Chapter 19
Dynamic Stackelberg Problems

19.1. History dependence

Previous chapters described decision problems that are recursive in what we can call "natural" state variables, i.e., state variables that describe stocks of capital, wealth, and information that helps forecast future values of prices and quantities that impinge on future utilities or profits. In problems that are recursive in the natural state variables, optimal decision rules are functions of the natural state variables.

This chapter is our first encounter with a class of problems that are not recursive in the natural state variables. Kydland and Prescott (1977), Prescott (1977), and Calvo (1978) gave macroeconomic examples of decision problems whose solutions exhibited *time inconsistency* because they are not recursive in the natural state variables. Those authors studied the decision problem of a large agent (a government) that confronts a competitive market composed of many small private agents whose decisions are influenced by their *forecasts* of the government's future actions. In such settings, the natural state variables of private agents at time t are partly shaped by past decisions that were influenced by their earlier forecasts of the government's action at time t. In a rational expectations equilibrium, the government on average confirms private agents' earlier forecasts of the government's time t actions. This requirement to confirm prior forecasts puts constraints on the government's time t decisions that prevent its problem from being recursive in natural state variables. These additional constraints make the government's decision rule at t depend on the entire history of the state from time 0 to time t.

It took some time for economists to figure out how to formulate policy problems of this type recursively. Prescott (1977) asserted that recursive optimal control theory does not apply to problems with this structure. This chapter and chapters 20 and 23 show how Prescott's pessimism about the inapplicability of optimal control theory has been overturned by more recent work.[1] An important finding is that if the natural state variables are augmented with additional state variables that measure costs in terms of the government's *current* continuation value of confirming *past* private sector expectations about its current behavior, this class of problems can be made recursive. This fact affords immense computational advantages and yields substantial insights. This chapter displays these within the tractable framework of linear quadratic problems.

19.2. The Stackelberg problem

To exhibit the essential structure of the problems that concerned Kydland and Prescott (1977) and Calvo (1979), this chapter uses the optimal linear regulator to solve a linear quadratic version of what is known as a dynamic Stackelberg problem.[2] For now we refer to the Stackelberg leader as the government and the Stackelberg follower as the representative agent or private sector. Soon we'll give an application with another interpretation of these two players.

Let z_t be an $n_z \times 1$ vector of natural state variables, x_t an $n_x \times 1$ vector of endogenous variables free to jump at t, and u_t a vector of government instruments. The z_t vector is inherited from the past. But x_t is *not* inherited from the past. The model determines the "jump variables" x_t at time t. Included in x_t are prices and quantities that adjust instantaneously to clear markets at time t. Let $y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}$. Define the government's one-period loss function[3]

$$r(y, u) = y' R y + u' Q u. \tag{19.2.1}$$

Subject to an initial condition for z_0, but not for x_0, a government wants to maximize

$$-\sum_{t=0}^{\infty} \beta^t r(y_t, u_t). \tag{19.2.2}$$

1 The important contribution by Kydland and Prescott (1980) helped to dissipate Prescott's initial pessimism.

2 In some settings it is called a Ramsey problem.

3 The problem assumes that there are no cross products between states and controls in the return function. There is a simple transformation that converts a problem whose return function has cross products into an equivalent problem that has no cross products. For example, see Hansen and Sargent (2008, chapter 4, pp. 72-73).

The government makes policy in light of the model

$$\begin{bmatrix} I & 0 \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \hat{B} u_t. \tag{19.2.3}$$

We assume that the matrix on the left is invertible, so that we can multiply both sides of the above equation by its inverse to obtain[4]

$$\begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + B u_t \tag{19.2.4}$$

or

$$y_{t+1} = A y_t + B u_t. \tag{19.2.5}$$

The government maximizes (19.2.2) by choosing sequences $\{u_t, x_t, z_{t+1}\}_{t=0}^{\infty}$ subject to (19.2.5) and the initial condition for z_0.

The private sector's behavior is summarized by the second block of equations of (19.2.3) or (19.2.4). These typically include the first-order conditions of private agents' optimization problem (i.e., their Euler equations). They summarize the forward-looking aspect of private agents' behavior. We shall provide an example later in this chapter in which, as is typical of these problems, the last n_x equations of (19.2.4) or (19.2.5) constitute *implementability constraints* that are formed by the Euler equations of a competitive fringe or private sector. When combined with a stability condition to be imposed below, these Euler equations summarize the private sector's best response to the sequence of actions by the government.

The certainty equivalence principle stated in chapter 5 allows us to work with a nonstochastic model. We would attain the same decision rule if we were to replace x_{t+1} with the forecast $E_t x_{t+1}$ and to add a shock process $C\epsilon_{t+1}$ to the right side of (19.2.4), where ϵ_{t+1} is an i.i.d. random vector with mean of zero and identity covariance matrix.

Let X^t denote the history of any variable X from 0 to t. Miller and Salmon (1982, 1985), Hansen, Epple, and Roberds (1985), Pearlman, Currie, and Levine (1986), Sargent (1987), Pearlman (1992), and others have all studied versions of the following problem:

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[4] We have assumed that the matrix on the left of (19.2.3) is invertible for ease of presentation. However, by appropriately using the invariant subspace methods described under step 2 below (see Appendix 19B), it is straightforward to adapt the computational method when this assumption is violated.

