

## On Determination of Dominated and Dominating Portfolios<sup>\*</sup>

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### Abstract

Under Second-Order Stochastic Dominance (SSD), Yitzhaki-Mayshar (1997) and Post (2003) identify the absence or existence of marginal portfolio allocation improvements. We highlight the marginal nature of these authors' SSD (MSSD) methods through a simple example, and through reassessment of Post's (2003) Fama-French size/book-to-market investment application. As currently implemented, the Yitzhaki-Mayshar (1997) and Post (2003) procedures don't identify an alternative portfolio that dominates the initial allocation under SSD. To identify a dominating portfolio choice, we follow Ang (1975) and Bawa (1978) to develop a First-Order Lower Partial Moment second-order stochastic dominance-based algorithm. Providing increased utility relative to MSSD, we call our approach Supra-Marginal second-order stochastic dominance (SMSSD). In applications, it is beneficial to use both methods in concert.

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## On Determination of Dominated and Dominating Portfolios

### Abstract

Under Second-Order Stochastic Dominance (SSD), Yitzhaki-Mayshar (1997) and Post (2003) identify the absence or existence of marginal portfolio allocation improvements. We highlight the marginal nature of these authors' SSD (MSSD) methods through a simple example, and through reassessment of Post's (2003) Fama-French size/book-to-market investment application. As currently implemented, the Yitzhaki-Mayshar (1997) and Post (2003) procedures don't identify an alternative portfolio that dominates the initial allocation under SSD. To identify a dominating portfolio choice, we follow Ang (1975) and Bawa (1978) to develop a First-Order Lower Partial Moment second-order stochastic dominance-based algorithm. Providing increased utility relative to MSSD, we call our approach Supra-Marginal second-order stochastic dominance (SMSSD). In applications, it is beneficial to use both methods in concert.

For investor's manifesting nonsatiation and risk aversion (i.e. the second-order stochastic dominance problem – SSD), dominated portfolio choices among discretely distributed investment returns have been investigated by Yitzhaki-Mayshar (YM-1997), and Post (2003).<sup>1</sup> YM follows a marginal distributional approach to SSD, while Post works with necessary and sufficient utility function restrictions.<sup>2</sup> Both methods reduce to large-scale linear programming problems.

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<sup>1</sup> Assuming only non-satiation, Quirk-Saposnik (1962) and Fishburn (1964) developed First Order Stochastic Dominance. Hadar-Russell (1969, 1971), Hanoch-Levy (1969), and Rothschild-Stiglitz (1970, 1971) provided alternative developments of second-order stochastic dominance. Whitmore (1970) introduced Third-Order Dominance, and Vickson (1975) treated decreasing absolute risk aversion. Levy (1992) provides a review of Stochastic Dominance. Fishburn (1974) developed Convex Stochastic Dominance (CSD) to identify optimal choices among mutually exclusive alternatives, and the Bawa et. al. (1985) algorithm determines First-, Second- and Third-Order CSD choices.

<sup>2</sup> Kuosmanen (2004) provides mixed integer and linear programming problem specifications for first-order stochastic dominance, an alternative linear program for second-order stochastic dominance and a linear program for the third-order stochastic dominance problem (as does Post.) As noted by Post (2003), Kuosmanen's second-order stochastic dominance specification will be relatively inefficient for large portfolio problems. However, transformations to his equation 8) should yield problems that scale to the same size as the Post specification.

Post takes his development one step further to identify utility functions associated with supramarginal utility. However, the Post algorithm doesn't fully generalize in this manner.<sup>3</sup> We show through two examples, one simple and the other using Post's sample, that his implementation identifies an alternative allocation that does not dominate the original allocation under test. Therefore, some investors will not prefer the Post portfolio reallocation from the dominated alternative to the identified alternative. Based in Ang (1975) and Bawa (1978), we develop an alternative algorithm that identifies dominating and supramarginal portfolio reallocations.

The paper is organized as follows: First, we summarize Post's (2003) efficiency test specifications. Next, we provide a simple example that highlights the marginal nature of the YM (1997) and Post (2003) tests. We call these tests marginal second-order stochastic dominance, MSSD. Third, we reexamine Post's market efficiency test relative to the Fama-French size/book-to-market portfolios, and show that his reallocation does not dominate the market portfolio. Nevertheless, MSSD implies that such a portfolio must exist.

Therefore, in our fourth section, we adapt Bawa's (1978) mean and first-order lower partial moment ( $LPM_1$ ) portfolio selection procedure to identify both admissible and dominating portfolio investment alternatives. Improving on the initial allocation, we call this method supramarginal second-order stochastic dominance (SMSSD). Our fifth section outlines a hybrid MSSD-SMSSD portfolio efficiency test, and applies a dual specification of Bawa et. al. (1985) to identify a supramarginal choice that minimizes the relative expected disutility function of the dominated alternative. Section six concludes.

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<sup>3</sup> Given YM's and Post's conditional-marginal implementations, only the existence of the conditionally optimal utility function follows from Afriat (1967), not a globally optimal choice. The optimal utility function follows for a globally optimal allocation.

