k_W(m_b) \geq k_W(m_c)$, so we know that $0 \leq k_W(m_b) - \epsilon$ and we can write

$$\begin{aligned} Z(m_b) &\geq u(m_b - k_W(m_b) + \epsilon) + \beta E_y u(y + Rk_w(m_b) - R\epsilon) \\ &> u(m_b - k_W(m_b) + \epsilon) + \beta E_y u(y + Rk_w(m_b) - \delta) = W(m_b), \end{aligned}$$

which is a contradiction. ∎

Finally, suppose that condition (iii) is violated, i.e., for some $i \in \mathbf{S}$,

$$\Psi_s \equiv b_s + R^{-1}e_s \neq b_{s-1} + R^{-1}e_{s-1} \equiv \Psi_{s-1}.$$

First, we can rule out $\Psi_s < \Psi_{s-1}$ because it would compel households with income shock $\overline{y}_s$ in the first period to lie downward. This is so because our condition (ii) implies that the nonnegative storage constraint binds for neither these households nor the households with the lower income shock $\overline{y}_{s-1}$. Hence, households with income shock $\overline{y}_s$ will only report truthfully if $Z(\overline{y}_s + \Psi_s) \geq Z(\overline{y}_s + \Psi_{s-1})$, where $Z(\cdot)$ is the value of the first savings problem defined in the lemma above. Thus, we conclude that $\Psi_s \geq \Psi_{s-1}$.

Second, we can rule out $\Psi_s > \Psi_{s-1}$ by constructing an alternative incentive feasible allocation that attains a higher *ex ante* utility. Compute the certainty equivalent $\tilde{\Psi}$ such that

$$\Pi_s Z(\overline{y}_s + \tilde{\Psi}) + \Pi_{s-1} Z(\overline{y}_{s-1} + \tilde{\Psi}) = \Pi_s Z(\overline{y}_s + \Psi_s) + \Pi_{s-1} Z(\overline{y}_{s-1} + \Psi_{s-1}).$$

Then change the transfer scheme so that households reporting $\overline{y}_s$ or $\overline{y}_{s-1}$ get the same present value of transfers equal to $\tilde{\Psi}$. Because of the strict concavity of the utility function, the new scheme frees up resources that can be used to improve *ex ante* utility. Also, the new scheme does not violate any incentive constraints. Households with income shock $\overline{y}_{s-1}$ are now better off when reporting truthfully, households with income shock $\overline{y}_s$ are indifferent to telling the truth, and households with income shock $\overline{y}_{s+1}$ will not lie because the present value of the transfers associated with lying has gone down. Since the planner satisfies the aggregate resource constraint at equality in our condition (i), we conclude that all households receive the same present value of transfers equal to zero.

By establishing conditions (i)–(iii), we have in effect shown that any solution to (P3) must satisfy equations (20.6.8) and (20.6.9). Thus, problem (P3) has a unique solution $(c^*, 0, b^*, K^*)$, where c^* is given by Proposition 2 and

$$\begin{aligned} b_t^*(h_t) &= c_t^*(h_t) - y_t(h_t), \\ K_t^* &= -\sum_{h_t} \pi(h_t) \sum_{j=1}^{t} R^{t-1} b_j^*(h_j(h_t)). \end{aligned}$$

Moreover, $(c^*, 0, b^*, K^*)$ is incentive compatible with respect to the unrestricted strategy set Ω^2. If a household tells the truth, its consumption is optimally smoothed. Hence, households weakly prefer to tell the truth and not store.

The proof of Proposition 2 for $T = 2$ is completed by noting that by construction, if some allocation $(c^*, 0, b^*, K^*)$ solves (P3), and $(c^*, 0, b^*, K^*)$ is incentive compatible with respect to Ω^2, then $(c^*, 0, b^*, K^*)$ solves (P1). Also, since equations (20.6.8) and (20.6.9) fully characterize the consumption allocation c^*, we have uniqueness with respect to c^* (but there exists a multitude of storage and transfer schemes that the planner can use to implement c^* in problem (P1)).