Problem S: The *Stackelberg problem* is to maximize (19.2.2) by choosing an x_0 and a sequence of decision rules, the time t component of which maps the time t history of the state z^t into the time t decision u_t of the Stackelberg leader. The Stackelberg leader commits to this sequence of decision rules at time 0. The maximization is subject to a given initial condition for z_0. But x_0 is among the objects to be chosen by the Stackelberg leader.

The optimal decision rule is history dependent, meaning that u_t depends not only on z_t but also on lags of z. History dependence has two sources: (a) the government's ability to commit[5] to a sequence of rules at time 0, and (b) the forward-looking behavior of the private sector embedded in the second block of equations (19.2.4). The history dependence of the government's plan is expressed in the dynamics of Lagrange multipliers μ_x on the last n_x equations of (19.2.3) or (19.2.4). These multipliers measure the costs today of honoring past government promises about current and future settings of u. It is appropriate to initialize the multipliers to zero at time $t = 0$, because then there are no past promises about u to honor. But the multipliers μ_x take nonzero values thereafter, reflecting future costs to the government of adhering to its commitment.

19.3. Solving the Stackelberg problem

This section describes a remarkable four-step algorithm for solving the Stackelberg problem.

[5] The government would make different choices were it to choose sequentially, that is, were it to select its time t action at time t.

19.3.1. Step 1: solve an optimal linear regulator

Step 1 seems to disregard the forward-looking aspect of the problem (step 3 will take account of that). If we temporarily ignore the fact that the x_0 component of the state $y_0 = \begin{bmatrix} z_0 \\ x_0 \end{bmatrix}$ is *not* actually a state vector, then superficially the Stackelberg problem (19.2.2), (19.2.5) has the form of an optimal linear regulator problem. It can be solved by forming a Bellman equation and iterating until it converges. The optimal value function has the form $v(y) = -y'Py$, where P satisfies the Riccati equation (19.3.5). The next steps note how the value function $v(y) = -y'Py$ encodes objects that solve the Stackelberg problem, then tell how to decode them.

A reader not wanting to be reminded of the details of the Bellman equation can now move directly to step 2. For those wanting a reminder, here it is. The linear regulator is

$$v(y_0) = -y_0'Py_0 = \max_{\{u_t, y_{t+1}\}_{t=0}^{\infty}} - \sum_{t=0}^{\infty} \beta^t \left(y_t'Ry_t + u_t'Qu_t \right) \tag{19.3.1}$$

where the maximization is subject to a fixed initial condition for y_0 and the law of motion[6]

$$y_{t+1} = Ay_t + Bu_t. \tag{19.3.2}$$

Associated with problem (19.3.1), (19.3.2) is the Bellman equation

$$-y'Py = \max_{u, y^*} \{-y'Ry - u'Qu - \beta y^{*\prime}Py^*\} \tag{19.3.3}$$

where the maximization is subject to

$$y^* = Ay + Bu \tag{19.3.4}$$

where y^* denotes next period's value of the state. Problem (19.3.3), (19.3.4) gives rise to the matrix Riccati equation

$$P = R + \beta A'PA - \beta^2 A'PB(Q + \beta B'PB)^{-1}B'PA \tag{19.3.5}$$

and the formula for F in the decision rule $u_t = -Fy_t$

$$F = \beta(Q + \beta B'PB)^{-1}B'PA. \tag{19.3.6}$$

6 In step 4, we acknowledge that the x_0 component is *not* given but is to be chosen by the Stackelberg leader.

Thus, we can solve problem (19.2.2), (19.2.5) by iterating to convergence on the difference equation counterpart to the algebraic Riccati equation (19.3.5), or by using a faster computational method that emerges as a by-product in step 2. This method is described in Appendix 19B.

19.3.2. Step 2: use the stabilizing properties of shadow price Py_t

At this point we decode the information in the matrix P in terms of shadow prices that are associated with a Lagrangian. We adapt a method described earlier in section 5.5 that solves a linear quadratic control problem of the form (19.2.2), (19.2.5) by attaching a sequence of Lagrange multipliers $2\beta^{t+1}\mu_{t+1}$ to the sequence of constraints (19.2.5) and then forming the Lagrangian:

$$\mathcal{L} = -\sum_{t=0}^{\infty} \beta^t \left[y_t' R y_t + u_t' Q u_t + 2\beta \mu_{t+1}' (A y_t + B u_t - y_{t+1}) \right]. \tag{19.3.7}$$

For the Stackelberg problem, it is important to partition μ_t conformably with our partition of $y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}$, so that $\mu_t = \begin{bmatrix} \mu_{zt} \\ \mu_{xt} \end{bmatrix}$, where μ_{xt} is an $n_x \times 1$ vector of multipliers adhering to the implementability constraints. For now, we can ignore the partitioning of μ_t, but it will be very important when we turn our attention to the specific requirements of the Stackelberg problem in step 3.

