EXPECTED UTILITY, BOUNDED PREFERENCES AND PARADOXES

Abstract

The common investment decision rules, Markowitz's Mean-Variance (MV) rule and the non parametric Stochastic Dominance (SD) rules, suffer from one sever drawback: there are pairs choices where 100% of the investors choose one option, yet these rules are unable to rank the two options under consideration, a paradoxical result. Thus, the MV and SD efficient sets are too large. It is shown that one needs to bound the preferences such that new decision rules corresponding to the restricted preference set emerge which theoretically eliminate the paradoxes and reduce the efficient set. What is considered as a paradox? What restrictions on preference are economically reasonable? We address this paper to this issue. We estimate experimentally the bounded preference set which solve the paradoxes, hence we define the economically relevant set of preference and the corresponding new decision rules. In particular, we derive the Mean-Variance efficient frontier with restricted preferences and compare it to Markowitz's efficient frontier.

1 Introduction

The most commonly employed investment decision rules are the mean-Variance (MV) rule and the non-parametric Stochastic Dominance (SD) rules. All these rules have one major deficiency: there are many cases where these rules are unable to rank two prospects even though it is obvious that in any sample of investors that one takes, practically all subjects will choose one of the prospects. To illustrate consider two prospects A and B. Prospect A yields \$900 with a probability of 1/2 and \$100,000 with a probability of 1/2. Prospect B yields \$1,000 with certainty. We suspect that all investors will choose A, yet A does not dominate B neither by MV rule nor by SD rules[1]. Thus, the standard MV and SD rules are too lose, because they take into account preferences that are unrealistic. The efficient set can be reduced substantially by ruling out such unrealistic preferences as those showing a preference for prospect B in the example above. This paper experimentally reveals the set of unrealistic preferences that can be ruled out, and the resulting reduction of the efficient investment set.

One may doubt the above claim and argue that in the above choice indeed some investors may prefer B, hence the no dominance is justified. For those doubtful investors let us make the following changes: A yields \$999 with a probability 1/100 and \$1 million with probability of 99/100 and B yields \$1,000 with certainty. We suspect that even those who choose B in the previous example would shift to A with the new figures. Yet, even with the second example there is no dominance by MV and by SD rules, a clear deficiency of these common investment decision criteria. Of course we present above an extreme paradox, but in practice,

[1]Let us demonstrate the no dominance relationship by First degree Stochastic Dominance (FSD). By FSD to have dominance of A over B the two cumulative distributions should not cross, and the cumulative distribution of A should be below the cumulative distribution of B. This is not the case in the above example as the two cumulative distributions intersect. The reason for the no dominance of A is that there is a mathematically legitimate utility function for which B is preferred. For example, take the utility function $u(x) = x$ for $x < 1,000$ and $u(x) \leq 1,000$ for $x > 1,000$. It is easy to verify that $E_B u > E_A u$, hence prospect A does not dominate prospect B by FSD.

as we shall see in this paper paradoxes exist with much less extreme choices. As the SD rules are optimal rules in expected utility paradigm, the paradoxes reveal a deficiency of expected utility paradigm that should be resolved.

The purpose of this study is to present new decision rules called Almost MV (AMV) and Almost SD (ASD), which on the one hand resolved the paradoxes. e.g. revealing a preference for A in the above, and similar, examples, and on the other hand these new rules are consistent with expected utility paradigm.

The reason for paradoxes like the once given above, is that the common investment decision rules correspond to a wide set of preferences which generally contain also *economically irrelevant* preferences, which in turn, induce the paradoxes. The economically irrelevant preference reveals either a very large variation in the marginal utility, u', corresponding to First degree Stochastic Dominance (FSD) or a very large variation in the change in u' and u'' corresponding to Mean-Variance (MV) rule or to Second degree Stochastic Dominance (SSD). (In the above example of $u(x)$, u' drops from 1 to zero at $x = \$1,000$, see footnote 1). These preferences are economically irrelevant as in practice such large variation in u' or u'' does not correspond to any of the investors. Thus, all investors will choose A though there is no FSD.

As we shall see below the decision rules corresponding to a given set of preferences with some bounds on preferences are different from the existing decision rules, but coincide with them when the bounds on preferences are relaxed. Therefore the decision rules employed in this paper refer theoretically to a subset of all investors. Yet, in practice these decision rules refer to all or "almost" all investors. To see this recall that the set of all rational preference is the set U_1 such that $u \in U_1$ if $u' \geq 0$ and there is some range where $u' > 0$. The set of all risk averse preferences is the set U_2 where $u \in U_2 \subset U_1$, if $u'' \leq 0$ and for some range $u'' < 0$. First degree stochastic dominance (FSD) and second degree stochastic dominance (SSD) corresponds to U_1 and U_2, respectively. If distributions are normal and

$u \in U_2$ the MV rule coincides with SSD (see Hanoch and Levy [1969]). Though the sets U_1 and U_2 are generally accepted in decision making models, sometimes they induce paradoxes, as illustrated above. Namely, in the above choices though it is reasonable to assume that experimentally *all* investors would choose option A, theoretically we cannot say that A dominates B for all $u \in U_1$ as A does not dominate B by FSD (note that option A does not dominate option B also by Markowitz's MV rule)[2].

As explained above, the above paradoxes stem from the fact that $u \in U_1$ includes also preferences which may reveal a sharp decline or a sharp increase in $u'(x)$ over various ranges of x. Recently, Leshno & Levy [2002] (hereafter L&L) define "almost" all rational investors where these investors are characterized by the set of non-decreasing monotone functions $u \in U_1^\star$ where $U_1^\star \subset U_1$, when some constraints are imposed on the variability of u'. Intuitively, a sharp decline (see example in footnote 1) or increase (see footnote 2) in u' is not allowed. By the same token, "almost" all risk aversion is defined by the set $U_2^\star$ where $U_2^\star \subset U_2$, when some constraints are imposed on the variability of u' and u''. Once again, the constraints are imposed to avoid paradoxes like the one illustrated above. Thus, instead of FSD corresponding to U_1 we have "almost" FSD (AFSD) corresponding to the bounded set $U_1^\star$, and "almost" SSD (ASSD) corresponding to the bounded set $U_2^\star$. By the same argument we have "almost" MV (AMV) corresponding to $U_2^\star$ and normal distribution of returns[3]. We denote these new rules by ASD, ASSD and AMV, respectively. Thus, we obtain three new rules with no paradoxes as illustrated above, when AFSD, ASSD and AMV substitute for

[2]The above example demonstrates a possible paradox due to a sharp decline in the range $x \geq 1,000$. As U_1 includes also convex functions, let us illustrate such a possible paradox also with a convex function with a sharp increase in u'. Suppose that option A provides $14,999 with certainty and B provides $1,000 with probability 0.99 and $15,000 with a probability 0.01. We suspect that in any experiment with these two choices one would find that all subjects would prefer option A. Yet option A does not dominate B by FSD as the two cumulative distributions cross. The reason for such a no-dominance result is that there is a mathematically legitimate convex function $u \in U_1$ with $u' = 0$ for $x \leq 14,999$ and $u' > 0$ and very large for $x > 14,999$. Thus, the possible extreme changes in u' assigns a very high utility weight to the last dollar in the $15,000 outcome, which in turn, creates such a paradox with this convex preference.

[3]The MV rule is optimal also under a set of Elliptic distributions and normality is not necessary (see for example Berk [1997]). For example for Logistic distributions MV rule is optimal. For quadratic utility function the MV rule is sufficient but not necessary (see Hanoch and Levy [1970]).

FSD, SSD and MV, respectively.

The mathematical relationship between U_i and $U_i^\star$ $(i = 1, 2)$ and SD and ASD have been developed by L&L, where, for any constraints imposed on the variability u' or u'', L&L assert whether there is ASD or ASSD. The paper of L&L is purely theoretical which relates to any arbitrary constraints on preference. However, they did not investigate what constraints are reasonable to impose from the investors' point of view. Namely, they did not investigate what are the economically relevant set of preference. We suggest in this study an experimental method to find the relevant constraints on preferences such that:

a) the new rules solve paradoxes by showing a preference to the option which seems to be preferred by all investors and,

b) that indeed all or "almost" all investors experimentally choose the preferred option[4].

What constraints on u' and u'' are reasonable? What preferences are economically irrelevant? These questions are related to the following question: In comparing two choices, what is considered by all or "almost" all investors as a paradox? To illustrate this notion let us consider the following example where option A yields \$1 or x =\$1 million with an equal probability and option B yields \$2 with certainty. We suppose that all investors prefer prospect A, and as there is no FSD, we have a paradox. Assume now that with option A rather than x =\$1 million we have x =\$10,000 with probability of 1/2. Is it still considered a paradox that A and B cannot be ranked by FSD? What if it is only x =\$1,000? We define a clear cut paradox when 100% of the subjects prefer one option over the other despite of the fact that by FSD (or SSD or MV) the two options cannot be ranked, hence are mistakenly included in the efficient set. In this study we experimentally find the economically called

[4] L&L [2002] impose constraints on u' for AFSD and on u'' for ASSD. We show in this paper that in addition to the constraint on u'' one may improve the ASSD rule by adding a constraint on u' accompanied by the assumption that $u'' < 0$. While the constraint on u'' is given in L&L, we focus in this study only on the constraint on u' (not given in L&L) which allows us to show that AFSD$\Rightarrow$ASSD.

for constraints on preferences, namely we find the set of preferences of $U_1^\star$ and $U_2^\star$ and the corresponding AFSD, ASSD (and AMV) investment criteria.

Finally, note that there is one fundamental paradox in expected utility paradigm pointed out by Allais [1953]. Prospect Theory (PT) and Rank-dependent expected utility (RDEU) may explain this paradox by employing decision weights rather than probabilities[5]. The paradoxes given this paper remain in expected utility, PT and RDEU alike, as we can present the above paradoxes in terms of either objective probabilities or decision weights. Hence, the investment criteria suggested here are also intact for improving Prospect Theory SD (PSD) and Markowitz's Stochastic Dominance (MSD)[6].

The structure of this paper is as follows: Section 2 provides the various decision rules with numerical example revealing paradoxes and explain the source of the paradox. We analyze MV, FSD and SSD as well as AMV, AFSD and ASSD. Section 3 provides the experiment and the results. Section 4 is devoted to the implication of the results to MV and AMV. We show in this section that the lower part of Markowitz's MV efficient frontier may contain *economically* inefficient portfolios. Concluding remarks are given in section 5.

2 Decision Rules and Paradoxes

2.1 The MV Rule

The most common rule employed in investment decision making under uncertainty is Markowitz's [1952] mean-variance (MV) rule. Indeed, the Sharpe-Lintner Capital Asset Pricing Model (CAPM) (see Sharpe [1964] and Lintner [1965]) and Sharpe's performance ratio (Sharpe, [1966]) are all based on the MV rule. Denote two investment options by F and G. Then F

[5]For PT see Kahneman and Tversky [1979] and Tversky and Kahneman [1992]. For RDEU see Quiggin [1982], [1993].

[6]For PSD and MSD see Levy & Levy [2002].

dominates G by MV rule if,

$$E_F(x) \geq E_G(x), \quad \sigma_F \leq \sigma_G$$

with at least one strict inequality. (If F and G are normally distributed and $u \in U_2$, SSD and MV coincide).