20.6.5. Role of the planner

Proposition 2 states that any allocation (c, k, b, K) that solves the planner's problem (P1) has the same consumption outcome $c = c^*$ as the solution to (P2), i.e., the market outcome when each household can lend *or* borrow at the risk-free interest rate R. This result has both positive and negative messages about the role of the planner. Because households have access only to a storage technology, the planner implements the efficient allocation by designing an elaborate transfer scheme that effectively undoes each household's nonnegativity constraint on storage while respecting solvency requirements. In this sense, the planner has an important role to play. However, the optimal transfer scheme offers no insurance across households and implements only a self-insurance scheme tantamount to a borrowing-and-lending outcome for each household. Thus, the planner's accomplishments as an insurance provider are very limited.

If we had assumed that households themselves have direct access to the credit market outside of the village, it would follow immediately that the planner would be irrelevant, since the households could then implement the efficient allocation themselves. Allen (1985) first made this observation. Given any transfer scheme, he showed that all households would choose to report the income that yields the highest present value of transfers regardless of what the actual income is. In our setting where the planner has no resources of his own, we get the zero net present value condition for the stream of transfers to any individual household.

20.6.6. Decentralization in a closed economy

Suppose that consumption allocation c^* in Proposition 2 satisfies

$$\sum_{h_t} \pi(h_t) \sum_{j=1}^{t} R^{t-j} \left[y_j(h_t) - c_j^*(h_j(h_t)) \right] \geq 0, \quad \forall t. \tag{20.6.13}$$

That is, aggregate storage is nonnegative at all dates. It follows that the efficient allocation in Proposition 2 would then also be the solution to a closed system where the planner has no access to outside borrowing. Moreover, c^* can then be decentralized as the equilibrium outcome in an incomplete markets economy where households competitively trade consumption and risk-free one-period bonds that are available in zero net supply in each period. Here we are assuming complete enforcement so that households must pay off their debts in every state of the world, and they cannot end their lives in debt.

In the decentralized equilibrium, let $a_t(h_t)$ and $k_t^d(h_t)$ denote bond holdings and storage, respectively, of a household indexed by its history h_t. The gross interest rate on bonds between periods t and $t+1$ is denoted $1+r_t$. We claim that the efficient allocation $(c^*, 0, b^*, K^*)$ can be decentralized by recursively defining

$$r_t \equiv R - 1, \tag{20.6.14}$$

$$k_t^d(h_t) \equiv K_t^*, \tag{20.6.15}$$

$$a_t(h_t) \equiv y_t(h_t) - c_t^*(h_t) - K_t^* + RK_{t-1}^* + Ra_{t-1}(h_{t-1}), \tag{20.6.16}$$

with $a_0 = 0$. First, we verify that households are behaving optimally. Note that we have chosen the interest rate so that households are indifferent between lending and storing. Because we also know that the household's consumption is smoothed at c^*, we need only to check that households' budget constraints hold with equality. By substituting (20.6.15) into (20.6.16), we obtain the household's one-period budget constraint. The consolidation of all one-period budget constraints yields

$$\begin{aligned} a_T(h_T) = & - k_T^d(h_T) + \sum_{t=1}^{T} R^{T-t} \left[y_t(h_T) - c_t^*(h_t(h_T)) \right] \\ & + R^{T-1}(k_0^d + a_0) = 0 \end{aligned}$$

where the last equality is implied by $K_T^* = K_0 = a_0 = 0$ and (20.6.9). Second, we verify that the bond market clears by summing all households' one-period budget constraints,

$$\sum_{h_t} \pi(h_t) a_t(h_t) = \sum_{h_t} \pi(h_t) \Big[y_t(h_t) - c_t^*(h_t) - k_t^d(h_t) \\ + R k_{t-1}^d(h_{t-1}(h_t)) + R a_{t-1}(h_{t-1}(h_t)) \Big].$$