We want to maximize (19.3.7) with respect to sequences for u_t and y_{t+1}. The first-order conditions with respect to u_t, y_t, respectively, are:

$$0 = Q u_t + \beta B' \mu_{t+1} \tag{19.3.8a}$$

$$\mu_t = R y_t + \beta A' \mu_{t+1}. \tag{19.3.8b}$$

Solving (19.3.8a) for u_t and substituting into (19.2.5) gives

$$y_{t+1} = A y_t - \beta B Q^{-1} B' \mu_{t+1}. \tag{19.3.9}$$

We can represent the system formed by (19.3.9) and (19.3.8b) as

$$\begin{bmatrix} I & \beta B Q^{-1} B' \\ 0 & \beta A' \end{bmatrix} \begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -R & I \end{bmatrix} \begin{bmatrix} y_t \\ \mu_t \end{bmatrix} \tag{19.3.10}$$

or

$$L^* \begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix} = N \begin{bmatrix} y_t \\ \mu_t \end{bmatrix}. \tag{19.3.11}$$

We seek a "stabilizing" solution of (19.3.11), i.e., one that satisfies

$$\sum_{t=0}^{\infty} \beta^t y_t' y_t < +\infty.$$

19.3.3. Stabilizing solution

By the same argument used in section 5.5 of chapter 5, a stabilizing solution satisfies $\mu_0 = Py_0$, where P solves the matrix Riccati equation (19.3.5). The solution for μ_0 replicates itself over time in the sense that

$$\mu_t = Py_t. \tag{19.3.12}$$

Appendix 19A verifies that the matrix P that satisfies the Riccati equation (19.3.5) is the same P that defines the stabilizing initial conditions (y_0, Py_0). In Appendix 19B, we describe how to construct P by computing generalized eigenvalues and eigenvectors.

19.3.4. Step 3: convert implementation multipliers into state variables

19.3.4.1. Key insight

We now confront the fact that the x_0 component of y_0 consists of variables that are not state variables, i.e., they are not inherited from the past but are to be determined at time t. In the optimal linear regulator problem, y_0 is a state vector inherited from the past; the multiplier μ_0 jumps at t to satisfy $\mu_0 = Py_0$ and thereby stabilize the system. But in the Stackelberg problem, pertinent components of *both* y_0 *and* μ_0 must adjust to satisfy $\mu_0 = Py_0$. In particular, partition μ_t conformably with the partition of y_t into $[\, z_t' \quad x_t' \,]'$:[7]

$$\mu_t = \begin{bmatrix} \mu_{zt} \\ \mu_{xt} \end{bmatrix}.$$

For the Stackelberg problem, the first n_z elements of y_t are predetermined but the remaining components are free. And while the first n_z elements of μ_t are free to jump at t, the remaining components are not. The third step completes the solution of the Stackelberg problem by acknowledging these facts. *After* we have performed the key step of computing the matrix P that solves the Riccati equation (19.3.5), we convert the last n_x Lagrange multipliers μ_{xt} into state variables by using the following procedure

Write the last n_x equations of (19.3.12) as

$$\mu_{xt} = P_{21} z_t + P_{22} x_t, \tag{19.3.13}$$

where the partitioning of P is conformable with that of y_t into $[\, z_t \quad x_t \,]'$. The vector μ_{xt} becomes part of the state at t, while x_t is free to jump at t. Therefore, we solve (19.3.13) for x_t in terms of (z_t, μ_{xt}):

$$x_t = -P_{22}^{-1} P_{21} z_t + P_{22}^{-1} \mu_{xt}. \tag{19.3.14}$$

Then we can write

$$y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{19.3.15}$$

and from (19.3.13)

$$\mu_{xt} = [\, P_{21} \quad P_{22} \,]\, y_t. \tag{19.3.16}$$

With these modifications, the key formulas (19.3.6) and (19.3.5) from the optimal linear regulator for F and P, respectively, continue to apply. Using (19.3.15), the optimal decision rule is

$$u_t = -F \begin{bmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \tag{19.3.17}$$

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[7] This argument just adapts one in Pearlman (1992). The Lagrangian associated with the Stackelberg problem remains (19.3.7), which means that the same section 5.5 logic implies that the stabilizing solution must satisfy (19.3.12). It is only in how we impose (19.3.12) that the solution diverges from that for the linear regulator.

Then we have the following complete description of the Stackelberg plan:[8]

$$\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_{21} & P_{22} \end{bmatrix} (A - BF) \begin{bmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{19.3.19a}$$

$$x_t = [\,-P_{22}^{-1} P_{21} \quad P_{22}^{-1}\,] \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \tag{19.3.19b}$$

The difference equation (19.3.19a) is to be initialized from the given value of z_0 and a value for μ_{x0} to be determined in step 4.

19.3.5. Step 4: solve for x_0 and μ_{x0}

The value function $V(y_0)$ satisfies

$$V(y_0) = -z_0' P_{11} z_0 - 2x_0' P_{21} z_0 - x_0 P_{22} x_0. \tag{19.3.20}$$

Now choose x_0 by equating to zero the gradient of $V(y_0)$ with respect to x_0:

$$-2P_{21} z_t - 2P_{22} x_t = 0,$$

which by virtue of (19.3.13) is equivalent with

$$\mu_{x0} = 0. \tag{19.3.21}$$

Then we can compute x_0 from (19.3.14) to arrive at

$$x_0 = -P_{22}^{-1} P_{21} z_0. \tag{19.3.22}$$

Setting $\mu_{x0} = 0$ means that at time 0 there are no past promises to keep.