The MV rule is based on a relatively confining assumption (e.g., a quadratic preference or normal distribution with risk aversion, see footnote 3). The MV rule suffers from the same drawbacks of FSD and SSD as discussed above. For example, the MV rule is unable to rank $x = \{(1, \frac{1}{2}), (2, \frac{1}{2})\}$ and $y = \{(2, \frac{1}{2}), (4, \frac{1}{2})\}$ (where 1,2,2,4 are outcomes all with a probability of $\frac{1}{2}$), while it is obvious that every rational investor would choose y.

One may claim that the MV rule fails in such a case because normality does not hold. This is not a valid claim as the MV possible paradoxical results may exist even if distributions are normal. A simple example is sufficient to make this point. Suppose that x and y are normally distributed and

$$E_x = 1, \sigma_x = 1, \quad E_y = 10^6, \sigma_y = 2.$$

It is easy to see that neither x nor y dominates the other by MV rule as well as by SD rules. For example, take the mathematically legitimate risk averse utility function as follows:

$$u(t) = \begin{cases} t & t \leq t_0 \\ t_0 & t > t_0 \end{cases}$$

with t_0 *very small* such that for $t < t_0$ the cdf of y is above the cdf of x. Such a utility function reveals a preference for x. Similarly, one can easily find many risk averse functions showing a preference for y[7]. Therefore, paradoxically both x and y are MV efficient. What is the remedy to such a paradox, where probably 100% of investors in practice would choose y,

[7]Denote the cumulative distribution of x by F, and of y by G. Then $\Delta \equiv E_F u - E_G u = \int_{-\infty}^{\infty} [G(z) - F(z)] u'(z) dz$. Two cumulative normal distributions intersect at most once. Denote the intersection point by t_0. Then for $t < t_0$, $G(z) < F(z)$ and as $u' \geq 0$, we have $\Delta > 0$, i.e. x is preferred over y for this specific utility function (recall that $u' = 0$ for $t > t_0$).

yet decision rules do not reveal this preference? Simply, derive adjusted MV rule denoted by AMV, which corresponds to a narrower set of preferences suitable for almost all investors, ruling out preferences like the one given above, and hence avoiding such paradoxes. We illustrate below how to bound the utility function such that a paradox like the one given above will not exist, and show the implication of these constraints on the MV efficient frontier (Section 4.2). We turn now to the traditional investment SD rules and the new Almost Stochastic Dominance (ASD) rules. Then, we provide the AMV as a substitute to the MV rule.

2.2 Stochastic Dominance Decision Rules

We first define SD rules and demonstrate intuitively the concept of ASD with two examples. We show that in these two examples there is no stochastic dominance relationship, yet, it seems that "most" decision makers prefer one prospect over the other. We say, "it seems," as we leave it to the reader to judge our assertion. In the examples below, we relate to First-degree Stochastic Dominance (FSD) and Second-degree Stochastic Dominance (SSD), respectively. Therefore, let us first define these decision rules. Let x and y be two random variables, and F and G denote the cumulative distribution functions of x and y, respectively.

1. **FSD:** F dominates G by FSD ($F \succeq_1 G$) if $F(x) \leq G(x)$ for all $x \in R$ and a strict inequality holds for at least some x. $F \succeq_1 G$ iff $E_F u \geq E_G u$ for all $u \in U_1$, where U_1 is the set of all non-decreasing differentiable real-valued functions.

2. **SSD:** F dominates G by SSD ($F \succeq_2 G$) if $\int_{-\infty}^{x}[G(t) - F(t)]dt \geq 0$, for all $x \in R$ and a strict inequality holds for at least some x. $F \succeq_2 G$) iff $E_F u \geq E_G u$ for all $u \in U_2$, where U_2 is the set of all non-decreasing real-valued functions such that $u'' < 0$.

   We also define a risk-seeking dominance (RSD) which will be employed in the interpretation of our results. This rule is defined as follows:

3. **<u>RSD:</u>** F dominates G for all risk-seeking utility function (i.e., $u \in U_{RS}$ if $u' \geq 0$ and $u'' \geq 0$) if and only if $\int_x^{\infty}[G(t) - F(t)]dt \geq 0$ for all values x.

For proofs and discussion of SD rules, see Fishburn [1964], Hanoch and Levy [1969], Hadar and Russell [1969], Rothschild and Stiglitz [1970][8]. For RSD and a survey of SD rules and further analysis, see Levy [1992, 1998][9].

We introduce the notion of Almost FSD (AFSD) and the intuitive difference between FSD and AFSD by means of the following example. Consider the following two distributions:

$$x = \begin{cases} 1 & \text{probability } 0.1 \\ 10^6 & \text{probability } 0.9, \end{cases} \quad \text{y} = 2 \text{ with probability } 1$$

The cumulative distribution F_x and F_y are given in Figure 1. Because of the small "area violation", denoted by "N" in Figure 1, $F(x)$ does not dominate $F(y)$ by FSD, SSD or MV rules. The mathematical reason for the no dominance is that there are some utility functions which assign a large utility weight to area "N" and very small or zero weight to area K (see Figure 1). As almost all investors would choose prospect x, we would like to establish another decision rule which reveals a dominance of x over y despite the fact that FSD does not prevail. With this new rule we eliminate preferences assigning a very large weight to area N relative to the utility weight given to area K. By allowing this FSD violation, we obtain Almost FSD rules (AFSD) corresponding to *almost* all investors. By a similar way we allow SSD area violation to derive ASSD rule.

Let us turn to the ASD criteria developed by L&L. First for two cumulative distribution functions under consideration (F and G), we assume that the distributions have a finite support, say $[a, b]$ $(-\infty < a < b < \infty)$. Define the following "area violation" regions that

[8] SD criteria are not as well developed as the MV rule, in particular in their application to portfolio diversification. However, recently, Post [2003] developed a linear programming procedure where he uses SD criteria to test whether one can rationalize the market portfolio efficiency by various set of preferences (see also Kuosmanen [2004].

[9] In all the above investment criteria it is assumed that F and G are known. When F and G are empirical distributions, one has to test also for significance. See for example Barrett and Donald [2003].

will be used later on:

$$S_1(F,G) = \{t \in [a,b] : G(t) < F(t)\}, \tag{1}$$

$$S_2(F,G) = \left\{ t \in S_1(F,G) : \int_a^t G(x)dx < \int_a^t F(x)dx \right\}, \tag{2}$$

where F and G are two cumulative distribution functions of the returns on two investments under consideration, respectively. As we shall see below, equations (1) and (2) describe the "area violation" regions of FSD and SSD, respectively. For two cumulative distribution functions F and G we define $\|F - G\| = \int_a^b |F(t) - G(t)|dt$. The integral over S_1 of $[|G(t) - F(t)|]$ and over its complement, $\overline{S_1}$, gives the total area enclosed between the two cumulative distributions. Before we turn to the new decision rules we need to distinguish between the following two "area violation" concepts:

- ε_1 = the *actual* FSD relative area violation, i.e. it is the integral over the range $S_1(F,G)$ divided by the total absolute area enclosed between F and G. It does not reflect preferences, and is purely determined by the characteristics of F and G.
- $\varepsilon_1^\star$ = the *allowed* FSD relative area violation which reflects the investors' preferences. As we shall see below if $\varepsilon_1 < \varepsilon_1^\star$, we have AFSD.

ε_2 and $\varepsilon_2^\star$ corresponding to ASSD are defined in a similar way.

To provide an intuitive explanation to the relationship between ε_1 and $\varepsilon_1^\star$, suppose that F is below G in most of the range of outcomes but it is above G in some range S_1, with a calculated $\varepsilon_1 = 1\%$. If all subjects allow even a larger violation area, say $\varepsilon_1^\star = 5\%$, and still all prefer F over G, such a preference *a fortiori* holds a lower actual area violation $\varepsilon_1 = 1\%$. Thus, if $\varepsilon_1 < \varepsilon_1^\star$ we have a dominance by AFSD.

Definition (ASD): Let x and y be two random variables, and F and G denote the cumulative distribution functions of x and y, respectively. For $0 < \varepsilon_1^\star, \varepsilon_2^\star < 0.5$ we define:

AFSD: F dominates G by $\varepsilon_1^\star$-Almost FSD if and only if,

$$\int_{S_1} [F(t) - G(t)]dt \leq \varepsilon_1^\star \|F - G\| \tag{3}$$

ASSD: F dominates G by $\varepsilon_2^\star$-Almost SSD if and only if,

$$\int_{S_2} [F(t) - G(t)]dt \leq \varepsilon_2^\star \|F - G\| \tag{4}$$

where $\varepsilon_1^\star$ and $\varepsilon_2^\star$ correspond to AFSD and ASSD, respectively. Note that $\varepsilon_1^\star$ and $\varepsilon_2^\star$ are the "allowed" area violation by all rational investors and by all risk averse investors, respectively, such that ASD exists. Therefore, for a given $\varepsilon_1^\star = \varepsilon_2^\star = \varepsilon^\star$, if (3) holds (4) holds, implying that AFSD$\Rightarrow$ASSD. Also note that the conditions $\varepsilon_1^\star < 0.5$ and $\varepsilon_2^\star < 0.5$ implies that $E_F(x) \geq E_G(x)$ is a necessary condition for AFSD and ASSD[10].

Thus, to have ASD one requires that the *actual* negative area where F is above G has to be smaller than the *allowed* area violation, i.e. $\varepsilon \leq \varepsilon^\star$. Therefore, by stating what are $\varepsilon_1^\star$ and $\varepsilon_2^\star$ we state the *allowed* proportion of area violation. The larger the $\varepsilon_i^\star$, the more preferences are eliminated or considered as irrelevant, or pathological. On the contrary, if $\varepsilon^\star \to 0$, no area violation is allowed, hence no preference is eliminated, and we cannot avoid the paradoxes shown above. L&L have shown that the amount of *actual* area violation ε is related to the relevant preferences as follows;

For FSD we have $u \in U_1$. For Almost FSD (AFSD) we define a set of preferences given by $U_1^\star$ as follows,

$$U_1^\star(\varepsilon_1) = \left\{ u \in U_1 : u'(x) \leq \inf\{u'(x)\} \left[\frac{1}{\varepsilon_1} - 1 \right], \forall x \in [a, b] \right\} \tag{5}$$

[10] To have dominance, we must have that $\varepsilon_i \leq \varepsilon_i^\star$ $(i = 1, 2)$. Hence, also $\varepsilon_1 < \frac{1}{2}$ and $\varepsilon_2 < \frac{1}{2}$. To see that this implies that $E_F(x) \geq E_G(x)$ is a necessary condition for dominance, note that $E_F(x) - E_G(x) = \int_a^b [G(x) - F(x)]dx = \int_{S_1} [G(x) - F(x)]dx + \int_{\overline{S_1}} [G(x) - F(x)]dx$. We have $\varepsilon_1 = \frac{\int_{S_1} [F(x) - G(x)]dx}{\int_{S_1} [F(x) - G(x)]dx + \int_{\overline{S_1}} [G(x) - F(x)]dx} < \frac{1}{2}$. Cross multiplied to see that $E_F(x) \geq E_G(x)$.

for $0 < \varepsilon_1 < 0.5$.