After invoking (20.6.15) and the fact that $b_t^*(h_t) = c_t^*(h_t) - y_t(h_t)$, we can rewrite this expression as

$$\begin{aligned} \sum_{h_t} \pi(h_t) a_t(h_t) = & - K_t^* + R K_{t-1}^* \\ & - \sum_{h_t} \pi(h_t) \Big[b_t^*(h_t) - R a_{t-1}(h_{t-1}(h_t)) \Big] \\ = & R \sum_{h_{t-1}} \pi(h_{t-1}) a_{t-1}(h_{t-1}) = 0 \,, \end{aligned}$$

where the second equality is implied by (20.6.1) holding with equality at the allocation $(c^*, 0, b^*, K^*)$, and the last equality follows from successive substitutions leading back to the initial condition $a_0 = 0$.

It is straightforward to make the reverse argument and show that if $1 + r_t = R$ for all t in our incomplete markets equilibrium, then the equilibrium consumption allocation is efficient and equal to c^*, as given in Proposition 2.

Cole and Kocherlakota note that seemingly ad hoc restrictions on the securities available for trade are consistent with the implementation of the efficient allocation in this setting, and they argue that their framework provides an explicit micro foundation for incomplete markets models such as Aiyagari's (1994) model that we studied in chapter 18.

20.7. Concluding remarks

The idea of using promised values as a state variable has made it possible to use dynamic programming to study problems with history dependence. In this chapter we have studied how using a promised value as a state variable helps to study optimal risk-sharing arrangements when there are incentive problems coming from limited enforcement or limited information. The next several chapters apply and extend this idea in other contexts. Chapter 21 discusses how to build a closed-economy, or general equilibrium, version of our model with imperfect enforcement. Chapter 22 discusses ways of designing unemployment insurance that optimally compromise between supplying insurance and providing incentives for unemployed workers to search diligently. Chapter 23 uses a continuation value as a state variable to encode a government's reputation. Chapter 25 discusses some models of contracts and government policies that have been applied to some enforcement problems in international trade.

A. Historical development

20.A.1. Spear and Srivastava

Spear and Srivastava (1987) introduced the following recursive formulation of an infinitely repeated, discounted repeated principal-agent problem: A *principal* owns a technology that produces output q_t at time t, where q_t is determined by a family of c.d.f.'s $F(q_t|a_t)$, and a_t is an action taken at the beginning of t by an *agent* who operates the technology. The principal has access to an outside loan market with constant risk-free gross interest rate β^{-1}. The agent has preferences over consumption streams ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, a_t).$$

The principal is risk neutral and offers a contract to the agent designed to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \{q_t - c_t\}$$

where c_t is the payment from the principal to the agent at t.

20.A.2. Timing

Let w denote the discounted utility promised to the agent at the beginning of the period. Given w, the principal selects three functions $a(w)$, $c(w,q)$, and $\tilde{w}(w,q)$ determining the current action $a_t = a(w_t)$, the current consumption $c = c(w_t, q_t)$, and a promised utility $w_{t+1} = \tilde{w}(w_t, q_t)$. The choice of the three functions $a(w)$, $c(w,q)$, and $\tilde{w}(w,q)$ must satisfy the following two sets of constraints:

$$w = \int \{u[c(w,q), a(w)] + \beta \tilde{w}(w,q)\} \; dF[q|a(w)] \tag{20.A.1}$$

and

$$\begin{aligned} &\int \{u[c(w,q), a(w)] + \beta \tilde{w}(w,q)\} \; dF[q|a(w)] \\ &\quad \geq \int \{u[c(w,q), \hat{a}] + \beta \tilde{w}(w,q)\} dF(q|\hat{a}), \qquad \forall\, \hat{a} \in A. \end{aligned} \tag{20.A.2}$$