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[8] When a random shock $C\epsilon_{t+1}$ is present, we must add

$$\begin{bmatrix} I & 0 \\ P_{21} & P_{22} \end{bmatrix} C\epsilon_{t+1} \tag{19.3.18}$$

to the right side of (19.3.19a).

19.3.6. Summary

In summary, we solve the Stackelberg problem by formulating a particular optimal linear regulator, solving the associated matrix Riccati equation (19.3.5) for P, computing F, and then partitioning P to obtain representation (19.3.19).

19.3.7. History-dependent representation of decision rule

For some purposes, it is useful to eliminate the implementation multipliers μ_{xt} and to express the decision rule for u_t as a function of z_t, z_{t-1}, and u_{t-1}. This can be accomplished as follows.[9] First represent (19.3.19a) compactly as

$$\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{19.3.23}$$

and write the feedback rule for u_t

$$u_t = f_{11} z_t + f_{12} \mu_{xt}. \tag{19.3.24}$$

Then where f_{12}^{-1} denotes the generalized inverse of f_{12}, (19.3.24) implies $\mu_{x,t} = f_{12}^{-1}(u_t - f_{11} z_t)$. Equate the right side of this expression to the right side of the second line of (19.3.23) lagged once and rearrange by using (19.3.24) lagged once to eliminate $\mu_{x,t-1}$ to get

$$u_t = f_{12} m_{22} f_{12}^{-1} u_{t-1} + f_{11} z_t + f_{12}(m_{21} - m_{22} f_{12}^{-1} f_{11}) z_{t-1} \tag{19.3.25a}$$

or

$$u_t = \rho u_{t-1} + \alpha_0 z_t + \alpha_1 z_{t-1} \tag{19.3.25b}$$

for $t \geq 1$. For $t = 0$, the initialization $\mu_{x,0} = 0$ implies that

$$u_0 = f_{11} z_0. \tag{19.3.25c}$$

By making the instrument feed back on itself, the form of (19.3.25) potentially allows for "instrument-smoothing" to emerge as an optimal rule under commitment.[10]

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[9] Peter Von Zur Muehlen suggested this representation to us.

[10] This insight partly motivated Woodford (2003) to use his model to interpret empirical evidence about interest rate smoothing in the United States.

19.3.8. Digression on determinacy of equilibrium

Appendix 19B describes methods for solving a system of difference equations of the form (19.2.3) or (19.2.4) with an arbitrary feedback rule that expresses the decision rule for u_t as a function of current and previous values of y_t and perhaps previous values of itself. The difference equation system has a unique solution satisfying the stability condition $\sum_{t=0}^{\infty} \beta^t y_t \cdot y_t$ if the eigenvalues of the matrix (19.B.1) split, with half being greater than unity and half being less than unity in modulus. If more than half are less than unity in modulus, the equilibrium is said to be indeterminate in the sense that there are multiple equilibria starting from any initial condition.

If we choose to represent the solution of a Stackelberg or Ramsey problem in the form (19.3.25), we can substitute that representation for u_t into (19.2.4), obtain a difference equation system in y_t, u_t, and ask whether the resulting system is determinate. To answer this question, we would use the method of Appendix 19B, form system (19.B.1), then check whether the generalized eigenvalues split as required. Researchers have used this method to study the determinacy of equilibria under Stackelberg plans with representations like (19.3.25) and have discovered that sometimes an equilibrium can be indeterminate.[11] See Evans and Honkapohja (2003) for a discussion of determinacy of equilibria under commitment in a class of equilibrium monetary models and how determinacy depends on how the decision rule of the Stackelberg leader is represented. Evans and Honkapohja argue that casting a government decision rule in a way that leads to indeterminacy is a bad idea.

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11 The existence of a Stackelberg plan is not at issue because we know how to construct one using the method in the text.

19.4. A large firm with a competitive fringe

As an example, this section studies the equilibrium of an industry with a large firm that acts as a Stackelberg leader with respect to a competitive fringe. Sometimes the large firm is called 'the monopolist' even though there are actually many firms in the industry. The industry produces a single nonstorable homogeneous good. One large firm produces Q_t and a representative firm in a competitive fringe produces q_t. The representative firm in the competitive fringe acts as a price taker and chooses sequentially. The large firm commits to a policy at time 0, taking into account its ability to manipulate the price sequence, both directly through the effects of its quantity choices on prices, and indirectly through the responses of the competitive fringe to its forecasts of prices.[12]

The costs of production are $\mathcal{C}_t = eQ_t + .5gQ_t^2 + .5c(Q_{t+1} - Q_t)^2$ for the large firm and $\sigma_t = dq_t + .5hq_t^2 + .5c(q_{t+1} - q_t)^2$ for the competitive firm, where $d > 0, e > 0, c > 0, g > 0, h > 0$ are cost parameters. There is a linear inverse demand curve

$$p_t = A_0 - A_1(Q_t + \overline{q}_t) + v_t, \tag{19.4.1}$$

where A_0, A_1 are both positive and v_t is a disturbance to demand governed by

$$v_{t+1} = \rho v_t + C_\epsilon \check{\epsilon}_{t+1} \tag{19.4.2}$$

and where $|\rho| < 1$ and $\check{\epsilon}_{t+1}$ is an i.i.d. sequence of random variables with mean zero and variance 1. In $(19.4.1)$, $\overline{q}_t$ is equilibrium output of the representative competitive firm. In equilibrium, $\overline{q}_t = q_t$, but we must distinguish between q_t and $\overline{q}_t$ in posing the optimum problem of a competitive firm.