Note that ε_1 of (5) and $\varepsilon_1^\star$ of equation (3) are related but not necessary equal. ε_1 is the *actual* area violation. For a given actual area violation, F has a higher expected utility than G for all $u \in U_1^\star$ as defined above. This is a purely technical relationship between ε_1 and $U_1^\star$ (see Appendix A). However, if $\varepsilon_1 < \varepsilon_1^\star$, for all subjects it implies that all subjects allow a larger area violation than actually exists, hence we can safely conclude that there is AFSD for all $u \in U_1^\star(\varepsilon_1)$, and all subjects have preferences which belong to $U_1^\star(\varepsilon_1)$.

As mentioned above, for Almost SSD (ASSD) one can employ restrictions on both u' and u''. In this paper we focus on the restrictions on u', though the results can be improved by adding restrictions on u''. Thus, for SSD we define for $0 < \varepsilon_2 < 0.5$, the following relevant set of risk averse utility functions,

$$U_2^\star(\varepsilon) = \left\{ u \in U_2 : u'(a) \leq u'(b) \} \left[\frac{1}{\varepsilon_2} - 1 \right] \right\} \tag{6}$$

We have, $U_2^\star \subset U_2$ where the set $U_2^\star$ corresponds to ASSD[11]. The relationship between ε_2 and $\varepsilon_2^\star$ is similar to the relationship between ε_1 and $\varepsilon_1^\star$ with the exception that these terms are related to ASSD rather than AFSD.

In Appendix A we elaborate on the formal relationship between ε_i and the constraints on $U_{(i=1,2)}^\star$, when only restrictions on u' are imposed.

We call the sets $U_1^\star$ and $U_2^\star$, bounded sets because we impose some bounds on the variability of u' (see equations (5) and (6)). Also to have a ASD dominance we do not allow more than 50% area violation because for $\varepsilon^\star > 0.5$, if F dominates G, obviously also G dominates F (see equations (3) and (4)). To avoid this property for AFSD and ASSD we

[11] Actually, one needs that the highest derivative u', will be smaller than the smallest derivative u' times $\left[\frac{1}{\varepsilon_2} - 1\right]$. However, with concave preferences $u'(a)$ is the highest value and $u'(b)$ is the smallest one. This formulation doest not exist in L&L. Also, L&L require that $E_F(x) \geq E_G(x)$ as a second condition for ASSD. This condition is actually not required as it follows from (4), accompanied with the constraint $\varepsilon_2 < 0.5$ (see footnote 10).

must have $\varepsilon^\star < 0.5$.

3 The Experiment and the Results

In the experiments reported below we first define *actual* relative area violation ε_1 and ε_2, corresponding to two hypothetical distributions F and G, which in turn, determine $U_1^\star$ and $U_2^\star$. Then, we establish FSD* and SSD* rules corresponding to $U_1^\star$ and $U_2^\star$, respectively.

There are two experiments, one with relatively low bets and one with much larger bets. We first measure experimentally the corresponding *allowed* relative area violation $\varepsilon_1^\star$ and $\varepsilon_2^\star$ by each subject, and the smallest $\varepsilon^\star$ across all subjects denotes an allowed area violation by 100% of the subjects. We have the following decision rules:

a) If $\min_i \varepsilon_{1,i}^\star > \varepsilon_1$ (where $i = 1, \ldots, n$ stands for the i^{th} subject), we conclude that there is AFSD for all $u \in U_1^\star(\varepsilon_1)$.

b) If $\min_i \varepsilon_{2,i}^\star > \varepsilon_2$ (where $i = 1, \ldots, n$ stands for the i^{th} subject), we conclude that there is ASSD for all $u \in U_2^\star(\varepsilon_1)$.

Experiment 1

There were 196 subjects of undergraduate students in the experiments. The subjects face two tasks, one designed to test the relationship of U_1 and $U_1^\star$ and one is designed to test the relationship between U_2 and $U_2^\star$. Thus, the reasonable constraints on u' and u'' that should be imposed in deriving FSD* and SSD* are studied from these two tasks.

3.1 Task I: Almost FSD (or AFSD)

Table 1 provides the choices in Task I. The subjects had to choose between prospects A and B five times. In the sixth choice they have to write what is the *minimum* value of $\$z$ such that they will choose prospect B. Note that names (option A or B) and the place (right and left columns) of the almost dominating option changes across the various choices to make sure that the subjects will not be biased in their selection by selecting one option mechanically. The choices in decisions 1 and 2 are conducted simply to check whether the subjects violate FSD. In the first choice A dominates B by FSD and in the second choice B dominates A by FSD. In the next three choices there is no FSD dominance. However, we use these 3 choices to check for consistency of the decisions. For example, if in choice 4 the subject prefers option B and in choice 6 the subjects write the minimum value $z = \$400$, these two choices are inconsistent because the minimum z should be $350 or less. Similarly, if in choice 4 the subject selects B and in choice 5, the subject selects also B we consider these choices inconsistent (see Table 1). We have about 8% inconsistent choices. All the cases of inconsistent choices were eliminated, as we suspect these subjects did not give serious consideration to the choices, and filled out the questionnaire just to get rid of it.

In problem 6, which is the heart of Task I of the experiment, the subjects had to write the *minimum* value $\$z$ such that B is preferred over A. From this minimum value we can learn about the subjects' preferences and in particular about the relationship between U_1 and $U_1^\star$.

Figure 2 demonstrates the cdf of A and B for a hypothetical value $\$z$ corresponding to decision number 6. We have to distinguish between various ranges of z where different implications regarding $U_1^\star$ can be drawn. First note that B can never dominate A by FSD. However, by the same token it is obvious that for a very large value z, e.g., $1 million, *in practice all* investors would prefer B. Thus, though B does not dominate A in U_1 such a dominance may hold for the bounded set $U_1^\star$ i.e. there is FSD* dominance. Finally note that the selected value by the i^{th} subject, z_i, determine her allowed area violation $\varepsilon_i^\star$, and for the highest selected z_i we have the smallest value $\varepsilon_i^\star$ (see Figure 2). Before we turn to the results of Table 1 let us investigate the various ranges of the possible selected value z corresponds to task 6 and the induced implication regarding preferences.

a) $\underline{z \leq \$200}$: Selecting B with $z < 200$ violates FSD as $F_A(x) \leq F_B(x)$ for all values x.

b) $\underline{200 < z \leq \$250}$: Such choices do not violate FSD. However, a selection of $\$200 < z < \$250$ cannot be consistent with risk averse preference. However, such a choice may be considered with risk seeking because for any choice in this range one can find $u_0 \in U_{RS}$ (where U_{RS} is the set of risk-seekers preferences, i.e., $u' > 0$ and $u'' > 0$) such that $E_B u_0(x) > E_A u_0(x)$. For example, for $z = 210$ take u_0 to be,

$$u_0(x) = \begin{cases} x & x \leq 200 \\ 5x & x > 200 \end{cases}$$

to have : $Eu_A(x) = 150$ and $Eu_B(z) = 550$, hence $Eu_B(x) > Eu_A(x)$. Moreover, for $z = \$250$ we can safely assert that B dominates A for *all* $u \in U_{RS}$. Thus, those who selected $z = \$250$ belong to $u_0 \in U_{RS}$ (see RSD rule given above). Finally, note that as z is the *minimum* required value such that B is preferred over A, and as for $z = \$250$, all risk-seeker prefer B over A (as there is a dominance by RSD criterion) all risk seeker must select $z \leq \$250$.

Let us turn now to the most interesting interpretation of preference which is related to the range $z > \$250$.

c) $\underline{z > 250}$: All those who selected $z >$\$250 are not risk-seeker (see b above). Risk neutral investors would select B (with $z =$\$251) and some risk averters may select B but certainly not all of them, because B does not dominate A by SSD as $\int_{-\infty}^{100}[F_A(x) - F_B(x)]dx < 0$ (see Figure 2). However, for large enough z we may have AFSD and therefore all investors would choose B. In other words the negative FSD violation area over the range $\$50 \leq x < \$100$ is small relative to the area in the range $\$200 \leq x < \$z$ (i.e. ε_1 is very small), hence all investors would choose B. To sum up, for any selected $z > \$250$ some but not all risk averters may choose B. As z increases, the group of risk averters who selects B also increases, and for a very large z (say, $z = \$10^6$) probably all investors[12] in any selected sample would prefer B. In the experiment we analyze what is the *minimum value* z such that, indeed, all subjects prefer B. We turn now to the results of Task I[13].

The Results of Experiment I

Out of the 196 subjects 16 subjects (i.e. about 8.2%) are characterized by inconsistent choices and in particular violated FSD dominance. Eliminating the inconsistent choices we are left with 180 subjects. Table 2a reports the results regarding the selected value z and the implied allowed relative area violation $\varepsilon_1^\star$ by the various subjects, such that we have a preference of option B over option A.

Table 2b focuses on the 180 relevant subjects and use the same information of Table 2a but the percentage figure are calculated based on $n = 180$ subjects. As we can see from Table 2b about 23% of the subjects selected $z = 250$, implying risk-seeking preferences, or risk neutrality. For all other choices with $z > 250$, risk-seeking preferences can not explain the choices, yet also there is no FSD or SSD dominance (see Figure 2). Moreover, as we see

[12] For z =\$250 all risk-seeker prefer B over A, and for a very large z also *all risk averters* as well as investors with varying risk attitude in different range of wealth (i.e. with concave as well as convex segments of preferences) would prefer B over A by AFSD or ASSD though we do not have FSD or SSD.

[13] Before we turn to discuss the results, an explanation of the unique feature of this experiment is called for. In equation (5) we have two objective distributions from which ε_1 can be objectively measured and compared to the allowed $\varepsilon_1^\star$ given in (3). Here, we do not have an objective ε_1, because by choosing different values z, the distribution of B is affected (see Figure 2) hence ε_1 is affected. Thus, for the largest selected value z we have a distribution B with the minimum ε_1, but it is also the *minimum* allowed area violation $\varepsilon_1^\star$. Thus, the experiment is designed such that z and distribution B varies across subjects and with the maximum value z we have the minimum value ε_1 which is equal the minimum allowed area violation $\varepsilon_1^\star$, because with this largest value z, 100% of the subjects select B. Therefore, unlike in equation (3) and equation (5), in our specific experiment $\varepsilon_1 = \varepsilon_1^\star$.

from Table 2b, many subjects required substantially a higher value z than \$250, indicating that we may have AFSD or ASSD of B over A. Thus, in practice as we increase z more subjects prefer B over A, and for z=\$1,000, 100% of the subjects prefer B over A, hence for this value we have AFSD of B over A. Thus though B never dominates A by FSD (regardless of the selected z), experimentally we find that for $z = \$1,000$, 100% of the population participating in the experiment prefer B over A. It is possible that with another sample of subjects the minimum value of z, which reveals 100% preference for B will be above or below $z = \$1,000$. Yet, it is clear that the same type of results would be obtained in any sample of subjects: namely, there is some finite value z for which 100% of the subjects would prefer B over A despite of the fact that B does not dominant A by FSD, and this is the main justification for the introduction of almost SD criteria.