Equation (20.A.1) requires the contract to deliver the promised level of discounted utility. Equation (20.A.2) is the *incentive compatibility* constraint requiring the agent to want to deliver the amount of effort called for in the contract. Let $v(w)$ be the value to the principal associated with promising discounted utility w to the agent. The principal's Bellman equation is

$$v(w) = \max_{a,c,\tilde{w}} \; \{q - c(w,q) + \beta \; v[\tilde{w}(w,q)]\} \; dF[q|a(w)] \tag{20.A.3}$$

where the maximization is over functions $a(w)$, $c(w,q)$, and $\tilde{w}(w,q)$ and is subject to the constraints (20.A.1) and (20.A.2). This value function $v(w)$ and the associated optimum policy functions are to be solved by iterating on the Bellman equation (20.A.3).

20.A.3. Use of lotteries

In various implementations of this approach, a difficulty can be that the constraint set fails to be convex as a consequence of the structure of the incentive constraints. This problem has been overcome by Phelan and Townsend (1991) by convexifying the constraint set through randomization. Thus, Phelan and Townsend simplify the problem by extending the principal's choice to the space of lotteries over actions a and outcomes c, w'. To introduce Phelan and Townsend's formulation, let $P(q|a)$ be a family of discrete probability distributions over discrete spaces of outputs and actions Q, A, and imagine that consumption and values are also constrained to lie in discrete spaces C, W, respectively. Phelan and Townsend instruct the principal to choose a probability distribution $\Pi(a, q, c, w')$ subject first to the constraint that for all fixed $(\bar{a}, \bar{q})$

$$\sum_{C\times W} \Pi(\bar{a}, \bar{q}, c, w') = P(\bar{q}|\bar{a}) \sum_{Q\times C\times W} \Pi(\bar{a}, q, c, w') \tag{20.A.4a}$$

$$\Pi(a, q, c, w') \geq 0 \tag{20.A.4b}$$

$$\sum_{A\times Q\times C\times W} \Pi(a, q, c, w') = 1. \tag{20.A.4c}$$

Equation $(20.A.4a)$ simply states that $\text{Prob}(\bar{a}, \bar{q}) = \text{Prob}(\bar{q}|\bar{a})\text{Prob}(\bar{a})$. The remaining pieces of $(20.A.4)$ just require that "probabilities are probabilities." The counterpart of Spear-Srivastava's equation $(20.A.1)$ is

$$w = \sum_{A\times Q\times C\times W} \{u(c, a) + \beta w'\}\, \Pi(a, q, c, w'). \tag{20.A.5}$$

The counterpart to Spear-Srivastava's equation $(20.A.2)$ for each $a, \hat{a}$ is

$$\begin{aligned} &\sum_{Q\times C\times W} \{u(c, a) + \beta w'\}\, \Pi(c, w'|q, a) P(q|a) \\ &\qquad \geq \sum_{Q\times C\times W} \{u(c, \hat{a}) + \beta w'\}\, \Pi(c, w'|q, a) P(q|\hat{a}). \end{aligned}$$

Here $\Pi(c, w'|q, a)P(q|\hat{a})$ is the probability of (c, w', q) if the agent claims to be working a but is actually working $\hat{a}$. Express

$$\begin{aligned} &\Pi(c, w'|q, a)P(q|\hat{a}) = \\ &\Pi(c, w'|q, a)P(q|a)\, \frac{P(q|\hat{a})}{P(q|a)} = \Pi(c, w', q|a) \cdot \frac{P(q|\hat{a})}{P(q|a)}. \end{aligned}$$

To write the incentive constraint as

$$\sum_{Q\times C\times W} \{u(c,a)+\beta w'\}\Pi(c,w',q|a)$$
$$\geq \sum_{Q\times C\times W} \{u(c,\hat{a})+\beta w'\}\ \Pi(c,w',q|\hat{a})\ \cdot\ \frac{P(q|\hat{a})}{P(q|a)}.$$

Multiplying both sides by the unconditional probability $P(a)$ gives expression (20.A.6).