[12] Hansen and Sargent (2008, ch. 16) use this model as a laboratory to illustrate an equilibrium concept featuring robustness in which at least one of the agents has doubts about the stochastic specification of the demand shock process.

19.4.1. The competitive fringe

The representative competitive firm regards $\{p_t\}_{t=0}^{\infty}$ as an exogenous stochastic process and chooses an output plan to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ p_t q_t - \sigma_t \right\}, \quad \beta \in (0,1) \tag{19.4.3}$$

subject to q_0 given, where $c > 0, d > 0, h > 0$ are cost parameters, and E_t is the mathematical expectation based on time t information. Let $i_t = q_{t+1} - q_t$. We regard i_t as the representative firm's control at t. The first-order conditions for maximizing (19.4.3) are

$$i_t = E_t \beta i_{t+1} - c^{-1} \beta h q_{t+1} + c^{-1} \beta E_t (p_{t+1} - d) \tag{19.4.4}$$

for $t \geq 0$. We appeal to the certainty equivalence principle stated on page 131 to justify working with a non-stochastic version of (19.4.4) formed by dropping the expectation operator and the random term $\check{\epsilon}_{t+1}$ from (19.4.2). We use a method of Sargent (1979) and Townsend (1983).[13] We shift (19.4.1) forward one period, replace conditional expectations with realized values, use (19.4.1) to substitute for p_{t+1} in (19.4.4), and set $q_t = \overline{q}_t$ for all $t \geq 0$ to get

$$i_t = \beta i_{t+1} - c^{-1} \beta h \overline{q}_{t+1} + c^{-1} \beta (A_0 - d) - c^{-1} \beta A_1 \overline{q}_{t+1} - c^{-1} \beta A_1 Q_{t+1} + c^{-1} \beta v_{t+1}. \tag{19.4.5}$$

Given sufficiently stable sequences $\{Q_t, v_t\}$, we could solve (19.4.5) and $i_t = \overline{q}_{t+1} - \overline{q}_t$ to express the competitive fringe's output sequence as a function of the (tail of the) monopolist's output sequence. The dependence of i_t on future Q_t's opens an avenue for the monopolist to influence current outcomes by committing to future actions today. It is this feature that makes the monopolist's problem fail to be recursive in the natural state variables $\overline{q}, Q$. The monopolist arrives at period $t > 0$ facing the constraint that it must confirm the expectations about its time t decision upon which the competitive fringe based its decisions at dates before t.

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13 They used this method to compute a rational expectations competitive equilibrium. The key step was to eliminate price and output by substituting from the inverse demand curve and the production function into the firm's first-order conditions to get a difference equation in capital.

19.4.2. The monopolist's problem

The monopolist views the competitive firm's sequence of Euler equations as constraints on its own opportunities. They are *implementability constraints* on the monopolist's choices. Including (19.4.5), we can represent the constraints in terms of the transition law impinging on the monopolist:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ A_0 - d & 1 & -A_1 & -A_1 - h & c \end{bmatrix} \begin{bmatrix} 1 \\ v_{t+1} \\ Q_{t+1} \\ \overline{q}_{t+1} \\ i_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \frac{c}{\beta} \end{bmatrix} \begin{bmatrix} 1 \\ v_t \\ Q_t \\ \overline{q}_t \\ i_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_t, \tag{19.4.6}$$

where $u_t = Q_{t+1} - Q_t$ is the control of the monopolist. The last row portrays the implementability constraints (19.4.5). Represent (19.4.6) as

$$y_{t+1} = A y_t + B u_t. \tag{19.4.7}$$

Although we have entered i_t as a component of the "state" y_t in the monopolist's transition law (19.4.7), i_t is actually a "jump" variable. Nevertheless, the analysis in earlier sections of this chapter implies that the solution of the large firm's problem is encoded in the Riccati equation associated with (19.4.7) as the transition law. Let's decode it.

To match our general setup, we partition y_t as $y_t' = [\, z_t' \quad x_t' \,]$ where $z_t' = [\, 1 \quad v_t \quad Q_t \quad \overline{q}_t \,]$ and $x_t = i_t$. The large firm's problem is

$$\max_{\{u_t, p_t, Q_{t+1}, \overline{q}_{t+1}, i_t\}} \sum_{t=0}^{\infty} \beta^t \left\{ p_t Q_t - \mathcal{C}_t \right\}$$

subject to the given initial condition for z_0, equations (19.4.1) and (19.4.5) and $i_t = \overline{q}_{t+1} - \overline{q}_t$, as well as the laws of motion of the natural state variables z. Notice that the monopolist in effect chooses the price sequence, as well as the quantity sequence of the competitive fringe, albeit subject to the restrictions imposed by the behavior of consumers, as summarized by the demand curve (19.4.1), and the implementability constraint (19.4.5) that summarizes the best responses of the competitive fringe.