Let us illustrate how the results reported in Table 2b are calculated. Suppose that indeed our 180 subjects represent the whole population, i.e., we would have the same percentage choice of z (and $\varepsilon_1^\star$) in the population as in our experiment. The subjects who selected, say, $z = \$300$ allow an area violation of up to 33.3%[14] of the area enclosed between F_A and F_B , (see $\varepsilon_1^\star$ reported in Table 2b) or they are ready to overlook this area violation of FSD and hence prefer B over A. Those subjects who selected $z = \$400$ are more severe as their allowed area violation in the cdf is only up to $\varepsilon_1^\star = 20\%$. Finally, the most severe group contains those who selected z =\$1,000, i.e., allowing no more than $\varepsilon_1^\star = 5.9\%$ area violation. Of course, $U_1^\star$ $(\varepsilon_1^\star = 20\%) \subset U_1^\star$ $(\varepsilon_1^\star = 5.9\%)$, as those who would allow 20% area violation in the cdf to prefer B over A would *a fortiori* prefer B with a lower violation $\varepsilon_1^\star = 5.9\%$ in FSD[15]. In short, those subjects who required $z = \$400$ will be more than happy to receive $z = \$1,000$. Therefore, in our experiment, for $z = \$1,000$ *all* investors would prefer B to A.

[14]To illustrate how $\varepsilon_1^\star$ is calculated, let us demonstrate with $z = \$300$. Looking at Figure 3, the area violation is given by $\$50 \cdot \frac{1}{2} = \$25$. With $z = \$300$, the positive area is given by $\$100 \cdot \frac{1}{2} = \$50$. As $\varepsilon_1^\star$ is defined as the area violation divided by the total absolute area enclosed between F and G, we have $\varepsilon_1^\star = \frac{\$25}{\$25+\$50} = 33.3\%$.

[15]We got robust results in the estimation of $\varepsilon_1^\star$ and $\varepsilon_2^\star$. However, more experiments are called for with a violation area located in various places and may be with more than one violation area, where the size of ε_1 is kept constant.

With $z = \$800$ (or $\varepsilon_1^\star = 7.7\%$) (see Table 2b) we can say that B dominates A for 97.8% of the population, and in this respect, B dominates A for "almost all" but not *"all"* investors. For $z = \$1,000$, B dominates A for all investors (in our sample with $u \in U_1^\star$) though such dominance is not intact for all $u \in U_1$.

If our subjects represent the population we have that all those preferences eliminated from U_1 are *economically irrelevant* though mathematically these preferences are included in U_1. Thus, in the specific example given in Table 1, decision 6, 5.9% area violation in the FSD is allowed by all subjects. Moreover, this 5.9% area violation must be allowed to avoid the paradox induced by the fact that 100% of the subjects prefer B over A, yet there is no FSD (see Table 2b). The bounded set $U_1^\star$ represents the sets of all *economically relevant* preference, while the set U_1 contains the set of all *mathematically relevant* preferences. Therefore, though FSD rule is mathematically correct, AFSD rule is economically more relevant in practice: on the one hand it applies to all investors and on the other hand it avoids paradoxes.

Let us now clarify the relation ship between ε_1 and $\varepsilon_1^\star$. Suppose that $\varepsilon_1^\star = 5.9\%$, as we found in the experiment and further suppose that this result is approximately robust for other similar experiments with different of choices. Now if we find with two actual prospects F and G with a lower ε_1, say, $\varepsilon_1 = 1\%$, we can safely conclude that F dominance G by AFSD. The reason is that all investors allow $\varepsilon_1^\star = 5.9\%$ or more area violation in order to prefer F over G, and such preference is *a fortiori* intact for a lower actual area violation of ε_1=1%. Thus, $\varepsilon_1^\star = 5.9\% = min_i \varepsilon_{1,i}$ is the minimum allowed area violation such that 100% of the subjects prefer one option over the other, and for any *actual* smaller area violation AFSD is intact. Of course for $\varepsilon_1 = 0$ we have FSD for all $u \in U_1$ regardless of the allowed relative area violation $\varepsilon_1^\star$.

Finally, in this study we stress that we need a decision rule e.g. FSD* which corresponds to 100% of the subjects. However, an investment consultant may decide to consider less sever restrictions on $U_1^\star$, and therefore will be satisfied with a rule corresponding to $p\%$ of the population, e.g. $p = 95\%$. Appendix B analyze this case.

Let us turn to the SSD* experimental results.

3.2 Task II: Almost SSD (ASSD)

In Task II we examine the relationship between SSD and almost SSD denoted by ASSD which reflects the relationship between U_2 and $U_2^\star$. Table 3 presents the five decisions of Task II. Once again, decisions 1-4 are used as the check for consistency of the decision process. For example, selecting B in decision 1 and B in decision 2 reflects inconsistency. Similarly, if one chooses B in decision 3 and writes $z = \$500$ in decision 5, it would be considered inconsistent behavior. We learn on the relationship between SSD and ASSD from decision 5, where z is the minimum value such that A is preferred (see Table 3).

Figure 3 presents the cdf of A and B, for a hypothetical value $z > \$300$. There is no FSD and no SSD dominance of either B over A or A over B. Moreover, even if we select z to be very large, e.g., $z = \$10,000$, there is no SSD of A over B because $\int_{100}^{200}[F_B(x) - F_A(x)]dx < 0$. Similarly, B does not dominate A because $\int_{100}^{125}[F_A(x) - F_B(x)]dx < 0$. Thus, in U_2 both A and B are in the efficient set. However, for a very large z, most if not all investors would select A because there is ASSD of A over B. Namely, the negative SSD violation area (see Figure 3, area "N") which is the reason for the no SSD dominance has much more smaller utility weight relative to the positive large area denoted by "+". Thus, unless u' changes very dramatically, A would be preferred over B. How far should z be to the right to overcome the effect of the negative area over the range $175 < x < 200$? What is the value z such that we have ASSD for all investors though there is no theoretical SSD? This is an experimental question to which we turn next.

<u>The Results:</u>

Table 3a presents the value z chosen by the subjects and the corresponding frequency distribution. We have relatively large frequency at $z = \$300, \$325, \$350, \$400$ and \$500 with the mode at $z = \$400$ with 58 out of the 200 subjects selecting this value. Table 3b focuses on the 180 subjects who did not violate the consistency test[16]. Let us analyze several relevant ranges of z and their implication regarding the subjects' preferences.

1. $\underline{z < \$350:}$ There is no SSD or RSD of A over B nor of B over A. However, one can find specific risk averse functions or specific risk seeking functions which show a preference of A over B. Thus, we do not have dominance of A over B, but a choice of A can be rationalized by a legitimate $u_0 \in U_2$ or even $u_0 \in U_{RS}$. For example, take the utility function $u(x) = x$ for $x < 125$ and $u(x) = \$125$ for $x \geq \$125$. With this utility function A is better than B for all $z > \$125$.

2. $\underline{z = \$350:}$ Here there are two possible interpretations for the preference of A:

   (a) investors are risk-seekers because for all $u \in U_{RS}$ A dominates B (see the RSD rule given above).

   (b) Investors are risk-averse and the selection of B is by ASSD, i.e., investors have preferences u such that $u \in U_2^\star$. We focus in this paper on case (b) and analyze the relationship between SSD and ASSD or $U_2^\star$ and U_2.

3. $\underline{z > \$350}$ Here there is only one interpretation: Investors prefer A over B as there is ASSD though there is no SSD. It is true that for $z > \$350$, A dominates B also by RSD. However, if these subjects are indeed risk-seeker they all should write $z = \$350$, because z is defined as the *minimum* value such that the decision A dominates B.

From Table 3b we see that if one selects $z = \$1,000$ with corresponding $\varepsilon_2^\star = 0.032$ (i.e., relatively very small area violation) than all investors, i.e., 100% of the subjects prefer A

[16] It incidently occurred that in the SSD test like in the FSD test exactly 180 subjects did not violate the consistency test. In the SSD test we have 200 subjects while in the FSD we have only 196 subjects.

over B. In such a case $U_2^\star \subset U_2$ but the utility functions eliminated from U_2 are economically irrelevant. Hence, we have ASSD rule which on the one hand corresponds to all relevant investors and on the other hand avoids paradoxes. Once again, recall that even for a very large z, say, $z = \$100^6$, A does not dominate B by SSD while it is obvious that in any sample of investors one takes all will choose A.

The most important lesson we learn from Task II is that some SSD area violation may be irrelevant in practice as there is some value z, not astronomically large such that A dominates B by ASSD for all risk-averse investors in $U_2^\star$. Namely, the negative utility due to the range where the cdf of A is above the cdf of B (and where SSD is violated, see area "N" in Figure 3) is much smaller than the positive utility induces from the fact that the cdf of A is below B. And if this does not occur, always one can increase z until it occurs. It is found in this experiment that with $z = \$500$, the ASSD covers 96.1% of the subjects (as only 1.11%+2.78% required a higher value z, see Table 3) and with $z = 600$ the ASSD covers 97.2% of the subjects[17]. Therefore, in practice the u' of most subjects does not change dramatically over the various ranges of x. If u' of the subjects would change dramatically, then we may need to have a very large z to have ASSD which avoids the paradox.

To see whether $\varepsilon_1^\star$ and $\varepsilon_2^\star$ depend on the magnitude of the payoffs, we conducted a similar experiment for ASD and ASSD where we multiplied the prices by 100. Table 4 and Table 5 provides the choices and the results with $n = 88$ subjects. We find that the dominance area violation $\varepsilon_1^\star$ and $\varepsilon_2^\star$ corresponding to AFSD and ASSD, respectively, are quite invariant to the size of the payoffs, at least for the range given in Tables 3 and 4.

For example, to get AFSD for 100% of the subjects we find with small bets that $\varepsilon_1^\star$=5.9% (see Table 2b), while with larger bets we have $\varepsilon_1^\star$=6.5%. For ASSD corresponding to 100% of the population we find that $\varepsilon_2^\star = 3.2\%$ in both, small and large bets.

[17]We define in Appendix B, ASSD which corresponds to p% of the subjects. However, in the text we define ASSD as the rule which corresponds to $p = 100\%$ of the subjects.

4 Almost MV (AMV)

So far, we analyzed in details AFSD and ASSD. Despite of the fact that SD criteria are distribution-free, still the most commonly employed investment decision rule is Markowitz's MV rule, hence we devote this section to MV and AMV. The MV rule may lead to paradoxical results similar to those obtained in FSD and SSD framework. To see this let us go back to the example given in section 2.1 with the following two investments x and y: $E_x = 1, \sigma_x = 1,\ E_y = 10^6, \sigma_y = 2$. By the traditional MV rule there is no dominance between x and y, yet we suspect that in practice 100% of the investors would choose y, hence the paradox. We can resolve the paradox by introducing AMV rule i.e. the almost MV rule. The MV rule can be justified either by assuming a quadratic utility function or by assuming risk aversion and normal distribution of returns. As the quadratic utility function has a few drawbacks we only briefly mention it here and focus on the normal distribution case in the face of risk aversion[18].