## 1. SSD Portfolio Dominance Specifications

For portfolio investment, we assume risk-averse,  $U_2$ , utility functions that are continuously differentiable, increasing and strictly concave. An investor's wealth is positive,  $w$ . There are  $N$  investment choices represented by return vectors,  $x \in \mathbb{R}^N$ . Investment returns are state-contingent,  $t$ , and the number of states is discrete and finite,  $T$ . The associated multinomial distributions are ordered by the returns on the portfolio being evaluated,  $(\Theta \equiv \{1, \dots, T\})$ . Initial portfolio allocations over investment returns,  $\alpha$ , sum to one.<sup>4</sup>

In his test application, Post treats discrete empirical distributions. Therefore, and without loss of generality, the probability of each unique state outcome is  $1/T$ .

The Post (2003) linear program (eq. P) is the following:

$$\begin{aligned}
 & \text{Min } \theta \\
 \text{s.t. } & \sum_{t=1}^T \beta_t (x_{\alpha t} - x_{it})/T + \theta \geq 0 \quad \forall i = \{1, \dots, N\} \quad [\lambda_i] \\
 & \beta_{t-1} - \beta_t \geq 0 \quad t = 1, \dots, T \quad [\rho_t] \\
 & \beta_t \geq 0 \quad t = 1, \dots, T \\
 & \beta_T = 1 \\
 & \theta \text{ free}
 \end{aligned} \tag{1}$$

If this LP optimand,  $\theta$ , is zero and the allocation is on the interior of the feasible set, then the allocation is efficient.<sup>5</sup> Otherwise, at least one positive artificial variable is needed for a feasible inequality set. In this case, the  $\alpha$  allocation under test is inefficient and dominated. To relate Post's primal and dual specifications, the constraint shadow prices are noted within brackets in 1).

<sup>4</sup> See both Yitzhaki-Mayshar (2003) and Post (2003) for technical details. Post (2003) also treats the no short sales case and utility functions that are not continuously differentiable.

<sup>5</sup> Post (2003) does not add the interior condition to his specification. We show in our example footnotes that the restriction is necessary.

Post's corresponding dual specification follows (Post, eq. D):

$$\begin{aligned}
& \text{Max}_{\lambda, \rho_1, \dots, \rho_{T-1}} \left( \sum_{i=1}^N \lambda_i x_{it} - x'_T \alpha \right) / T + \rho_{T-1} \\
& \text{s.t.} \quad \left( x'_1 \alpha - \sum_{i=1}^N \lambda_i x_{i1} \right) / T + \rho_1 \leq 0 \quad [\beta_1] \\
& \quad \left( x'_t \alpha - \sum_{i=1}^N \lambda_i x_{it} \right) / T + \rho_t - \rho_{t-1} \leq 0 \quad t = 1, \dots, T-1 \quad [\beta_t] \\
& \quad \sum_{i=1}^N \lambda_i = 1 \quad [\theta] \\
& \quad \lambda_i \geq 0 \quad i = 1, \dots, N \\
& \quad \rho_t \geq 0 \quad t = 1, \dots, T-1
\end{aligned} \tag{2}$$

The dual problem optimand and return-based constraints match Bowden's (2000) Ordered Mean Difference (OMD) Criterion. The dual specification provides the portfolio allocations associated with the primal maximal utility. Post's algorithm tests for expected utility function improvements.<sup>6</sup>

Though both the Post and YM specifications may separate dominated and locally efficient allocations, neither seeks to identify preferred allocations nor either is certain to identify such alternatives without additional constraints and analysis. Most importantly, Post's supramarginal extension for separating dominated and conditionally efficient allocations can identify reallocations that do not dominate the evaluated alternative. We highlight this issue in two examples: One example is simple, and the other follows Post's (2003) market portfolio efficiency test.

## 2. A Simple Example

We consider two investment alternatives (N=2) and three state outcomes (T=3). Each state is equally likely. The investor is initially allocated investments with returns of

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<sup>6</sup> When the investment reallocation is in the interior of feasible investment allocations, YM's test is equivalent to that of Post. Specifically, the YM Accumulated Concentration Curve (ACC) aggregates the OMD. These results are available on request. Shalit-Yitzhaki (2004) conduct an efficiency test over three investment choices and ten initial allocations. All allocations are shown to be efficient. However, YM don't seek to identify supramarginal alternatives or the associated portfolio allocations.

{1,2,3}. This ordered investment outcome set defines the conditional allocation under test,  $\alpha=[1,0]$ . The alternative that may be diversified into has returns of {7,0,0}. The expected return of the initial allocation is 2, and the expected return of the alternative is 2 1/3.

It is beneficial to diversify into the alternative, and the Post LP formulation solution points in the right direction. Nevertheless, the solution for this investor over steps: All investment is shifted into the alternative,  $\lambda=[0,1]$ .<sup>7</sup> The Post LP solution is equal to the mean difference, 1/3.