By substituting (19.4.1) into the above objective function, the monopolist's problem can be expressed as

$$\max_{\{u_t\}} \sum_{t=0}^{\infty} \beta^t \left\{ (A_0 - A_1(\overline{q}_t + Q_t) + v_t)Q_t - eQ_t - .5gQ_t^2 - .5cu_t^2 \right\} \tag{19.4.8}$$

subject to (19.4.7). This can be written

$$\max_{\{u_t\}} - \sum_{t=0}^{\infty} \beta^t \left\{ y_t' R y_t + u_t' Q u_t \right\} \tag{19.4.9}$$

subject to (19.4.7) where

$$R = - \begin{bmatrix} 0 & 0 & \frac{A_0 - e}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{A_0 - e}{2} & \frac{1}{2} & -A_1 - .5g & -\frac{A_1}{2} & 0 \\ 0 & 0 & -\frac{A_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $Q = \frac{c}{2}$.

19.4.3. Equilibrium representation

We can use (19.3.19) to represent the solution of the monopolist's problem (19.4.9) in the form:

$$\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{x,t} \end{bmatrix} \tag{19.4.10}$$

or

$$\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = m \begin{bmatrix} z_t \\ \mu_{x,t} \end{bmatrix}. \tag{19.4.11}$$

The monopolist is constrained to set $\mu_{x,0} \leq 0$, but will find it optimal to set it to zero. Recall that $z_t = [1 \quad v_t \quad Q_t \quad \overline{q}_t]'$. Thus, (19.4.11) includes the equilibrium law of motion for the quantity $\overline{q}_t$ of the competitive fringe. By construction, $\overline{q}_t$ satisfies the Euler equation of the representative firm in the competitive fringe, as we elaborate in Appendix 19C.

19.4.4. Numerical example

We computed the optimal Stackelberg plan for parameter settings $A_0, A_1, \rho, C_\epsilon$, $c, d, e, g, h, \beta = 100, 1, .8, .2, 1, 20, 20, .2, .2, .95$.[14] For these parameter values the decision rule is

$$u_t = (Q_{t+1} - Q_t) = \begin{bmatrix} 19.78 & .19 & -.64 & -.15 & -.30 \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{19.4.12}$$

which can also be represented as

$$u_t = 0.44 u_{t-1} + \begin{bmatrix} 19.7827 \\ 0.1885 \\ -0.6403 \\ -0.1510 \end{bmatrix}' z_t + \begin{bmatrix} -6.9509 \\ -0.0678 \\ 0.3030 \\ 0.0550 \end{bmatrix}' z_{t-1}. \tag{19.4.13}$$

Note how in representation (19.4.12) the monopolist's decision for $u_t = Q_{t+1} - Q_t$ feeds back negatively on the implementation multiplier.[15]

19.5. Concluding remarks

This chapter is our first brush with a class of problems in which optimal decision rules are history dependent. We shall confront many more such problems in chapters 20, 21, and 23 and shall see in various contexts how history dependence can be represented recursively by appropriately augmenting the natural state variables with counterparts to our implementability multipliers. A hint at what these counterparts are is gleaned by appropriately interpreting implementability multipliers as derivatives of value functions. In chapters 20,21, and 23, we make dynamic incentive and enforcement problems recursive by augmenting the state with continuation values of other decision makers.[16]

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[14] These calculations were performed by the Matlab program `oligopoly5.m`

[15] We also computed impulse responses to the demand innovation ϵ_t. The impulse responses show that a demand innovation pushes the implementation multiplier down and leads the monopolist to expand output while the representative competitive firm contracts output in subsequent periods. The response of price to a demand shock innovation is to rise on impact but then to decrease in subsequent periods in response to the increase in total supply $\overline{q} + Q$ engineered by the monopolist.

[16] In chapter 20, we describe Marcet and Marimon's (1992, 1999) method of constructing recursive contracts, which is closely related to the method that we have presented in this chapter.

A. The stabilizing $\mu_t = Py_t$

We verify that the P associated with the stabilizing $\mu_0 = Py_0$ satisfies the Riccati equation associated with the Bellman equation. Substituting $\mu_t = Py_t$ into (19.3.9) and (19.3.8b) gives

$$(I + \beta BQ^{-1}BP)y_{t+1} = Ay_t \tag{19.A.1a}$$

$$\beta A'Py_{t+1} = -Ry_t + Py_t. \tag{19.A.1b}$$

A matrix inversion identity implies

$$(I + \beta BQ^{-1}B'P)^{-1} = I - \beta B(Q + \beta B'PB)^{-1}B'P. \tag{19.A.2}$$

Solving (19.A.1a) for y_{t+1} gives

$$y_{t+1} = (A - BF)y_t \tag{19.A.3}$$

where

$$F = \beta(Q + \beta B'PB)^{-1}B'PA. \tag{19.A.4}$$

Premultiplying (19.A.3) by $\beta A'P$ gives

$$\beta A'Py_{t+1} = \beta(A'PA - A'PBF)y_t. \tag{19.A.5}$$

For the right side of (19.A.5) to agree with the right side of (19.A.1b) for any initial value of y_0 requires that

$$P = R + \beta A'PA - \beta^2 A'PB(Q + \beta B'PB)^{-1}B'PA. \tag{19.A.6}$$

Equation (19.A.6) is the algebraic matrix Riccati equation associated with the optimal linear regulator for the system A, B, Q, R.