4.1 Quadratic utility function:

Having $u(x) = x - bx^2$ (b >0) we have $Eu(x) = Ex - bEx^2$ and $Eu(y) = Ey - bEy^2$. Suppose that 100% of the subjects choose y (as in the above example) than, we would have with quadratic preference,

$$Ey - bEy^2 > Ex - bEx^2 \text{ for all relevant } b > 0$$

Namely,

$$b < \frac{Ey - Ex}{Ey^2 - Ex^2} = \frac{Ey - Ex}{(\sigma_y^2 - \sigma_x^2) + (Ey)^2 - (Ex)^2} \equiv k$$

Thus, by restricting the parameter b, we resolve the above paradox as for all quadratic utility functions with $b < k$ indeed $Eu(y) > Eu(x)$ which is consistent with most if not

[18]The MV rule can be used also for all risk averse utility functions without normal distribution as long as the variability of returns are not too large. In this case the MV rule is an excellent approximation for expected utility. For more details see Tsaing [1972], Markowitz [1991].

all, individual choices. Thus, with the above example, all quadratic utility functions with $b \geq k$ are considered as pathological or economically irrelevant as by these utility functions $Eu(y) < Eu(x)$.

Like with AFSD and ASSD also here one can run an experiment and find a relationship between the proportion of subjects who select y and the implied restrictions on the parameter b (rather than $\varepsilon_1^\star$ and $\varepsilon_2^\star$ corresponding to AFSD and ASSD). To see this let us go back to the above example, but this time plug various values for Ey. For example, with $Ey = 5$ we may find that $p_1\%$[19] of the population selecting y, which imposes a constraint $b \leq k_1$, where k_1 corresponds to p_1. Now, put instead of $Ey = 5$ the value $Ey = 10, Ey = 100, Ey = 1000$ and so on. Then we get restrictions of the type $b \leq k_i$ corresponding to p_i proportion of the subjects. Note that if for, say, Ey=1000 all subjects choose y, we have AMV dominance of y over x for all *relevant investors* (p_i=100%). Thus, we say that for all $u \in U_q^*$, $Eu(y) \geq Eu(x)$, where U_q^* includes all non-pathological quadratic utility functions.

Finally, like in the previous experiments one can ask the subjects to select a value Ey such that y will be preferred to x. This way one can analyze the relationship between the proportion of the population p_i which select y and corresponding the restriction imposed on the parameter b.

4.2 AMV with normal distribution

Let us turn now the more interesting case of normal distribution in the face of risk aversion. In this case two cumulative normal distributions of returns on two options intersect at most once. Figure 4 illustrates possible intersections of two distributions denoted by F and G. We draw F and G with some specific parameters but the analysis is the same for any two distributions which fulfill to conditions given below. In Figure 4a, F intersects G from below with $E_F > E_G$, $\sigma_F < \sigma_G$, hence for every risk averter F dominate G and no paradox prevails.

[19]On decision rules corresponding to $p\%$ of the population see Appendix B (see also footnote 17).

Figure 4b illustrates a case where there is no SSD or MV dominance, but we may have AMV dominance. We have $E_F > E_G$ but also $\sigma_F > \sigma_G$ and due to the violation area denoted by A there is no MV. Yet, if the area denoted by A is relatively small we may have AMV dominance of F over G.

To analyze the AMV rule first note that the intersection point x_0 of F and G is given by (see Hanoch & Levy [1969]):

$$x_0 = \frac{E_F\sigma_G - E_G\sigma_F}{\sigma_G - \sigma_F}$$

As $E_F > E_G$ and $\sigma_F > \sigma_G$ we (see Figure 4b) can write $E_F = E_G + \Delta\mu$ and $\sigma_F = \sigma_G + \Delta\sigma$ where $\Delta\mu = E_F - E_G$ and $\Delta\sigma = \sigma_F - \sigma_G$ and $\Delta\mu > 0$, $\Delta\sigma > 0$. Plugging these terms in the above formula we get

$$x_0 = \frac{(E_G + \Delta\mu)\sigma_G - E_G(\sigma_G + \Delta\sigma)}{\sigma_G - (\sigma_G + \Delta\sigma)} = \frac{\Delta\mu\sigma_G - E_G\Delta\sigma}{-\Delta\sigma}$$

or

$$x_0 = E_G - \sigma_G\frac{\Delta\mu}{\Delta\sigma}$$

From this we can see that the larger the ratio $\frac{\Delta\mu}{\Delta\sigma}$ the smaller the intersection point x_0. It can be easily shown that for two distributions F and G with $\Delta\mu > 0$ and $\Delta\sigma > 0$, for a given distribution G, the smaller x_0 the smaller is ε. To see this refer to Figure 4c. In Figure 4c we have $\varepsilon^{(A)} = \frac{A}{A+B}$. With Figure 4d with $x_0^* < x_0$, we have $\varepsilon^{(A^*)} = \frac{A^*}{A^*+B^*}$ where, $\varepsilon^{(A*)} < \varepsilon^{(A)}$. Thus, the smaller x_0 the smaller ε, which, as we shall see below, has a direct implication to the systematic reduction in the MV efficient set by AMV rule. Finally, note that as two cumulative normal distributions cross at most once we have AFSD $\sim$ ASSD. The reason is that if we have $\mu_F > \mu_G$ and $\sigma_F > \sigma_G$, the left tail of F is above the left tail of G, and FSD area violation is identical to the SSD area violation (see Figure 4b), hence for normal distributions $\varepsilon_1 = \varepsilon_2$ and AFSD $\sim$ ASSD.

Having the MV efficient set and starting from the minimum variance portfolio (MVP), the ratio $\frac{\Delta\mu}{\Delta\sigma}$ is declining as we shift to higher mean higher variance portfolios. This means that for any portfolio G on the frontier as we compare it to a portfolio F on the frontier, the more F is shifted to the right the smaller $\frac{\Delta\mu}{\Delta\sigma}$ hence, the larger x_0 and the larger ε. Therefore, if F does not dominate G by AMV (for a given ε) any other portfolio F^* shifted to the right will not dominate G. This means that there is a systematic reduction in the efficient set by AMV. To be more specific, for a given ε, AMV eliminates from the efficient set the lower part of the efficient set including the MVP (recall that a necessary condition for dominant of F over G by AMV is that $E_F > E_G$).

Let us demonstrate the MV efficient set reduction by AMV with normally distributions assets. In this demonstration we used three assets with expected return of 10%, 20% and 15% respectively, and the following hypothetical covariance and correlation matrices, respectively,

$$\Sigma = \begin{pmatrix} 0.0100 & -0.0061 & 0.0042 \\ -0.0061 & 0.0400 & -0.0252 \\ 0.0042 & -0.0252 & 0.0225 \end{pmatrix}, \ \rho = \begin{pmatrix} 1.0 & -0.305 & 0.28 \\ -0.305 & 1.0 & -0.84 \\ 0.28 & -0.84 & 1.0 \end{pmatrix}.$$

We then calculated (using the function "frontcon" in Matlab 7) the mean and standard deviation of 100 points (portfolios) on the efficient set starting with the MVP and with a linear increase of the mean (see Figure 5a). We then calculated ε_1 for each pair of point on the efficient set and deleted points which are AMV dominated for a given ε_1. Obviously, for allowed area violation $\varepsilon_1^\star = 0$, we have MV $\sim$ AMV and the MV efficient frontier includes all points to the right of MVP including the MVP portfolio. For $\varepsilon_1^\star > 0$ the efficient set is reduced. Starting with 100 points on the efficient set we ended for $\varepsilon_1^\star$ =5% with 70 points on the efficient set, deleting about 30% of the points that are located left to point a on the efficient set (see Figure 5b). Thus, all portfolios right to point a are AMV efficient. When we used $\varepsilon_1^\star = 3\%$, a smaller segment of the AMV efficient set is eliminated and the efficient set includes all portfolios right to point b (see Figure 5b). When we used other numerical examples we got *similar results*: the lower segment of MV efficient frontier is relegated to the inefficient set by AMV. The relegation of the lower part of the MV efficient set by AMV is similar to the results of Baumol [1963] and Levy & Levy [2004], though these two studies employed completely different framework, i.e. the expected gain confidence approach and Prospect Theory, respectively.

5 Concluding Remarks

Allais' paradox and other similar paradoxes were resolved in CPT or RDEU paradigms, namely by employing decision weights. However, even after taking into account these decision weights, the commonly employed investment decision rules in expected utility paradigm as well as CPT and RDEU paradigms suffer from a severe drawback: they may be unable to rank two options when in practice 100% of the investors prefer one option over the other. Thus, the efficient set implied by these rules may be too large. In particular, Markowitz's MV frontier contain an inefficient segment. For example, the Mean-Variance (MV) rule and Stochastic Dominance (SD) rules cannot rank x (\$1 with probability 0.1 and $\$10^6$ with probability 0.99) and y (\$2 with certainty), while in practice all investors would choose x. And this paradox exists regardless of whether the probabilities are objective or distorted. The reason why the existing decision rules are unable to rank such a pair of options is because they correspond to *all* preferences in a given utility class (e.g., all risk-averse functions), including utility functions with derivatives (u' or u''), which change wildly over various regions of the financial outcome and therefore, in practice do not correspond to any investor. Though these preferences are *mathematically* legitimate they are *economically* irrelevant. If one bounds the volatility of u' or of u'', or both, one reduces the class of preferences to which the decision rules correspond by eliminating preferences which induce the paradoxes such as the one given above. Thus, we employ in this study new decision rules AMV and ASD, which differ from the common MV and SD rules. The advantage of these new rules is that they avoid paradoxes, and allow to decrease the investment efficient set.

What restrictions on preference are legitimate? What is considered as a reasonable class of bounded utilities? We tried in this study to answer these questions experimentally. The paradoxes are induced by some actual "area violation" (ε) which avoid the the theoretical dominance. We provide 288 subjects with a few tasks of choosing from two options, where such a paradox exists. We find experimentally the maximum amount of *allowed* "area violation" ($\varepsilon^{\star}$) in the cumulative distribution function (cdf) such that 100% of the subjects would prefer one option over the other, despite the fact that there is no theoretical MV or SD dominance due to this actual "area violation" (ε). The calculated minimum allowed "area violation" in the cdf is determined experimentally by the subjects, and this, in turn dictates what preferences are economically irrelevant, though mathematically they are relevant. We find that with AFSD that the minimum allowed area violation across all subjects is $\varepsilon_1^{\star} = 5.9\%$, while the corresponding figure for ASSD is $\varepsilon_2^{\star} = 3.2\%$. FSD is a rule for all $u_1 \in U_1$ as long as $u' \geq 0$. We define a new set $U_1^{\star}$ such that $U_1^{\star} \subset U_1$, when some bounds are imposed on the volatility of u'. This bound allows us to create AFSD situation though there is no FSD, which in turn, avoids the paradox like the one given above. Hence, to avoid paradoxes we suggest AFSD corresponding to $U_1^{\star}$, instead of the common pair (FSD, U_1). The same is true for risk aversion: (ASSD, $U_2^{\star}$) replaces (SSD, U_2) and (AMV, $U_2^{\star}$) replaces (MV, U_2) in the cases of normal distributions. Thus, the size of $U_1^{\star}$ and $U_2^{\star}$ is found experimentally. In the case of MV and AMV we find that AMV eliminates from the efficient set the lower part of Markowitz's MV efficient frontier. It is interesting that this result is similar to the results of Baumol [1963] and Levy & Levy [2004], who used a completely different approaches. In particular, Levy & Levy [2004] used Prospect Theory to obtain a similar relegation of lower segment in the MV efficient frontier to the inefficient set.