To further examine this solution, we calculate the second-order stochastic dominance lower partial moments for both alternatives. The relevant lower partial moment is of the first-order. From Bawa et. al. (1985 – eq. 8), the discrete distribution first-order lower partial moment to a point t in the domain,  $L_{it}$ , is the following:

$$L_{it} = \sum_{s|x_{is} \leq t} (t - x_{is})/T = tF_{it} - \sum_{s=1}^t x_{is}/T \quad 3)$$

$F_{it}$  is the cumulative distribution up to return t.<sup>8</sup>

Bawa (1975) shows that this first-order  $LPM_1$  is a necessary and sufficient statistic for determining second-order stochastic dominance. Specifically, an alternative is dominated by another feasible investment if and only if the  $LPM_1$  of the second investment is everywhere equal to or below that of the first investment under evaluation, and below at minimally one point.

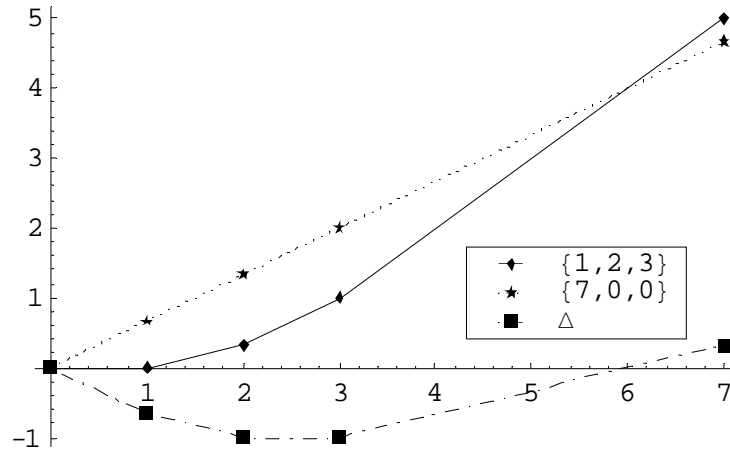
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<sup>7</sup> This {7,0,0} return vector allocation is in itself conditionally efficient. A relatively less risk-averse investor would choose this investment and not diversify. Nevertheless, a more risk-averse investor will prefer a more diversified alternative from both return vectors {1,2,3} and {7,0,0}. In the sequel, we identify an alternative to the initial {1,2,3} investment that is preferred for all investors.

<sup>8</sup> Shorrocks' (1983) Absolute Lorenz Curve-based analysis yields the same dominance and efficiency conclusions.

Figure 1 plots both the  $LPM_1$  for the two example investments, and the LPMs difference:

**Figure 1**  
 $LPM_1$



We see that the  $LPM_1$ s cross. Were we testing an allocation that was on the interior of the feasible allocation set, then the fact that the Post algorithm moves away from the zero optimand indicates that some allocation dominates the initial allocation for all investors. Any investor seeks such a specific improved alternative.<sup>9</sup>

### 3. The Fama-French Size/Book-to-Market and Market Portfolio Example

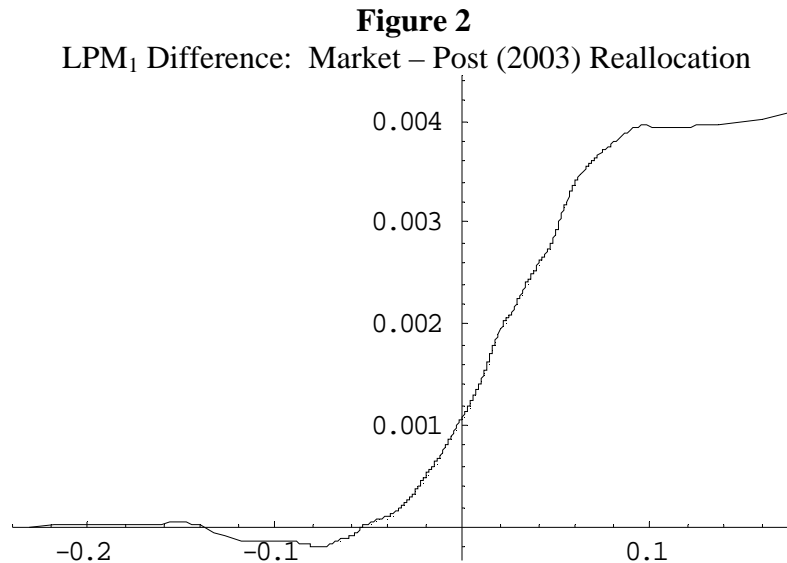
Post (2003) evaluates the CRSP market value portfolio. He tests its efficiency against reallocations into any or all of the Fama-French size and book-to-market

<sup>9</sup> Such an initial allocation is all investment in only one alternative. In this case, the Post LP can solve for a zero optimand to indicate admissibility, yet the choice is SSD dominated. The following multinomial return set has this result: An initial allocation over four states is  $\{10, 1, 0, 0\}$ , and diversification alternatives are  $\{1, 2, 3, 4\}$  and  $\{0, 1/2, 0, 17\}$ . Portfolios of these two alternatives dominate the initial allocation for a wide range of investment weights. Yet, the Post LP optimal solution is zero indicating that  $\{10, 1, 0, 0\}$  is an admissible initial allocation.

portfolios and the market portfolio itself.<sup>10</sup> Based on non-zero solution of his LP, equations 1) or 2), he states that the market portfolio is dominated. Nevertheless and as in our simple example, the market portfolio is not dominated by the allocation that Post identifies.<sup>11</sup> Therefore, an investor holding the market has no direction on how they should reallocate their dominated portfolio.