B. Matrix linear difference equations

This appendix generalizes some calculations from chapter 5 for solving systems of linear difference equations. Returning to system (19.3.11), let $L = L^* \beta^{-.5}$ and transform the system (19.3.11) to

$$L \begin{bmatrix} y^*_{t+1} \\ \mu^*_{t+1} \end{bmatrix} = N \begin{bmatrix} y^*_t \\ \mu^*_t \end{bmatrix}, \tag{19.B.1}$$

where $y^*_t = \beta^{t/2} y_t, \mu^*_t = \mu_t \beta^{t/2}$. Now $\lambda L - N$ is a symplectic pencil,[17] so that the generalized eigenvalues of L, N occur in reciprocal pairs: if λ_i is an eigenvalue, then so is λ_i^{-1}.

We can use Evan Anderson's Matlab program `schurg.m` to find a stabilizing solution of system (19.B.1).[18] The program computes the ordered real generalized Schur decomposition of the matrix pencil. Thus, `schurg.m` computes matrices $\bar{L}, \bar{N}, V$ such that $\bar{L}$ is upper triangular, $\bar{N}$ is upper block triangular, and V is the matrix of right Schur vectors such that for some orthogonal matrix W, the following hold:

$$\begin{aligned} WLV &= \bar{L} \\ WNV &= \bar{N}. \end{aligned} \tag{19.B.2}$$

Let the stable eigenvalues (those less than 1) appear first. Then the stabilizing solution is

$$\mu^*_t = P y^*_t \tag{19.B.3}$$

where

$$P = V_{21} V_{11}^{-1},$$

V_{21} is the lower left block of V, and V_{11} is the upper left block.

If L is nonsingular, we can represent the solution of the system as[19]

$$\begin{bmatrix} y^*_{t+1} \\ \mu^*_{t+1} \end{bmatrix} = L^{-1} N \begin{bmatrix} I \\ P \end{bmatrix} y^*_t. \tag{19.B.4}$$

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17 A *pencil* $\lambda L - N$ is the family of matrices indexed by the complex variable λ. A pencil is *symplectic* if $LJL' = NJN'$, where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. See Anderson, Hansen, McGratten, and Sargent (1996).

18 This program is available at <http://www.math.niu.edu/~anderson>.

19 The solution method in the text assumes that L is nonsingular and well conditioned. If it is not, the following method proposed by Evan Anderson will work. We want to solve for a solution of the form

$$y^*_{t+1} = A^*_o y^*_t.$$

The solution is to be initialized from $(19.B.3)$. We can use the first half and then the second half of the rows of this representation to deduce the following recursive solutions for y^*_{t+1} and μ^*_{t+1}:

$$\begin{aligned} y^*_{t+1} &= A^*_o y^*_t \\ \mu^*_{t+1} &= \psi^* y^*_t. \end{aligned} \tag{19.B.5}$$

Now express this solution in terms of the original variables:

$$\begin{aligned} y_{t+1} &= A_o y_t \\ \mu_{t+1} &= \psi y_t, \end{aligned} \tag{19.B.6}$$

where $A_o = A^*_o \beta^{-.5}, \psi = \psi^* \beta^{-.5}$. We also have the representation

$$\mu_t = P y_t. \tag{19.B.7}$$

The matrix $A_o = A - BF$, where F is the matrix for the optimal decision rule.

Note that with $(19.B.3)$,

$$L[I;P]y^*_{t+1} = N[I;P]y^*_t$$

The solution A^*_o will then satisfy

$$L[I;P]A^*_o = N[I;P].$$

Thus A^{o*} can be computed via the Matlab command

$$A^*_o = (L*[I;P])\backslash(N*[I;P]).$$

C. Forecasting formulas

The decision rule for the competitive fringe incorporates forecasts of future prices from (19.4.11) under m. Thus, the representative competitive firm uses equation (19.4.11) to forecast future values of (Q_t, q_t) in order to forecast p_t. The representative competitive firm's forecasts are generated from the jth iterate of (19.4.11):[20]

$$\begin{bmatrix} z_{t+j} \\ \mu_{x,t+j} \end{bmatrix} = m^j \begin{bmatrix} z_t \\ \mu_{x,t} \end{bmatrix}. \tag{19.C.1}$$

The following calculation verifies that the representative firm forecasts by iterating the law of motion associated with m. Write the Euler equation for i_t (19.4.4) in terms of a polynomial in the lag operator L and factor it: $(1 - (\beta^{-1} + (1+c^{-1}h))L + \beta^{-1}L^2) = -(\beta\lambda)^{-1}L(1-\beta\lambda L^{-1})(1-\lambda L)$ where $\lambda \in (0,1)$ and $\lambda = 1$ when $h = 0$.[21] By taking the nonstochastic version of (19.4.4) and solving an unstable root forward and a stable root backward using the technique of Sargent (1979 or 1987a, chap. IX), we obtain

$$i_t = (\lambda - 1)q_t + c^{-1}\sum_{j=1}^{\infty}(\beta\lambda)^j p_{t+j}, \tag{19.C.2}$$

or

$$i_t = (\lambda - 1)q_t + c^{-1}\sum_{j=1}^{\infty}(\beta\lambda)^j [(A_0 - d) - A_1(Q_{t+j} + q_{t+j}) + v_{t+j}], \tag{19.C.3}$$