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Table 1:

Task I: In each decision you have to choose between Prospect A and B

Circle your choice

Decision	Prospect A		Prospect B	
	Probability	Outcome	Probability	Outcome
1				
	0.5	\$100	0.5	\$50
	0.5	\$200	0.5	\$150
2				
	0.5	\$50	0.5	\$100
	0.5	\$200	0.5	\$200
3				
	0.5	\$50	0.5	\$100
	0.5	\$250	0.5	\$200
4				
	0.5	\$100	0.5	\$50
	0.5	\$200	0.5	\$350
5				
	0.5	\$50	0.5	\$100
	0.5	\$400	0.5	\$200

What is the minimal value z such that will make you prefer B over A? Please write $z =\$$___.

Decision	Prospect A		Prospect B	
	Probability	Outcome	Probability	Outcome
6				
	0.5	\$100	0.5	\$50
	0.5	\$200	0.5	$\$z$

Table 2a

The Distribution of the 196 Choices including those with FSD violation

Selected z	Number of Choices
1	1
50	1
100	7
200	6
201	1
250	41
251	10
255	1
275	1
300	63
301	1
325	3
350	9
400	27
401	1
500	15
600	2
750	1
800	1
1000	4
Total	196

Table 2b:

The Distribution of the selected value z and implied AFSD $\varepsilon_1^\star$ of the consistent 180 choices[a]

Selected z	Number of choices	In percent	The implied FSD violation, $\varepsilon_1^\star$
250	41	22.8	0.500
251-275	12	6.7	0.400- 0.495
300	63	35.0	0.333
301-350	13	7.2	0.250 - 0.331
400	27	15.0	0.20
401-800	20	11.1	0.077-0.199
1,000	4	2.2	0.059
Total	180	100%	

a) We obtain in this experiment that $\max z = \$1,000 \Rightarrow \min_i \varepsilon_{1,i}^\star = 5.9\%$.

Table 3: The Choices in Task II: Decision Choose between A and B in the following five decisions

Decision	Option A		Option B	
	Probability	Outcome	Probability	Outcome
1	$\frac{1}{3}$	\$100	$\frac{1}{3}$	\$125
	$\frac{1}{3}$	\$200	$\frac{1}{3}$	\$150
	$\frac{1}{3}$	\$300	$\frac{1}{3}$	\$300
2	$\frac{1}{3}$	\$125	$\frac{1}{3}$	\$100
	$\frac{1}{3}$	\$150	$\frac{1}{3}$	\$200
	$\frac{1}{3}$	\$325	$\frac{1}{3}$	\$300
3	$\frac{1}{3}$	\$100	$\frac{1}{3}$	\$125
	$\frac{1}{3}$	\$200	$\frac{1}{3}$	\$150
	$\frac{1}{3}$	\$300	$\frac{1}{3}$	\$400
4	$\frac{1}{3}$	\$125	$\frac{1}{3}$	\$100
	$\frac{1}{3}$	\$200	$\frac{1}{3}$	\$150
	$\frac{1}{3}$	\$300	$\frac{1}{3}$	\$400

What is the minimum value z for which you prefer A over B? Please write $z = \$\_\_\_$.

Decision	Prospect A		Prospect B	
	Probability	Outcome	Probability	Outcome
5	$\frac{1}{3}$	\$125	$\frac{1}{3}$	\$100
	$\frac{1}{3}$	\$150	$\frac{1}{3}$	\$200
	$\frac{1}{3}$	$\$z$	$\frac{1}{3}$	\$300

Table 3a

The Choices in Task II including the FSD Violations

Selected z	Number of Choices
1	2
100	7
150	2
175	4
200	5
250	6
275	1
300	16
310	1
325	26
326	7
350	36
362	1
375	1
400	58
425	1
430	1
450	3
451	1
500	14
600	2
1000	5
Total	200

Table 3b: Choices with no SSD Violation and the implied $\varepsilon_2^\star$

Selected Value z	Number of Choices	In Percent	The Implied $\varepsilon_2^\star$
250	6	3.3	1.0
275	1	0.56	0.50
300	16	8.9	0.33
310	1	0.56	0.294
325	26	14.4	0.250
326	7	3.9	0.248
350	36	20.0	0.20
362	1	0.56	0.182
375	1	0.56	0.167
400	58	32.2	0.143
425	1	0.56	0.125
430	1	0.56	0.122
450	3	1.67	0.111
451	1	0.56	0.111
500	14	7.78	0.091
600	2	1.11	0.067
1000	5	2.78	0.032

Table 4:

What is the minimum value z for which you prefer A over B? Please write $z =$\$

($n = 88$ subjects)

Decision 5	Prospect A		Prospect B	
	Probability	Outcome	Probability	Outcome
	$\frac{1}{3}$	\$12,500	$\frac{1}{3}$	\$10,000
	$\frac{1}{3}$	\$15,000	$\frac{1}{3}$	\$20,000
	$\frac{1}{3}$	\$$z$	$\frac{1}{3}$	\$30,000

Table 5: Choices with no Violation and to implied $\varepsilon_1^\star$ and $\varepsilon_2^\star$

Selected Value z	Number of Choices	In Percent	The Implied $\varepsilon_1^\star$	The Implied $\varepsilon_2^\star$
30,000	7	8.05	0.667	0.333
32,500	25	28.74	0.500	0.250
32,501	3	3.45	0.500	0.250
32,510	1	1.15	0.499	0.250
33,000	3	3.45	0.476	0.238
33,250	1	1.15	0.465	0.233
34,000	1	1.15	0.435	0.217
35,000	19	21.84	0.400	0.200
36,000	1	1.15	0.370	0.185
37,500	1	1.15	0.333	0.167
40,000	12	13.79	0.286	0.143
42,500	2	2.30	0.250	0.125
45,000	1	1.15	0.222	0.111
100,000	1	1.15	0.065	0.032

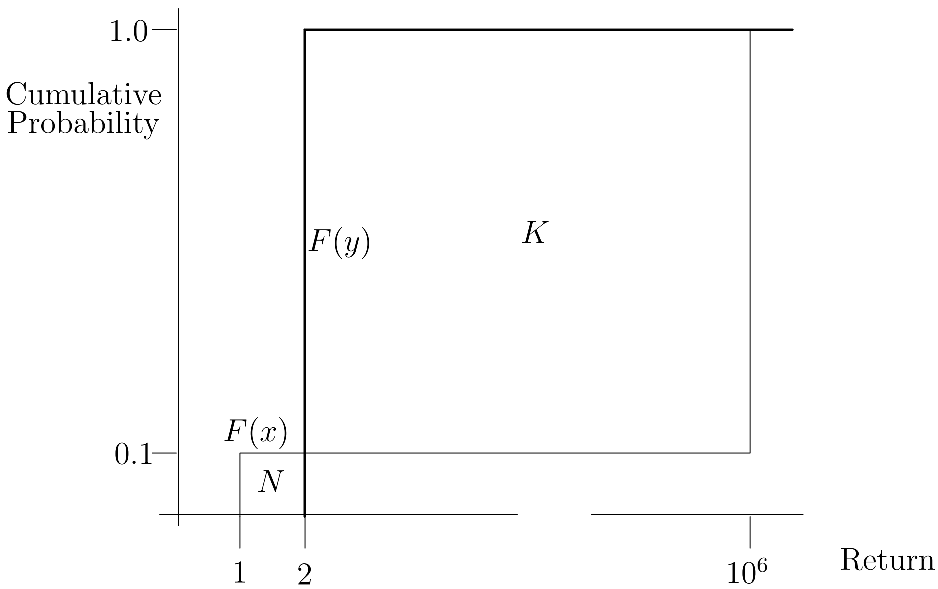

Figure 1: Two hypothetical cumulative distribution functions

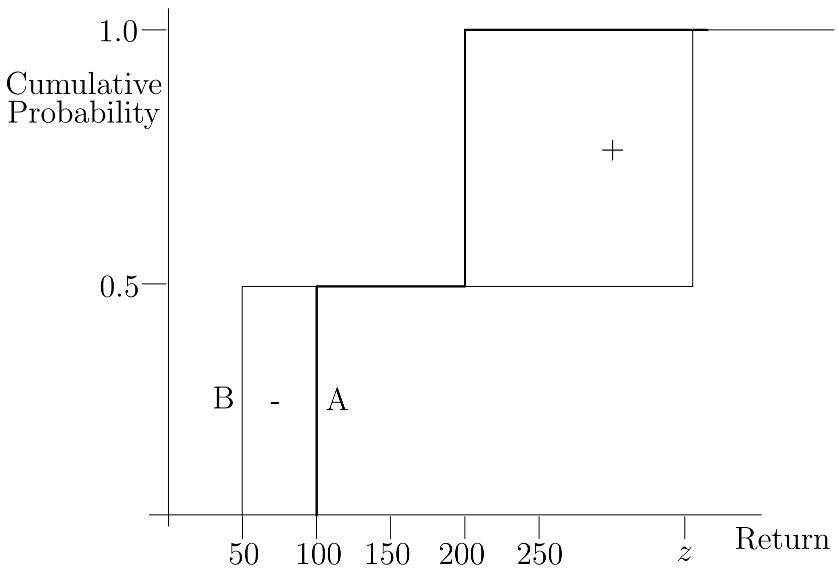

Figure 2: Cumulative distributions of A and B in Task I

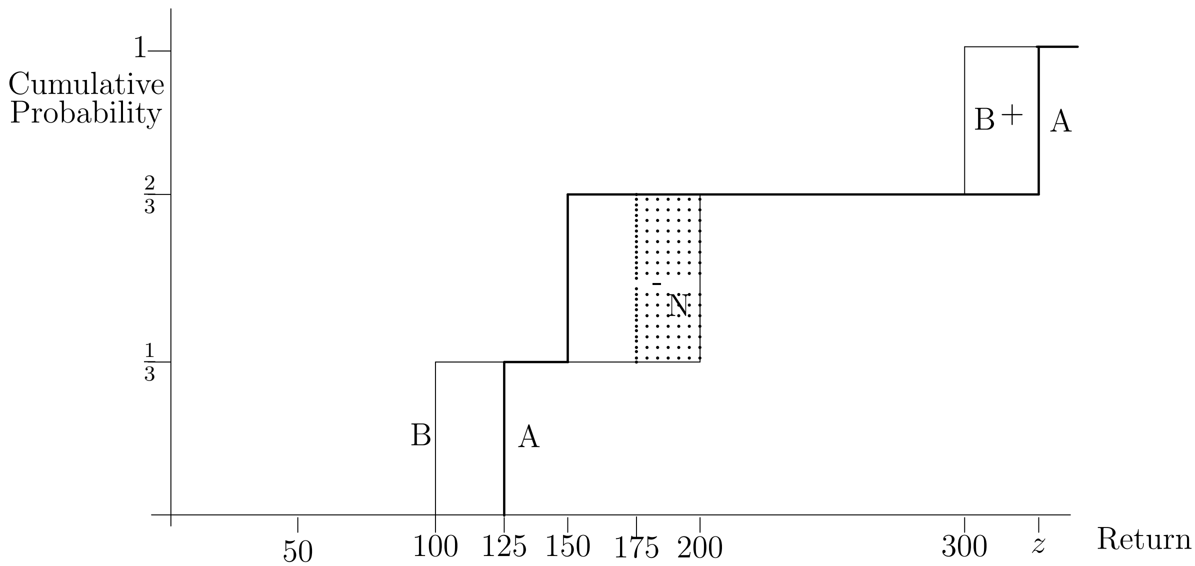

Figure 3: The Cumulative distribution functions of A and B in Task II

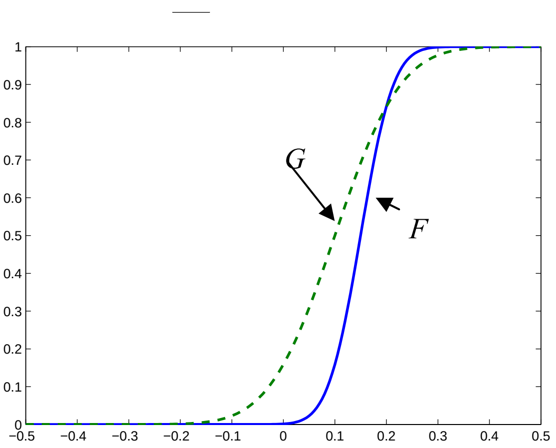

Figure 4a: The cdf of two normal distributions F ($E_F = 15\%, \sigma_F = 5\%$) and G ($E_G = 10\%, \sigma_G = 10\%$)