The Post algorithm identifies a market portfolio dominating reallocation of 31.6% into the first size quintile and fifth book-to-market quintile portfolio with the remaining 68.4% into the fourth size quintile and fourth book-to-market quintile portfolio.

Figure 2 plots the  $LPM_1$  for the market portfolio and this reallocation. The  $LPM_1$  difference crosses zero, and dominance for all investors is not manifest. Some  $U_2$  investors would prefer the initial allocation.

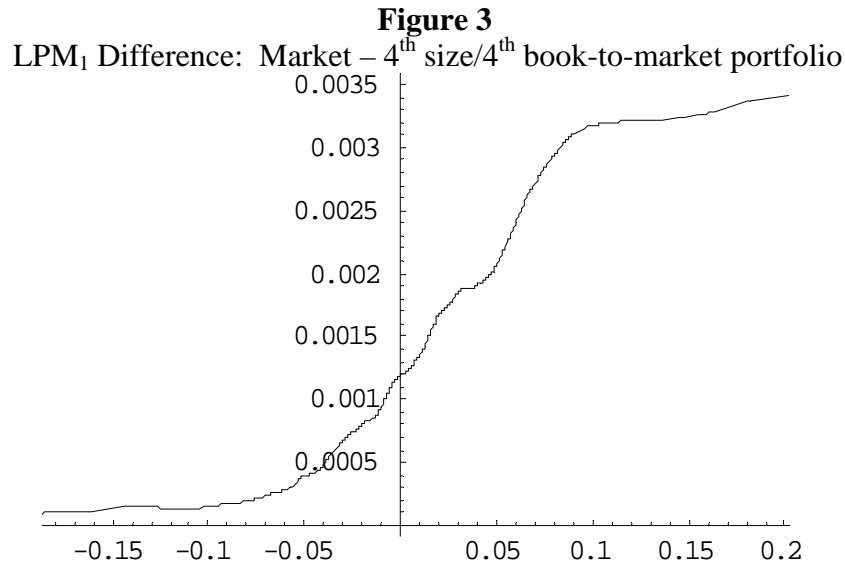


<sup>10</sup> [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/ftp/25\\_Portfolios\\_5x5\\_.zip](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/ftp/25_Portfolios_5x5_.zip). The CRSP market portfolio and the Ibbotson SBBI one-month riskless rates are also used.

<sup>11</sup> As mentioned, since the market portfolio is not an aggregate of the Fama-French portfolios, it is a corner allocation. In the general corner allocation case, a non-zero optimand doesn't guarantee that a dominating allocation exists. However in this sample, many do exist.



Interestingly, the fourth-size quintile and fourth book-to-market quintile portfolio alone dominates the market portfolio. Figure 3 plots the differential  $LPM_1$  for this portfolio and the market portfolio:



In addition to identifying a non-dominating alternative, Post’s algorithm does not identify an allocation that dominates the market for all investors. To identify dominating-supramarginal allocations, we propose an  $LPM_1$ -based efficiency test.

#### 4. Supramarginal Portfolios: An $LPM_1$ -based Test

Bawa (1978) extends Ang (1975) to specify an  $LPM_1$ -based optimal portfolio allocation procedure. Given an expected return target, the optimal portfolio is the one that minimizes a particular  $LPM_1(t)$ . The choice of the risk tolerance point,  $t$ , is critical to the algorithm. Bawa selects the riskless rate as the tolerance level, and Ang chooses the mean. In Ang’s case, his procedure is an approximation to Hogan-Warren’s (1972) semi-variance minimization for a given mean.

By following the Post/YM problem logic, risk-tolerance level choice is finessed: Given an initial allocation, we only identify improved portfolio allocations. As in their

case, there is no guarantee that an improved allocation will be globally admissible or optimal. Nevertheless, dominating portfolios with significant performance improvements may be so identified. In the context of the expected return-risk tradeoff, we maximize expected return for a particular LPM<sub>1</sub>-defined risk vector.

Our LPM<sub>1</sub>-based test is evaluated at some or all of the T discrete points observed. Therefore, we will test necessary, but not sufficient, dominance conditions. An alternative that is dominated by our test may be admissible. For particular alternatives, we eliminate this possibility by using SSD to compare the original allocation directly against the alternative reallocation.

The LPM necessary conditions are the following:

$$L_{it} = \sum_{s|x_{is} \leq t} (t - x_{is})/T \geq L_{\lambda t} = \sum_{s|\sum_{i=1}^N \lambda_i x_{is} \leq t} \left( t - \sum_{i=1}^N \lambda_i x_{is} \right) / T \quad \forall t \in \Theta \quad 4)$$

Hogan-Warren (1972) shows that the portfolio LPM,  $L_{it}$ , is convex.<sup>12</sup> Therefore, this set of inequalities may be evaluated with convex programming methods. Importantly, the convex program may also be formulated as a linear program.