$$\sum_{Q\times C\times W} \{u(c,a)+\beta w'\}\ \Pi(a,q,c,w')$$
$$\geq \sum_{Q\times C\times W} \{u(c,\hat{a})+\beta w'\}\ \frac{P(q|\hat{a})}{P(q|a)}\ \Pi(a,q,c,w') \tag{20.A.6}$$

The Bellman equation for the principal's problem is

$$v(w)=\max_{\Pi}\{(q-c)+\beta v(w')\}\Pi(a,q,c,w'), \tag{20.A.7}$$

where the maximization is over the probabilities $\Pi(a,q,c,w')$ subject to equations (20.A.4), (20.A.5), and (20.A.6). The problem on the right side of equation (20.A.7) is a linear programming problem. Think of each of (a,q,c,w') being constrained to a discrete grid of points. Then, for example, the term $(q-c)+\beta v(w')$ on the right side of equation (20.A.7) can be represented as a *fixed* vector that multiplies a vectorized version of the probabilities $\Pi(a,q,c,w')$. Similarly, each of the constraints (20.A.4), (20.A.5), and (20.A.6) can be represented as a linear inequality in the choice variables, the probabilities Π. Phelan and Townsend compute solutions of these linear programs to iterate on the Bellman equation (20.A.7). Note that at each step of the iteration on the Bellman equation, there is one linear program to be solved for each point w in the space of grid values for W.

In practice, Phelan and Townsend have found that lotteries are often redundant in the sense that most of the $\Pi(a,q,c,w')$'s are zero, and a few are 1.

Exercises

Exercise 20.1 **Thomas and Worrall meet Markov**

A household orders sequences $\{c_t\}_{t=0}^{\infty}$ by

$$E \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad \beta \in (0,1)$$

where u is strictly increasing, twice continuously differentiable, and strictly concave with $u'(0) = +\infty$. The good is nondurable. The household receives an endowment of the consumption good of y_t that obeys a discrete-state Markov chain with $P_{ij} = \text{Prob}(y_{t+1} = \overline{y}_j | y_t = \overline{y}_i)$, where the endowment y_t can take one of the I values $[\overline{y}_1, \ldots, \overline{y}_I]$.

a. Conditional on having observed the time t value of the household's endowment, a social insurer wants to deliver expected discounted utility v to the household in the least costly way. The insurer observes y_t at the beginning of every period, and contingent on the observed history of those endowments, can make a transfer τ_t to the household. The transfer can be positive or negative and can be enforced without cost. Let $C(v,i)$ be the minimum expected discounted cost to the insurance agency of delivering promised discounted utility v when the household has just received endowment $\overline{y}_i$. (Let the insurer discount with factor β.) Write a Bellman equation for $C(v,i)$.

b. Characterize the consumption plan and the transfer plan that attains $C(v,i)$; find an associated law of motion for promised discounted value.

c. Now assume that the household is isolated and has no access to insurance. Let $v^a(i)$ be the expected discounted value of utility for a household in autarky, conditional on current income being $\overline{y}_i$. Formulate Bellman equations for $v^a(i), i = 1, \ldots, I$.

d. Now return to the problem of the insurer mentioned in part b, but assume that the insurer cannot enforce transfers because each period the consumer is free to walk away from the insurer and live in autarky thereafter. The insurer must structure a history-dependent transfer scheme that prevents the household from ever exercising the option to revert to autarky. Again, let $C(v,i)$ be the minimum cost for an insurer that wants to deliver promised discounted utility v to a household with current endowment i. Formulate Bellman equations for $C(v,i), i = 1,\ldots,I$. Briefly discuss the form of the law of motion for v associated with the minimum cost insurance scheme.