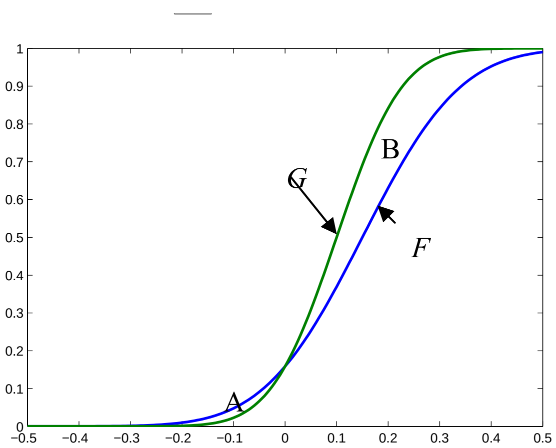

Figure 4b: The cdf of two normal distributions F ($E_F = 15\%, \sigma_F = 15\%$) and G ($E_G = 10\%, \sigma_G = 10\%$)

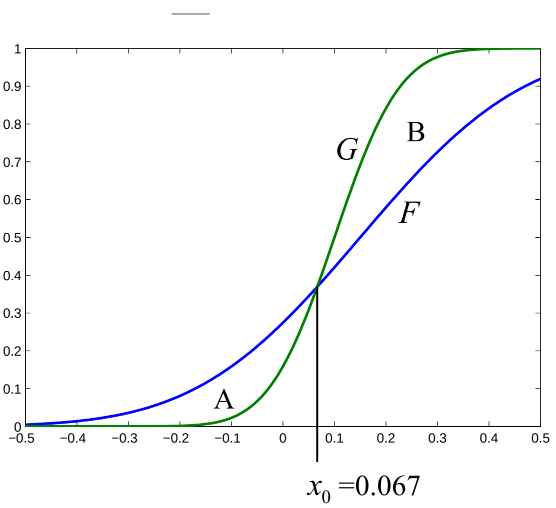

Figure 4c: The cdf of two normal distributions F ($E_F = 15\%, \sigma_F = 25\%$) and ($E_G = 10\%, \sigma_G = 10\%$)

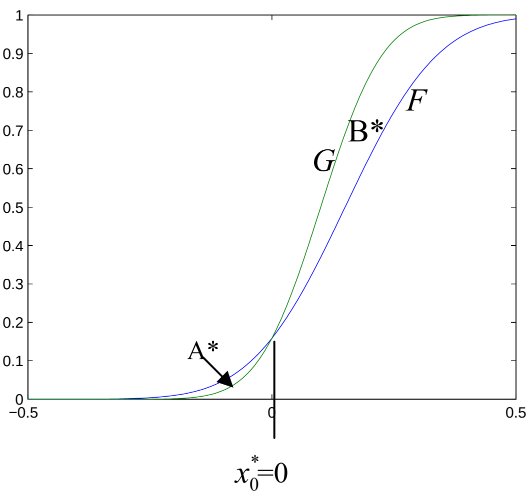

Figure 4d: The cdf of two normal distributions F ($E_F = 15\%, \sigma_F = 15\%$) and G ($E_G = 10\%, \sigma_G = 10\%$)

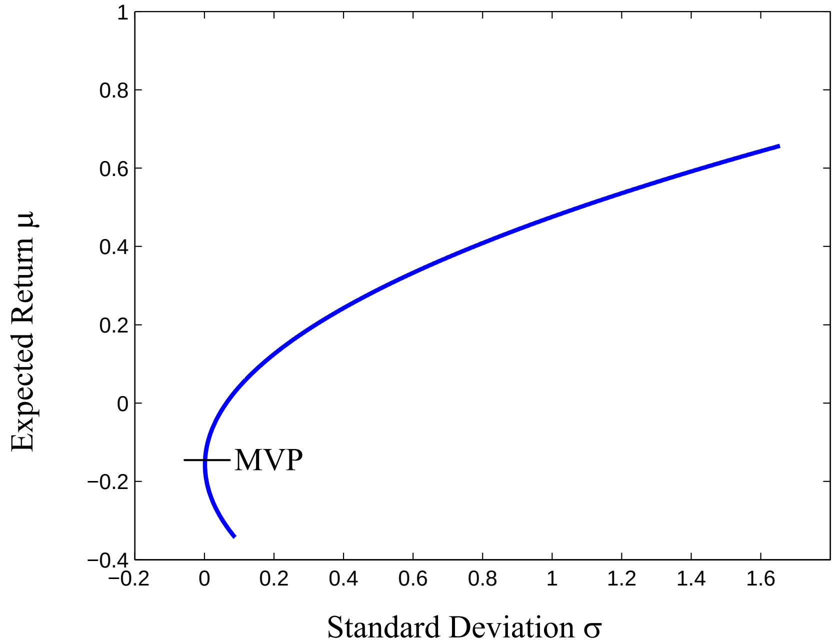

Figure 5a: The MV efficient frontier.

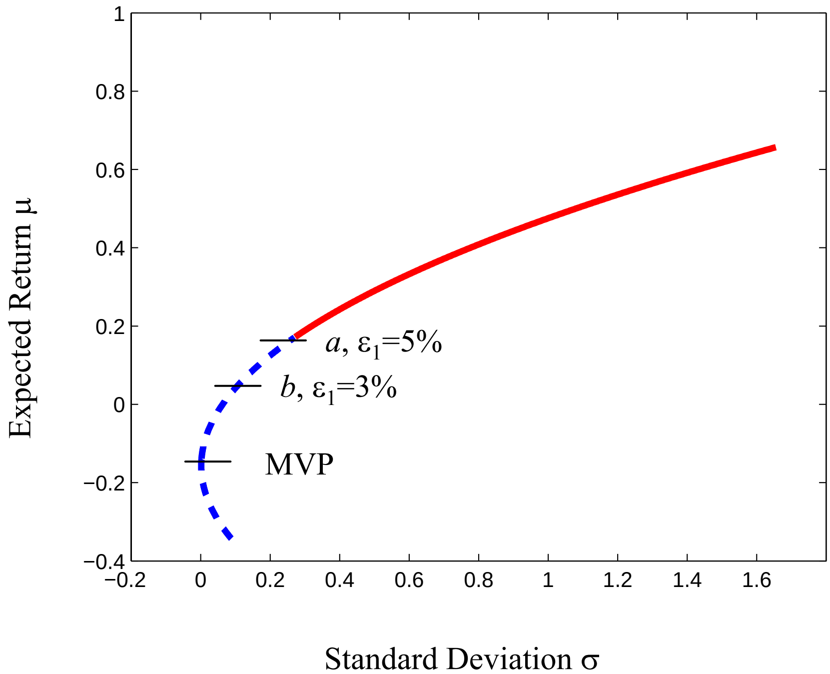

Figure 5b: The AMV efficient frontier: Illustrating the reduction of the MV efficient set.

Appendix A

In this Appendix we show the relationship between ε_i and $U_i^\star$ $(i = 1, 2)$ for AFSD and ASSD respectively, where ε_i $(i = 1, 2)$ are the *actual* relative area violation corresponding to AFSD and ASSD, respectively.

Almost FSD (AFSD)

For any two cdf F and G we have,

$$E_F(u) = \int_a^b u(x)dF(x) = [u(x)F(x)]_a^b - \int_a^b u'(x)F(x)dx = u(b) - \int_a^b u'(x)F(x)dx.$$

Therefore

$$\begin{aligned} \triangle \equiv E_F(u) - E_G(u) &= \int_a^b u'(x)[G(x) - F(x)]dx = \\ &= \int_{S_1} u'(x)[G(x) - F(x)]dx + \int_{\overline{S_1}} u'(x)[G(x) - F(x)]dx, \end{aligned}$$

where over S_1, $F(x) > G(x)$ and $\overline{S_1}$ is the compliment of S_1. Thus, $\triangle > 0$ iff

$$-\int_{S_1} u'(x)[G(x) - F(x)]dx = \int_{S_1} u'(x)[F(x) - G(x)]dx < \int_{\overline{S_1}} u'(x)[G(x) - F(x)]dx$$

This inequality holds if,

$$\sup\{u'(x)\} \int_{S_1} [F(x) - G(x)]dx < \inf\{u(x)\} \int_{\overline{S_1}} [G(x) - F(x)]dx$$

or

$$\sup\{u'(x)\} < \inf\{u(x)\} \frac{\int_{\overline{S_1}} [G(x) - F(x)]dx}{\int_{S_1} [F(x) - G(x)]dx}$$

Define

$$\varepsilon_1 = \frac{\int_{S_1} [F(x) - G(x)]dx}{\int_{S_1} [F(x) - G(x)]dx + \int_{\overline{S_1}} [G(x) - F(x)]dx}$$

To obtain the condition for $\triangle > 0$ if

$$\sup\{u'(x)\} < \inf\{u'(x)\}\left[\frac{1}{\varepsilon_1} - 1\right]$$

Where ε_1 is the actual relative area violation that is determined solely by F and G.

For a given area violation ε_1 all utility functions obeying the above inequality belongs to $U_1^{\star} \subset U_1$. Finally in equation (5) we write $u'(x)$ rather than $\sup\{u'(x)\}$ because the above inequality holds with $\sup\{u'(x)\}$ if and only if its holds for all $u'(x)$.

Almost SSD (ASSD)

Leshno & Levy [2002] derive ASSD by imposing constraints on u''. In this paper we derive ASSD by imposing constraint on u'. We can add the constraint on u'', hence with u' and u'' on can obtained a better theoretical results relative to the one reported by Leshno & Levy [2002].