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<sup>12</sup> Their proof is for the Semi-Variance, a special-case of the LPM, and their proof generalizes directly.



#### 4.1 Linear Programming Specification

To specify the linear program, Max functions are defined by sets of linear inequalities. In the simple example, we had two assets (N=2) and three outcomes (T=3).

The associated linear program is the following:<sup>13</sup>

$$\begin{array}{l}
 \text{Min} \\
 \left. \begin{array}{l} y=[y_1, y_2, y_3] \\ \Lambda=[\lambda_1, \lambda_2] \\ M=[m_{11}, \dots, m_{33}] \end{array} \right\} e'y \\
 \\
 \text{s.t.} \\
 \left[ \begin{array}{cccccccccccccccc}
 1 & 0 & 0 & \bar{x}_1 & \bar{x}_2 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & \bar{x}_1 & \bar{x}_2 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & \bar{x}_1 & \bar{x}_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\
 0 & 0 & 0 & x_{11} & x_{21} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & x_{12} & x_{22} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & x_{13} & x_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & x_{11} & x_{21} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & x_{12} & x_{22} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & x_{13} & x_{23} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & x_{11} & x_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & x_{12} & x_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & x_{13} & x_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & \bar{x}_1 & \bar{x}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \right] \geq \left[ \begin{array}{l} -L_{11}+x_{11} \\ -L_{12}+x_{12} \\ -L_{13}+x_{13} \\ -x_{11} \\ -x_{11} \\ -x_{11} \\ -x_{12} \\ -x_{12} \\ -x_{12} \\ -x_{13} \\ -x_{13} \\ -x_{13} \\ 1 \\ -1 \\ -1 \\ -1 \\ \bar{x}_1+\mu \end{array} \right] \quad 7)
 \end{array}$$

In this specification constraint set, the first three rows correspond to the T LPM<sub>1</sub>-based constraints in the nonlinear constraint set of 5). The next nine constraint rows introduce artificial variables to ensure that (N x T) Max conditions are satisfied. The last five constraints again follow specification 5): The allocation sum equals one, and each of N allocations are greater than zero and less than or equal to one. The last constraint ensures that an alternative allocation has a mean greater than or equal to the mean of the allocation under test.

If this minimization problem is infeasible for a zero  $\mu$  parameter, then the allocation under test is not dominated. Otherwise, a dominating portfolio with allocation

<sup>13</sup> In our applications, it is computationally more efficient to switch the signs of the y variables from positive to negative. In this case, a non-positive optimand value indicates a dominated allocation.

$[\lambda_1, \lambda_2]$  is preferred for all risk-averse investors. Relative to the Post and Yitzhaki-Mayshar algorithms, we identify our specifications as supramarginal second-order stochastic dominance, SMSSD.

For our simple example, the largest possible reallocation is 40%. This allocation results in returns of {3.4, 1.2, 1.8} and 2.133 expected return. These returns compare with the initial allocation of {1, 2, 3} and 2.0 expected return.

In general, the size of the constraint set is  $(T+N+TxT) \times (T+TxT+N+3)$ . For Post's (2003) data set, the specification has  $N=26$  and  $T=460$ , and the constraint matrix dimension is  $212089 \times 212086$ . Though the matrices are sparse, the calculation load is significant.

To lighten the computation load, an approximation suggested by Bawa-Goroff-Whitt (1977) and Goroff-Whitt (1980) is very useful. Financial return data is often characterized by close to two parameter-location/scale distributions. Location/scale distributions include the normal, T and stable. If the two-parameter distribution approximation is close, then only a few LPM may be needed to ascertain dominance. Though this simplification is heuristic for general distributions, it works well for both of our examples.

For the simple three-outcome example, the middle data point need not be treated. In specification 7), we drop the  $y_2, m_{21}, m_{22}, m_{23}$  variables, as well as the second, seventh, eighth and ninth constraint rows. Again, the 40% reallocation is the largest expected return dominating-supramarginal portfolio choice.

For the Post data set, we specifically augment our algorithm to constrain dominance in subsets of the domain. First, we solve LP problem 7) for only the least and

greatest returns among all 460 multinomial returns. We check for the worst dominance violation, and this occurs for the 18<sup>th</sup> ordered market return.

Next, we add variables and constraints for this, the 18<sup>th</sup> ordered, return point. We rerun the algorithm, and the 6<sup>th</sup> and 7<sup>th</sup> points have the largest dominance violation. We add variables and constraints for the 6<sup>th</sup> point, but the LPM difference is still negative at the 7<sup>th</sup> point. We treat this point, and the LP solution with this constraint set provides SSD over the market portfolio.

The allocation identified is 95.4% in the 4<sup>th</sup> size quintile and 4<sup>th</sup> book-to-market quintile portfolio, with 4.6% in the 5<sup>th</sup> size quintile and 4<sup>th</sup> book-to-market quintile portfolio. There is a 0.34% average monthly return increase,  $\mu$ , over the 0.45% monthly market average return.