Exercise 20.2 **Wealth dynamics in moneylender model**

Consider the model in the text of the village with a moneylender. The village consists of a large number (e.g., a continuum) of households, each of which has an i.i.d. endowment process that is distributed as

$$\text{Prob}(y_t = \overline{y}_s) = \frac{1-\lambda}{1-\lambda^S}\lambda^{s-1}$$

where $\lambda \in (0,1)$ and $\overline{y}_s = s+5$ is the sth possible endowment value, $s = 1,\ldots,S$. Let $\beta \in (0,1)$ be the discount factor and β^{-1} the gross rate of return at which the moneylender can borrow or lend. The typical household's one-period utility function is $u(c) = (1-\gamma)^{-1}c^{1-\gamma}$, where γ is the household's coefficient of relative risk aversion. Assume the parameter values $(\beta, S, \gamma, \lambda) = (.5, 20, 2, .95)$. **Tom XXXX: I changed β from .95 to .5 following Isaac's advice. Double check my matlab programs to verify that $\beta = .5$ in the programs.**

Hint: The formulas given in the section 20.3.3 will be helpful in answering the following questions.

a. Using Matlab, compute the optimal contract that the moneylender offers a villager, assuming that the contract leaves the villager indifferent between refusing and accepting the contract.

b. Compute the expected profits that the moneylender earns by offering this contract for an initial discounted utility that equals the one that the household would receive in autarky.

c. Let the cross-section distribution of consumption at time $t \geq 0$ be given by the c.d.f. $\text{Prob}(\mathrm{c_t} \leq \overline{\mathrm{C}}) = F_t(\overline{C})$. Compute F_t. Plot it for $t = 0$, $t = 5$, $t = 10$, $t = 500$.

d. Compute the moneylender's savings for $t \geq 0$ and plot it for $t = 0,\ldots,100$.

e. Now adapt your program to find the initial level of promised utility $v > v_{\text{aut}}$ that would set $P(v) = 0$.

Exercise 20.3 **Thomas and Worrall (1988)**

There is a competitive spot market for labor always available to each of a continuum of workers. Each worker is endowed with one unit of labor each period that he supplies inelastically to work either permanently for "the company" or each period in a new one-period job in the spot labor market. The worker's productivity in either the spot labor market or with the company is an i.i.d. endowment process that is distributed as

$$\text{Prob}(w_t = \overline{w}_s) = \frac{1-\lambda}{1-\lambda^S}\lambda^{s-1}$$

where $\lambda \in (0,1)$ and $\overline{w}_s = s+5$ is the sth possible marginal product realization, $s = 1,\dots,S$. In the spot market, the worker is paid w_t. In the company, the worker is offered a history-dependent payment $\omega_t = f_t(h_t)$ where $h_t = w_t,\dots,w_0$. Let $\beta \in (0,1)$ be the discount factor and β^{-1} the gross rate of return at which the company can borrow or lend. The worker cannot borrow or lend. The worker's one-period utility function is $u(\omega) = (1-\gamma)^{-1}w^{1-\gamma}$ where ω is the period wage from the company, which equals consumption, and γ is the worker's coefficient of relative risk aversion. Assume the parameter values $(\beta, S, \gamma, \lambda) = (.5, 20, 2, .95)$. **Tom XXXX: again, I changed β from .95 to .5, following Isaac's advice. Check matlab programs.**

The company's discounted expected profits are

$$E\sum_{t=0}^{\infty}\beta^t\left(w_t - \omega_t\right).$$

The worker is free to walk away from the company at the start of any period, but must then stay in the spot labor market forever. In the spot labor market, the worker receives continuation value

$$v_{\text{spot}} = \frac{Eu(w)}{1-\beta}.$$

The company designs a history-dependent compensation contract that must be sustainable (i.e., self-enforcing) in the face of the worker's freedom to enter the spot labor market at the beginning of period t *after* he has observed w_t but before he receives the t period wage.