For any two cdf F and G we have,

$$E_F(u) = \int_a^b u(x)dF(x) = [u(x)F(x)]_a^b - \int_a^b u'(x)F(x)dx = u(b) - \int_a^b u'(x)F(x)dx.$$

Therefore

$$\begin{aligned}\triangle \equiv E_F(u) - E_G(u) &= \int_a^b u'(x)[G(x) - F(x)]dx = \\ &= \int_{S_2} u'(x)[G(x) - F(x)]dx + \int_{\overline{S_2}} u'(x)[G(x) - F(x)]dx,\end{aligned}$$

where S_2 is defined by equation (2) (the region which if eliminated we would have SSD of F over G), and $\overline{S_2}$ is the compliment of S_2. Obviously S_2 is a subset of S_1 (see equations (1) and (2)). Note that as $u \in U_2$ the integral over $\overline{S_2}$ is non-negative (see equation (2) and

Figure 3). However, if $u \notin U_2$ than this claim is incorrect and the integral over S_2 may be negative. Thus, for $u \in U_2$ $\triangle > 0$ iff,

$$-\int_{S_2} u'(x)[G(x) - F(x)]dx = \int_{S_2} u'(x)[F(x) - G(x)]dx < \int_{\overline{S_2}} u'(x)[G(x) - F(x)]dx$$

This inequality holds if and only if,

$$\sup\{u'(x)\} \int_{S_2} [F(x) - G(x)]dx < \inf\{u(x)\} \int_{\overline{S_2}} [G(x) - F(x)]dx.$$

As $u \in U_2$ (i.e. $u'' < 0$) the above condition is reduced to

$$u'(a) < u'(b) \frac{\int_{\overline{S_2}} [G(x) - F(x)]dx}{\int_{S_2} [F(x) - G(x)]dx}$$

Define the relative *actual* SSD area violation ε_2 as follows,

$$\varepsilon_2 = \frac{\int_{S_2} [F(x) - G(x)]dx}{\int_{S_2} [F(x) - G(x)]dx + \int_{\overline{S_2}} [G(x) - F(x)]dx}$$

To obtain the condition for $\triangle > 0$ if

$$u'(a) < u'(b) \left[\frac{1}{\varepsilon_2} - 1 \right]$$

Appendix B: The p% Almost FSD

From Task 6 in experiment 1, it is obvious that for a very large z, say, $z = \$10^6$ all investors would prefer B over A. However, an investment consultant may be interested to serve his clients also with actual prospects with lower and more realistic value z. If there is a prospect with, say, $z = \$800$, she may recommend to her clients to invest in B and not in A, though B dominates A by AFSD only for 97.8% of the population (see Table 2b). Thus, there is a trade off between the minimum required value z and the percentage of the population for which the AFSD is intact. To illustrate these relationships suppose that indeed our sample represents the whole population of decision makers. We can define a decision rule corresponding to 95% of the population but not all of them. This rule will correspond to $\varepsilon_{1,95\%}$ and the corresponding bounded utility class is denoted by $U^{\star}_{1,95\%}$. Similarly, we may decide that covering p =90% of the population is sufficient. In this case we will have two values $U^{\star}_{1,90\%}$ and $\varepsilon_{1,90\%}$ corresponding to p =90% of the decision makers. Obviously, the higher p, the lower the corresponding ε_1 and the larger the class $U^{\star}_{1,p}$. Let us demonstrate this idea with the results of Table 2b. For simplicity, and without loss of generality we illustrate this idea with a piece-wise linear utility function, denoted by u, given in Figure B-1. We denote the set of all utilities showing a preference of B over A by $U^{\star}_1$. First note that the utility u_1 (see Figure B-1) does not belong to $U^{\star}_1$ as regardless of the selected value z, B can not be preferred to A with such a utility function[1]. Thus, there is no $0 < \varepsilon^{\star} < 0.5$ that reveals a preference for B with u_1 and therefore $u_1 \notin U^{\star}_1$.

Now suppose that we have a utility function like u_2 in Figure B-1 where the slope of u_2 corresponding to segment c is a, $0 < a < 1$. In this case $Eu(B) > Eu(A)$ if the following holds for $z > \$200$:

$$\frac{1}{2}(50) + \frac{1}{2}[200 + a(z - 200)] > \frac{1}{2}100 + \frac{1}{2}200$$

[1]Namely, with u_1 we have $\frac{1}{2}u(100) + \frac{1}{2}u(200) > \frac{1}{2}u(50) + \frac{1}{2}u(z)$ for any selected value z.

Namely,

$$a(z - 200) > 50$$

or

$$a > \frac{50}{z - 200}$$

Thus, the slope of segment c (see Figure B-1) which guarantees dominance of B over A is a function of the selected z as calculated from the above inequality (and of the value ε_1 which is determined also by z). Also in order to have $a < 1$ we must have that $z \geq 250$. Table B-1 presents the selected value z, the corresponding slope a and the percentage of the subjects selecting value z or less.

We see from Table B-1 that if we want *all* investors (p =100%) to prefer B over A we need to have $a \geq 0.06$. Suppose that the slope of line c is 0.06 (see Figure B-1). Then all utility functions with a slope below 0.06 are not allowed. Hence $U^{\star}_{1,p=100\%} \subset U_1$ as $U^{\star}_{1,p=100\%}$ excludes all utility functions enclosed between u_1 and u_2 (see Figure B-1), and U_1 contains all these functions because $u' \geq 0$ as required. With this respect we say that $U^{\star}_1$ is a bounded set, which in this specific example the restriction is that all utility functions with a slope of less than 0.06 are not included in $U^{\star}_1$.

Now suppose we settle for a decision rule corresponding to 97.8% of the population. Namely, we would like to have a decision rule (AFSD) suitable for $U^{\star}_{1,97.8\%}$, i.e., suitable for 97.8% of the population. Then, if in option B we have z = \$800 we would assert that B dominates A by AFSD, which is suitable for 97.8% of the population. By the same token, if p =86.7%, we would say that B is preferred over A for all $u \in U^{\star}_{1,86.7\%}$ and p =86.7% corresponds to z = \$400[2]. By employing the two options A and B with z = \$1000 or more it is obvious that *all* (i.e. 100%) investors participating in our experience would prefer B over A. The subjects' behavior in our experiment lends support to the hypothesis that, in our specific example of piecewise preferences that utility functions with a slope of segment c

[2]Note that 11.1%+2.2%=13.3% of the subjects selected $z > 400$, hence for z = \$400, B dominates A by 100%-13.3%=86.7% of the population (see Table 2b).

lower than 0.06, although mathematically allowed, are irrelevant economically because they do not conform to any of the investors' preferences. Therefore, in the above specific example we allow risk aversion but we do not allow an extreme risk aversion as illustrated by utility function, u_1 or any utility function with a slope given by the shaded area in Figure B-1. Thus, with FSD we allow all functions $u \in U_1$ and with AFSD corresponding to our experiment, we disallow functions with a slope smaller than u_2 (see Figure B-1). If we would like to have a decision rule for say p% of the population we disallow more functions, hence we obtain that all functions with a slope below line p are disallowed. The smaller the percentage of the population, p, for which there is a dominance, the higher the slope of line p (see Figure B-1), and more utility functions are disallowed. Therefore, we have the following relationship:

$$U^{\star}_{1,p_2} \subset U^{\star}_{1,p_1} \subset U_1 \quad \text{for } 100\% > p_1\% > p_2\%$$

Thus, the smaller the allowed $\varepsilon^{\star}_1$ by switching from FSD to AFSD, the closer $U^{\star}_1$ to U_1. We demonstrated the relationship between p, $U^{\star}_1(p)$ with the above piecewise linear utility function. Obviously, the same relationship holds for any smooth utility function, e.g., $u(x) = \frac{x^\alpha}{\alpha}$. In this case the smaller α the more risk averse are the investors, hence it is possible, that for p =95%, say values of $\alpha \leq 0.01$ are not allowed and for p =90%, values of $\alpha \leq 0.1$ are not allowed. The same arguments given with the piecewise linear utility holds also in the case $u(x) = \frac{x^\alpha}{\alpha}$ or for that matter any other utility functions. Table B-1 presents the *minimum* value α for which B is preferred over A for the utility function $u(x) = \frac{x^\alpha}{\alpha}$[3].

[3] As FSD corresponds to concave as well as convex functions, it can be easy shown that also some convex functions are eliminated from U_1 to avoid paradoxes. Thus $U^{\star}_1 \subset U_1$ where some concave as well as convex functions which may create paradoxes are eliminated

Table B-1: The Value z, a, α where $p\%$ of the subjects prefer B over A in Task 1[a]

Selected z	The minimum slope of the utility, a (piece-wise linear utility)	The minimum value α with $(u(x) = \frac{x^\alpha}{\alpha})$	Percentage of subjects selecting z or (less) (in %)
250	1	-	22.8
275	0.67	0.76	29.5
300	0.50	0.60	64.5
350	0.33	0.43	71.7
400	0.25	0.33	86.7
800	0.08	0.10	97.8
1000	0.06	0.07	100.0

a) Let us explain how the values α given in Table B-1 are calculated. Given $\varepsilon_1^\star$ and z we have that in range $[50, z]$ $\sup u'(x) = 50^{\alpha-1}$ and $\inf u'(x) = z^{\alpha-1}$. The smallest value of α that obey inequality given in (5) is when we have an equality in (5). Thus, we have that $(\alpha - 1)\log(50) = (\alpha - 1)\log(z) + \log\left(\frac{1}{\varepsilon_1^\star} - 1\right)$ or $\alpha = 1 + \frac{\log\left(\frac{1}{\varepsilon_1^\star} - 1\right)}{\log\left(\frac{z}{50}\right)}$. Thus, for a given z, $\varepsilon_1^\star$ is determined, hence α can be calculated.

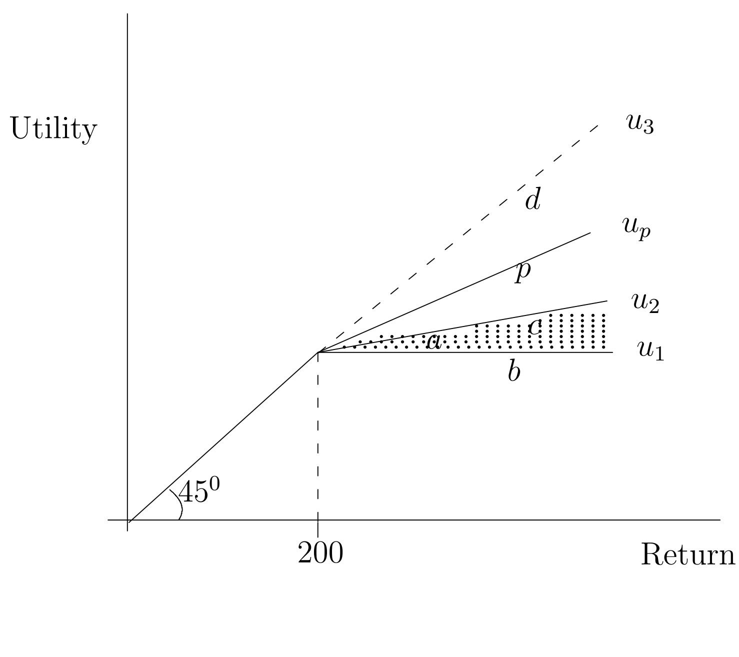

Figure B-1: The Cumulative distribution functions of A and B in Task II