To find the best supramarginal allocation, we increase the target mean parameter,  $\mu$ , from 0.34% monthly. This parameter may be increased to 0.3905%, and the associated reallocation is 84.9% in the 4<sup>th</sup> size quintile and 4<sup>th</sup> book-to-market quintile portfolio, and 15.1% in the 1<sup>st</sup> size quintile and 5<sup>th</sup> book-to-market quintile portfolio.<sup>14</sup>

Importantly, our mean estimate is also a statistically admissible dominance criterion. In the Bayesian framework of Ferguson (1974) and Bawa-Brown-Klein (1979), our estimate rejects market portfolio efficiency. This framework begins with a diffuse Dirichlet process prior for the multinomial discrete portfolio return outcomes. Since this prior is conjugate to the multinomial empirical distribution of return observations that we treat, the positive posterior optimizing estimated mean return improvement indicates

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<sup>14</sup> Like Post (2003), we ascertain sufficient transaction costs to result in market portfolio admissibility. A 0.29% or greater transaction cost results in no dominance of the market portfolio. In this situation, an additional return, the 393<sup>rd</sup>, and the associate variables and constraints must be added to the LP simplex to ensure SSD admissibility.

dominance.<sup>15</sup>

Though it is beyond the scope of this work, frequentist significance tests of our estimates may be implemented. Developing an LPM admissibility-based bootstrap specification is analogous to Post's (2003) tests.<sup>16</sup>

## 4.2 A Relative Expected Utility Function

Post's primal MSSD algorithm, 1), identifies a piecewise linear utility function associated with an undominated choice. In line with this construct, we identify a utility function that minimizes the expected disutility of the dominated choice relative to the dominating-supramarginal alternative.

Fishburn-Vickson (1975, pg. 76, eq. 2.42) provides an expected utility function construction for the case in which the minimum of one discrete point provides admissibility. Given more than a singleton undominated point, multiple utility functions will choose the undominated choice.<sup>17</sup>

As a specific alternative, the dual of the Bawa et. al. (1985) Convex Second-Order Stochastic Dominance (CSSD) algorithm will identify a relative maximum expected

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<sup>15</sup> The result follows Ferguson (1974) Theorem 1 and Example 2. Bawa (1980) provides an alternative proof which also generalizes beyond the risk-averse case that we treat.

<sup>16</sup> For his bootstrap tests, Post draws on the related stochastic dominance and income inequality/poverty index tests of Beach-Davidson (1983), Dardanoni-Forcina (1999), Davidson-DuClos (2000), Barret-Donald (2003), and Linton-Maasoumi-Whang (2005). A major difference between these income equality/stochastic tests and portfolio efficiency tests is in identifying alternatives. For the cited income inequality/stochastic dominance tests, the indices under test are generally pre-specified and are assumed independent. Efficient portfolio allocations must be estimated from the data, and are not independent from the alternative allocation under test. Efron and Tibshirani (1993) provide guidance on bootstrap methods. Nelson-Pope (1991) motivates bootstraps over the empirical distribution function test that we use.

<sup>17</sup> There are many such functions. Vickson-Brumelle (1975) and Russell-Seo (1989) specify variants of second-order stochastic dominance directly in utility function terms. Building on Russel-Seo (1989), Post (2005) proposes an alternative LPM-based dominating portfolio procedure. Analogous to MSSD, utility function identification methods began with Afriat (1967), and were further developed by Varian (1982, 1983), Green-Srivastava (1985, 1986), Dybvig-Ross (1982), and others.

utility function. As standard for SSD, necessary Lower Partial Moment restrictions follow from second-order integration by parts of the utility function. CSSD is assessed over the domain-union of both the initial and supramarginal allocation return sets. Define the domain,  $D = \bigcup_1^T (x_\alpha, x_\lambda)$ , and the domain element count,  $d = |D|$ .

With  $u'(x_d) \geq 0$  and  $u''(x) \leq 0$ , integration by parts yields the following equivalent expected utility definitions:

$$\int_{x_1}^{x_d} u(y) dF_i(y) = u(x_d) - u'(x_d) \int_{x_1}^{x_d} F_i(y) dy - \int_{x_1}^{x_d} |u''(x)| \int_{x_1}^x F_i(y) dy dx, \quad i = \{\alpha, \lambda\} \quad 8)$$

As  $L_{ix} = \int_{x_1}^x F_i(y) dy$ , the Bawa (1975) SSD necessary LPM conditions follow. Clearly, maximizing expected utility is equivalent to minimizing the negative terms on the right-hand side of equation 8).

For discrete data, the distribution function is a step function. For the risk-averse utility function, we also specify a second derivative step function.