Hint: Do these questions ring a bell? See exercise *20.2*.

a. Using Matlab, compute the optimal contract that the company offers the worker, assuming that the contract leaves the worker indifferent between refusing and accepting the contract.

b. Compute the expected profits that the firm earns by offering this contract for an initial discounted utility that equals the one that the worker would receive by remaining forever in the spot market.

c. Let the distribution of wages that the firm offers to its workers at time $t \geq 0$ be given by the c.d.f. $\text{Prob}(\omega_\text{t} \leq \overline{\text{w}}) = F_t(\overline{w})$. Compute F_t. Plot it for $t = 0$, $t = 5$, $t = 10$, $t = 500$.

d. Plot an expected wage-tenure profile for a new worker.

e. Now assume that there is competition among companies and free entry. New companies enter by competing for workers by raising initial promised utility with the company. Adapt your program to find the initial level of promised utility $v > v_{\text{spot}}$ that would set expected profits from the average worker $P(v) = 0$.

Exercise 20.4 **Thomas-Worrall meet Phelan-Townsend**

Consider the Thomas Worrall environment and denote $\Pi(y)$ the density of the i.i.d. endowment process, where y belongs to the discrete set of endowment levels $Y = [\overline{y}_1, \ldots, \overline{y}_S]$. The one-period utility function is $u(c) = (1-\gamma)^{-1}(c-a)^{1-\gamma}$ where $\gamma > 1$ and $\overline{y}_S > a > 0$.

Discretize the set of transfers B and the set of continuation values W. We assume that the discrete set $B \subset (a - \overline{y}_S, \overline{b}]$. Notice that with the one-period utility function above, the planner could never extract more than $a - \overline{y}_S$ from the agent. Denote $\Pi^v(b, w|y)$ the joint density over (b, w) that the planner offers the agent who reports y and to whom he has offered beginning-of-period promised value v. For each $y \in Y$ and each $v \in W$, the planner chooses a set of conditional probabilities $\Pi^v(b, w|y)$ to satisfy the Bellman equation

$$P(v) = \max_{\Pi^v(b,w,y)} \sum_{B \times W \times Y} [-b + \beta P(w)]\, \Pi^v(b, w, y) \tag{1}$$

subject to the following constraints:

$$v = \sum_{B \times W \times Y} [u(y+b) + \beta w]\, \Pi^v(b, w, y) \tag{2}$$

$$\sum_{B \times W} [u(y+b) + \beta w]\, \Pi^v(b, w|y) \geq \sum_{B \times W} [u(y+b) + \beta w]\, \Pi^v(b, w|\tilde{y}) \qquad \forall (y, \tilde{y}) \in Y \times Y \tag{3}$$

$$\Pi^v(b, w, y) = \Pi(y)\Pi^v(b, w|y) \qquad \forall (b, w, y) \in B \times W \times Y \tag{4}$$

$$\sum_{B \times W \times Y} \Pi^v(b, w, y) = 1. \tag{5}$$

Here (2) is the promise-keeping constraint, (3) are the truth-telling constraints, and (4), (5) are restrictions imposed by the laws of probability.

a. Verify that given $P(w)$, one step on the Bellman equation is a linear programming problem.

b. Set $\beta = .94, a = 5, \gamma = 3$. Let S, N_B, N_W be the number of points in the grids for Y, B, W, respectively. Set $S = 10$, $N_B = N_W = 25$. Set $Y = [6 \quad 7 \quad \ldots 15]$, $\text{Prob}(y_t = \overline{y}_s) = S^{-1}$. Set $W = [w_{\min}, \ldots, w_{\max}]$ and $B = [b_{\min}, \ldots, b_{\max}]$, where the intermediate points in W and B, respectively, are equally spaced. Please set $w_{\min} = \frac{1}{1-\beta}\frac{1}{1-\gamma}(y_{\min} - a)^{1-\gamma}$ and $w_{\max} = w_{\min}/20$ (these are negative numbers, so $w_{\min} < w_{\max}$). Also set $b_{\min} = (1 - y_{\max} + .33)$ and $b_{\max} = y_{\max} - y_{\min}$.