This can be expressed as

$$i_t = (\lambda - 1)q_t + c^{-1}e_p\beta\lambda m(I - \beta\lambda m)^{-1}\begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix} \tag{19.C.4}$$

where $e_p = [\,(A_0 - d) \quad 1 \quad -A_1 \quad -A_1 \quad 0\,]$ is a vector that forms $p_t - d$ upon postmultiplication by $\begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}$. It can be verified that the solution procedure builds in (19.C.4) as an identity, so that (19.C.4) agrees with

$$i_t = -P_{22}^{-1}P_{21}z_t + P_{22}^{-1}\mu_{xt}. \tag{19.C.5}$$

20 The representative firm acts as though (q_t, Q_t) were exogenous to it.

21 See Sargent (1979 or 1987a) for an account of the method we are using here.

Exercises

Exercise 19.1 There is no uncertainty. For $t \geq 0$, a monetary authority sets the growth of the (log) of money according to

$$m_{t+1} = m_t + u_t \tag{1}$$

subject to the initial condition $m_0 > 0$ given. The demand for money is

$$m_t - p_t = -\alpha(p_{t+1} - p_t), \alpha > 0, \tag{2}$$

where p_t is the log of the price level. Equation (2) can be interpreted as the Euler equation of the holders of money.

a. Briefly interpret how equation (2) makes the demand for real balances vary inversely with the expected rate of inflation. Temporarily (only for this part of the exercise) drop equation (1) and assume instead that $\{m_t\}$ is a given sequence satisfying $\sum_{t=0}^{\infty} m_t^2 < +\infty$. Please solve the difference equation (2) "forward" to express p_t as a function of current and future values of m_s. Note how future values of m influence the current price level.

At time 0, a monetary authority chooses a possibly history-dependent strategy for setting $\{u_t\}_{t=0}^{\infty}$. (The monetary authority commits to this strategy.) The monetary authority orders sequences $\{m_t, p_t\}_{t=0}^{\infty}$ according to

$$-\sum_{t=0}^{\infty} .95^t \left[(p_t - \overline{p})^2 + u_t^2 + .00001 m_t^2\right]. \tag{3}$$

Assume that $m_0 = 10, \alpha = 5, \bar{p} = 1$.

b. Please briefly interpret this problem as one where the monetary authority wants to stabilize the price level, subject to costs of adjusting the money supply and some implementability constraints. (We include the term $.00001 m_t^2$ for purely technical reasons that you need not discuss.)

c. Please write and run a Matlab program to find the optimal sequence $\{u_t\}_{t=0}^{\infty}$.

d. Display the optimal decision rule for u_t as a function of u_{t-1}, m_t, m_{t-1}.

e. Compute the optimal $\{m_t, p_t\}_t$ sequence for $t = 0, \ldots, 10$.

Hint: The optimal $\{m_t\}$ sequence must satisfy $\sum_{t=0}^{\infty} (.95)^t m_t^2 < +\infty$. You are free to apply the Matlab program `olrp.m`.

Exercise 19.2 A representative consumer has quadratic utility functional

$$\sum_{t=0}^{\infty} \beta^t \left\{ -.5(b - c_t)^2 \right\} \tag{1}$$

where $\beta \in (0,1)$, $b = 30$, and c_t is time t consumption. The consumer faces a sequence of budget constraints

$$c_t + a_{t+1} = (1 + r) a_t + y_t - \tau_t \tag{2}$$

where a_t is the household's holdings of an asset at the beginning of t, $r > 0$ is a constant net interest rate satisfying $\beta(1 + r) < 1$, and y_t is the consumer's endowment at t. The consumer's plan for (c_t, a_{t+1}) has to obey the boundary condition $\sum_{t=0}^{\infty} \beta^t a_t^2 < +\infty$. Assume that y_0, a_0 are given initial conditions and that y_t obeys

$$y_t = \rho y_{t-1}, \quad t \geq 1, \tag{3}$$

where $|\rho| < 1$. Assume that $a_0 = 0$, $y_0 = 3$, and $\rho = .9$.

At time 0, a planner commits to a plan for taxes $\{\tau_t\}_{t=0}^{\infty}$. The planner designs the plan to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{ -.5(c_t - b)^2 - \tau_t^2 \right\} \tag{4}$$

over $\{c_t, \tau_t\}_{t=0}^{\infty}$ subject to the implementability constraints (2) for $t \geq 1$ and

$$\lambda_t = \beta(1 + r) \lambda_{t+1} \tag{5}$$

for $t \geq 1$, where $\lambda_t \equiv (b - c_t)$.

a. Argue that (5) is the Euler equation for a consumer who maximizes (1) subject to (2), taking $\{\tau_t\}$ as a given sequence.

b. Formulate the planner's problem as a Stackelberg problem.

c. For $\beta = .95, b = 30, \beta(1 + r) = .95$, formulate an artificial optimal linear regulator problem and use it to solve the Stackelberg problem.

d. Give a recursive representation of the Stackelberg plan for τ_t.