The recursion for the revised Bawa et. al. (1985) linear program variables is the following:<sup>18</sup>

$$\bar{L}_{id} = - \sum_{k=1}^{d-1} F_i(d-k)(x_{d+1-k} - x_{d-k})$$

$$\bar{L}_{i,d-k} = (x_{d+1-k} - x_{d-k}) \left[ \frac{(x_{d+1-k} - x_{d-k}) F_i(x_{d-k})}{2} + \sum_{k=1}^{d-1} F_i(x_{d-k}) (x_{d+1-k} - x_{d-k}) \right], \forall i = \{1, \dots, d-1\} \quad 9)$$

For our two alternatives, we restate our adapted Bawa et. al. (1985) algorithm to identify pair-wise admissible choices:

$$\begin{array}{ll} \text{Max} & [1]z \\ z & \\ \text{s.t.} & \begin{bmatrix} \bar{L}_{\lambda,1} \\ \vdots \\ \bar{L}_{\lambda,d} \end{bmatrix} z \leq \begin{bmatrix} \bar{L}_{\alpha,1} \\ \vdots \\ \bar{L}_{\alpha,d} \end{bmatrix} \end{array} \quad 10)$$

For the broader CSSD application, an optimal alternative,  $\alpha$ , will have a range of LPM values below all convex combinations of alternative distribution choices. Therefore, the optimand value will be less than one. For our pair-wise admissible application, the initial allocation,  $\alpha$ , is dominated and at least one of the associated adjusted LPM points is above that of the supramarginal alternative. Therefore, the program solves with a maximand greater than one.

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<sup>18</sup> This step function may be regularized to generate continuity. Among many alternatives, one may follow the Third-Order Stochastic Dominance specification of Bawa et. al. (1985) without restrictions on the third derivatives, but, with additional constraints for negative second derivatives and a positive first derivative at the upper support. Our bar L notation differentiates these discrete and adjusted lower partial moments from those defined previously, 3). In this case, the bar LPM indicates a rectangular approximation to the Stieltjes integral.

The dual specification of the CSSD linear program 10) follows:

$$\begin{aligned} \text{Min} \quad & [\bar{L}_{\lambda,1}, \dots, \bar{L}_{\lambda,d}] y \\ \text{s.t.} \quad & [\bar{L}_{\alpha,1}, \dots, \bar{L}_{\alpha,d}] y \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \end{aligned} \quad 11)$$

Given our utility function second derivative step function specification, the dual optimand, 11), is the negative of the expansion terms in the expected utility integration by parts, equation 8). The first  $d-1$  dual parameter estimates correspond to second derivatives of the associated utility function, and the  $d^{\text{th}}$  parameter estimate is marginal utility at the upper support. We must aggregate these parameters to generate the associated utility function.

We set the utility function value at the upper support,  $d$ , to zero. The recursion that calculates a maximal relative utility function choosing the alternative portfolio,  $\lambda$ , builds the negative of marginal utilities and, then, subtracts the marginal utility decrements,  $u_1(x_k)$ , recursively from the utility function value at the upper support:<sup>19</sup>

$$\begin{aligned} u(x_d) &= 0 \\ u(x_k) &= u(x_{k+1}) - u_1(x_k), \quad \forall k = \{1, \dots, d-1\} \\ u_1(x_k) &= (x_{d+1-k} - x_{d-k}) \left[ -y_d - \frac{y_{d-k}(x_{d+1-k} - x_{d-k})}{2} - \sum_{j=1}^{k-1} y_{d-j}(x_{d+1-j} - x_{d-j}) \right], \quad \forall k = \{1, \dots, d-1\} \end{aligned} \quad 12)$$

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<sup>19</sup> Yitzhaki-Mayshar (1997 – revised 2001, pg. 15) link their marginal utility coefficient estimates to the efficient random variable prices of Peleg-Yaari (1975). Also, see Dybvig-Ross (1982). Our marginal utility estimates may be related to these prices in an analogous manner.

Therefore, the minimum of the Bawa et. al. (1985) dual LP is the maximum relative expected utility associated with the primal.<sup>20</sup> Like all of our results, this expected utility level is conditional on the expected utility of the initial allocation.

The maximum expected utility for the dominated alternative is 1.00362. As was the case for our estimate of the mean dominating portfolio performance improvement, Ferguson's (1974) Bayesian perspective implies market portfolio efficiency rejection. Our minimal dominated portfolio disutility estimate provides an admissible testing measure. An estimate greater than one implies dominance.<sup>21</sup>

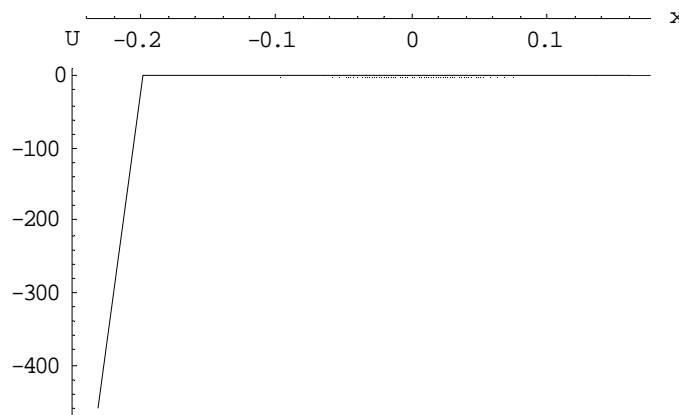
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<sup>20</sup> From equation 8) and with zero utility at the upper support, our LP structure assigns the dominating alternative expected utility of minus 1. We may also evaluate optimality of the alternative improved allocation. In this case, the initial allocation expected utility is minus one, and the SMSSD improved allocation expected utility is zero.