For these parameter values, compute the optimal contract by formulating a linear program for one step on the Bellman equation, then iterating to convergence on it.

c. Notice the following probability laws:

$$\begin{aligned}
\text{Prob}(b_t, w_{t+1}, y_t | w_t) &\equiv \Pi^{w_t}(b_t, w_{t+1}, y_t) \\
\text{Prob}(w_{t+1} | w_t) &= \sum_{b \in B, y \in Y} \Pi^{w_t}(b, w_{t+1}, y) \\
\text{Prob}(b_t, y_t | w_t) &= \sum_{w_{t+1} \in W} \Pi^{w_t}(b_t, w_{t+1}, y_t).
\end{aligned}$$

Please use these and other probability laws to compute $\text{Prob}(w_{t+1}|w_t)$. Show how to compute $\text{Prob}(c_t)$, assuming a given initial promised value w_0.

d. Assume that $w_0 \approx -2$. Compute and plot $F_t(c) = \text{Prob}(c_t \leq c)$ for $t = 1, 5, 10, 100$. Qualitatively, how do these distributions compare with those for the simple village and moneylender model with no information problem and one-sided lack of commitment?

Exercise 20.5 **The IMF**

Consider the problem of a government of a small country that has to finance an exogenous stream of expenditures $\{g_t\}$. For time $t \geq 0$, g_t is i.i.d. with $\text{Prob}(g_t = \overline{g}_s) = \pi_s$ where $\pi_s > 0, \sum_{s=1}^{S} \pi_s = 1$ and $0 < \overline{g}_1 < \cdots < \overline{g}_S$. Raising revenues by taxation is distorting. In fact, the government confronts a deadweight loss function $W(T_t)$ that measures the distortion at time t. Assume that

W is an increasing, twice continuously differentiable, strictly convex function that satisfies $W(0) = 0, W'(0) = 0, W'(T) > 0$ for $T > 0$ and $W''(T) > 0$ for $T \geq 0$. The government's intertemporal loss function for taxes is such that it wants to minimize

$$E_{-1} \sum_{t=0}^{\infty} \beta^t W(T_t), \quad \beta \in (0,1)$$

where E_{-1} is the mathematical expectation before g_0 is realized. If it cannot borrow or lend, the government's budget constraint is $g_t = T_t$. In fact, the government is unable to borrow and lend *except* through an international coalition of lenders called the IMF. If it does not have an arrangement with the IMF, the country is in autarky and the government's loss is the value

$$v_{\text{aut}} = E \sum_{t=0}^{\infty} \beta^t W(g_t).$$

The IMF itself is able to borrow and lend at a constant risk-free gross rate of interest of $R = \beta^{-1}$. The IMF offers the country a contract that gives the country a net transfer of $g_t - T_t$. A *contract* is a sequence of functions for $t \geq 0$, the time t component of which maps the history g^t into a net transfer $g - T_t$. The IMF has the ability to commit to the contract. However, the country cannot commit to honor the contract. Instead, at the beginning of each period, after g_t has been realized but before the net transfer $g_t - T_t$ has been received, the government can default on the contract, in which case it receives loss $W(g_t)$ this period and the autarky value ever after. A contract is said to be *sustainable* if it is immune to the threat of repudiation, i.e., if it provides the country with the incentive not to leave the arrangement with the IMF. The present value of the contract to the IMF is

$$E \sum_{t=0}^{\infty} \beta^t (T_t - g_t).$$

a. Write a Bellman equation that can be used to find an optimal sustainable contract.

b. Characterize an optimal sustainable contract that delivers initial promised value v_{aut} to the country (i.e., a contract that renders the country indifferent between accepting and not accepting the IMF contract starting from autarky).

c. Can you say anything about a typical pattern of government tax collections T_t and distortions $W(T_t)$ over time for a country in an optimal sustainable contract with the IMF? What about the average pattern of government surpluses $T_t - g_t$ across a panel of countries with identical g_t processes and W functions? Would there be a "cohort" effect in such a panel (i.e., would the calendar date when the country signed up with the IMF matter)?

d. If the optimal sustainable contract gives the country value v_{aut}, can the IMF expect to earn anything from the contract?