<sup>21</sup> Post-Versijp (2004) have proposed a GMM-based test for the case of short sales and unlimited borrowing and lending. Discussion of implementation issues with this test, and, particularly, the feasibility of implementing the test in a non-Bayesian setting, is available from the authors' on request.

For the Post (2003) application with the market portfolio and Fama-French size- and book-to-market-ranked portfolios, Figure 4 depicts the estimated utility function. Relative to the market portfolio, investors manifesting this utility function invest 84.9% in the 4<sup>th</sup> size quintile and 4<sup>th</sup> book-to-market quintile portfolio and 15.1% in the 1<sup>st</sup> size quintile and 5<sup>th</sup> book-to-market quintile portfolio:

**Figure 4**  
**Expected Utility Function**



From Figure 4 and equation 12), the utility function is concave.

## 5. Hybrid MSSD and SMSSD Portfolio Efficiency Tests

To ascertain portfolio efficiency for a discrete multinomial return distribution, both the MSSD and SMSSD methods should be used together. As long as the portfolio allocation under test is in the interior of the feasible allocation space, MSSD separates dominated and undominated allocations. As currently developed, MSSD doesn't identify dominating and supramarginal allocations.<sup>22</sup> In our applications, SMSSD identifies

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<sup>22</sup> Solving a sequence of the MSSD problems with adaptive bounds on the marginal changes may identify supramarginal gains. Nevertheless, iteration-by-iteration checks of the SSD dominating criterion will have to be conducted to ensure that the MSSD local optimality conditions identify discrete efficiency gains over the entire domain. Furthermore, a step-wise MSSD procedure requires iteration-by-iteration reordering of the state set. Additionally, SMSSD and MSSD may identify different efficiency improving allocations, and both allocations may be admissible. Preliminary analysis of the MSSD method for the Post Fama-French application is not promising, and these results are available on request.

dominating portfolios with significant efficiency gains.

From the algorithmic perspective, MSSD will often be a smaller problem than SMSSD. MSSD doesn't require the incremental identification of points in the domain to apply the  $LPM_1$  constraints of SMSSD. Nevertheless, our market portfolio SMSSD example required only five market return points.

An outline of a hybrid MSSD and SMSSD method combines Post's algorithm 2) and the SMSSD linear program 7):

Step 1: Test MSSD Algorithm, if optimand equals zero, stop. Initial portfolio allocation is undominated. 18)

Step 2: Define constraints at the least and greatest returns, and a small positive target mean increment,  $\mu$ .

Step 3: Execute SMSSD algorithm 7) with initial allocation  $\alpha$ . If infeasible, then stop and allocation is undominated. Otherwise, proceed to step 4.

Step 4: Test for SSD of identified portfolio relative to allocation. If allocation is dominated, go to step 5. If not, then add variables and constraints for the ordered return with worst dominance violation and go to Step 3.

Step 5: If optimand equals zero and  $\lambda$  does not equal  $\alpha$ , increase the target mean parameter,  $\mu$ ,

$$\sum_{i=1}^N \lambda_i \bar{x}_{iT} \geq \bar{x}_{\lambda T} + \mu \text{ and go to step 3. If not, then stop and the previous to last}$$

Step 3 allocation ( $\lambda$ ) is preferred.

As with Post's and YM's MSSD discrete multinomial applications, it is important to emphasize that SMSSD will not generally identify optimal SSD portfolio choices. As treated in Bodurtha (2003), optimal portfolios among multivariate continuous distributions of returns have only been identified for the Bawa (1975) location-scale

family of distributions.<sup>23</sup> Levy-Levy (2002) extends the SSD treatment to Prospect Theory.

## 6. Conclusion

We show that the Post (2003) and Yitzhaki-Mayshar (1997) linear programming methods identify dominated alternatives. As currently specified and implemented, the methods only identify the existence of marginal performance improvements. Dominating reallocations for all investors are not identified.

To identify dominating portfolio allocations, we specialize Bawa's (1978) first-order lower partial moment-based ( $LPM_1$ ) portfolio selection method. By conditioning portfolio reallocation on a particular initial allocation, we identify dominating supramarginal second-order stochastic dominance (SMSSD) portfolio efficiency improvements for all risk-averse investors. For portfolio efficiency applications, we outline a hybrid MSSD-SPMSSD procedure.

Finally, the admissible allocations that we identify are not optimal in the sense of Fishburn (1974) and Bawa et. (1985). Though it has been shown that admissible and efficient allocations are optimal for certain continuous location-scale distributions, this question remains open for the general distributions treated in Post (2003), Yitzhaki-Mayshar (1997) and here.

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<sup>23</sup> Bodurtha-Shen (2001) and YM (1997 - Propositions 9 and Proposition 11). Peleg and Yaari (1975), Peleg (1975), Bawa-Goroff-Whit (1978), Goroff-Whitt (1980) and Bawa-Goroff (1982) address this issue in broader contexts. See YM (1997- revised 2001, pg. 15) specifically regarding Peleg and Yaari (1975).

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