The Economics and the Econometrics of Human Development: Dynamic Models Part IIIA

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A Bare-Bones Model of Parenting as Investment

- To focus ideas, I present a simple model of family investment and skill development based on Cunha (2007) and Cunha and Heckman (2007).

The Problem of the Parent

- Life is assumed to last four periods:
  - Two periods as a passive child who makes no economic decisions (and whose consumption is ignored) but who receives investment in the form of goods
  - Two periods as a parent.
- When the parent dies, she is replaced by the generation of her grandchild.
- Denote by θ_1 the initial capability level of a child drawn from the distribution $J(\theta_1)$.
- The evolution of child skills depends on parental investments in the first and second period, I_1 and I_2.

- The productivity of parental investment depends on parental human capital, $\theta_{P,t}$.
- (For notational simplicity, we set $\theta_{P,t} = \theta_P$.) Equate scalar human capital with skill for both parents and children.
- Denoting by θ_3 the human capital of the child when he reaches adulthood
- Recursive substitution of the technology of skill formation using a CES specification gives the following representation:

$$\theta_3 = \delta_2 \left[\theta_1, \theta_P, \left(\gamma \left(I_1 \right)^{\phi} + (1 - \gamma) \left(I_2 \right)^{\phi} \right)^{\frac{\rho}{\phi}} \right], \tag{1}$$

  for $0 < \rho \leq 1$, $\phi \leq 1$ and $0 \leq \gamma \leq 1$.
- γ is a skill multiplier.

How to a Get Simple Representation

- Consider the following parameterization of the stage-specific production functions:

$$\theta_{t+1} = \delta_t \left\{ \gamma_{1,t}\theta_t^{\phi_t} + \gamma_{2,t} I_t^{\phi_t} + \gamma_{3,t}\theta_P^{\phi_t} \right\}^{\frac{\rho_t}{\phi_t}}$$

with $0 < \gamma_{1,t}, \gamma_{2,t},$ and $\gamma_{3,t}$; $\rho_t \leq 1$; $\phi_t \leq 1$; and $\sum_{k=1}^{3} \gamma_{k,t} = 1$.

- Substitute recursively. If $T = 2$, $\rho_1 = \rho_2 = 1$, $\delta_1 = 1$, and $\phi_1 = \phi_2 = \phi \leq 1$, skills at adulthood, $\theta_3 = \theta_{T+1}$, can be expressed as

$$\theta_3 = \delta_2 \left[\gamma_{1,2}\gamma_{1,1}\theta_1^{\phi} + \underbrace{\gamma_{1,2}\gamma_{2,1}}_{\textbf{“Multiplier”}} I_1^{\phi} + \gamma_{2,2} I_2^{\phi} + \left(\gamma_{3,2} + \gamma_{1,2}\gamma_{3,1}\right)\theta_P^{\phi} \right]^{\frac{1}{\phi}}.$$

- **Multiplier**: $\gamma = \gamma_{1,2}\gamma_{2,1}$.
- Arises from the conjunction of self-productivity ($\gamma_{2,1} \neq 0$) and the productivity of investment ($\gamma_{1,2} \neq 0$).
- Self-productivity joined with the productivity of investment generates dynamic complementarity.
- $\gamma_{2,1}$ characterizes how much of the investment in period $t = 1$ propagates into skills at adulthood, θ_3.
- The parameter ϕ captures the substitutability/complementarity of investments.

- If $\phi = 1$, investments at different periods are (almost) perfect substitutes.
- They are perfect one-to-one substitutes if $\gamma_{1,2}\gamma_{2,1} = \gamma_{2,2}$, in which case the timing of investment in skills does not matter for the developmental process.
- This is the only circumstance in which collapsing childhood into one period as in Becker–Tomes is without loss of generality.

- Polar opposite case: $\theta_3 = \delta_2 \left(\theta_1, \theta_P, \min\left(I_1, I_2\right)\right)$.
- Closer to the empirical truth than perfect substitution.

- Complementarity has a dual face.
- Early investment is essential but ineffective unless later investments are also made.
- In this extreme case, there is no possibility of remediation.
- If parents are poor and unable to borrow against the future earnings of their children and, as a result, I_1 is low, there is no amount of investment at a later age, I_2, that can compensate for early neglect.

- The parameters of the technology determine whether early and later investments are complements or substitutes.
- "Direct" complementarity for Equation (1) holds if $\rho > \phi$, whereas substitutability holds otherwise.
- Another definition of complementarity in the literature distinguishes (in the case of $\rho = 1$) whether $\phi > 0$ (*gross* substitutes; the elasticity of substitution is greater than 1) or $\phi < 0$ (*gross* complements; the elasticity of substitution is less than 1).
- Cobb-Douglas ($\phi = 0$) is the boundary case.

- Given ρ, the smaller ϕ, the harder it is to remediate low levels of early investment I_1 by increasing later investments.
- At the same time, the stronger the complementarity (the lower ϕ).
- The more important it is to follow high volumes of early investments with high volumes of late investments to achieve high levels of production of adult human capital.

Parent's Problem

- The parent allocates resources across household consumption in both periods of the child's life, c_1 and c_2; early and late investments, I_1 and I_2; and bequests, b'.
- Assets at the end of the first period, period a, may be constrained to be non-negative.
- Bequests are received when entering adulthood and may be positive or negative.
- The state variables for the parent are her initial wealth, b; human capital level, θ_P; and the initial skill level of the child, θ_1.
- Human capital is rewarded in the labor market according to the wage rate, w.
- Economy is characterized by one risk-free asset with return r.

- $u(\cdot)$: parental utility function
- β: discount factor
- υ: parental altruism given by the weight assigned to the utility of future generations
- θ_1': uncertain initial endowment of the child's child
- Goal of the parent: optimize

$$V\left(\theta_P, b, \theta_1\right) = \max_{c_1, c_2, l_1, l_2} \left\{u\left(c_1\right) + \beta u\left(c_2\right) + \beta^2 \upsilon \mathbb{E}\left[V\left(\theta_3, b', \theta_1'\right)\right]\right\} \tag{2}$$

- subject to (1), (3) and (4)

- Denote parental financial assets by a
- Parental labor market productivity grows at exogenous rate g
- One can represent the stage-of-childhood-specific budget constraints:

$$c_1 + l_1 + \frac{a}{(1+r)} = w\theta_P + b \tag{3}$$

and

$$c_2 + l_2 + \frac{b'}{(1+r)} = w\left(1+g\right)\theta_P + a \tag{4}$$

- Allow for the possibility of borrowing constraints
- $a \geq \underline{a}$ (intragenerational)
- $b' \geq 0$ (intergenerational)

- If no intra- and intergenerational credit constraints are assumed, a key property of the Becker and Tomes (1986) model persists in this framework.
- There is no role for initial financial wealth b, parental income, parental utility, or the magnitude of parental altruism v (above zero) in determining the optimal level of investment because parents can borrow freely in the market to finance the wealth-maximizing level of investment.

- Even if the altruism parameter is zero ($\upsilon = 0$), if the parents can make binding commitments, selfish parents ($\upsilon = 0$) will still invest in the child, as long as the economic return in doing so is positive.
- However, even in this setup, returns to parental investments depend on parental skills, θ_P, as they affect the productivity of investments.
- The returns to investments are higher for children of parents with higher θ_P.

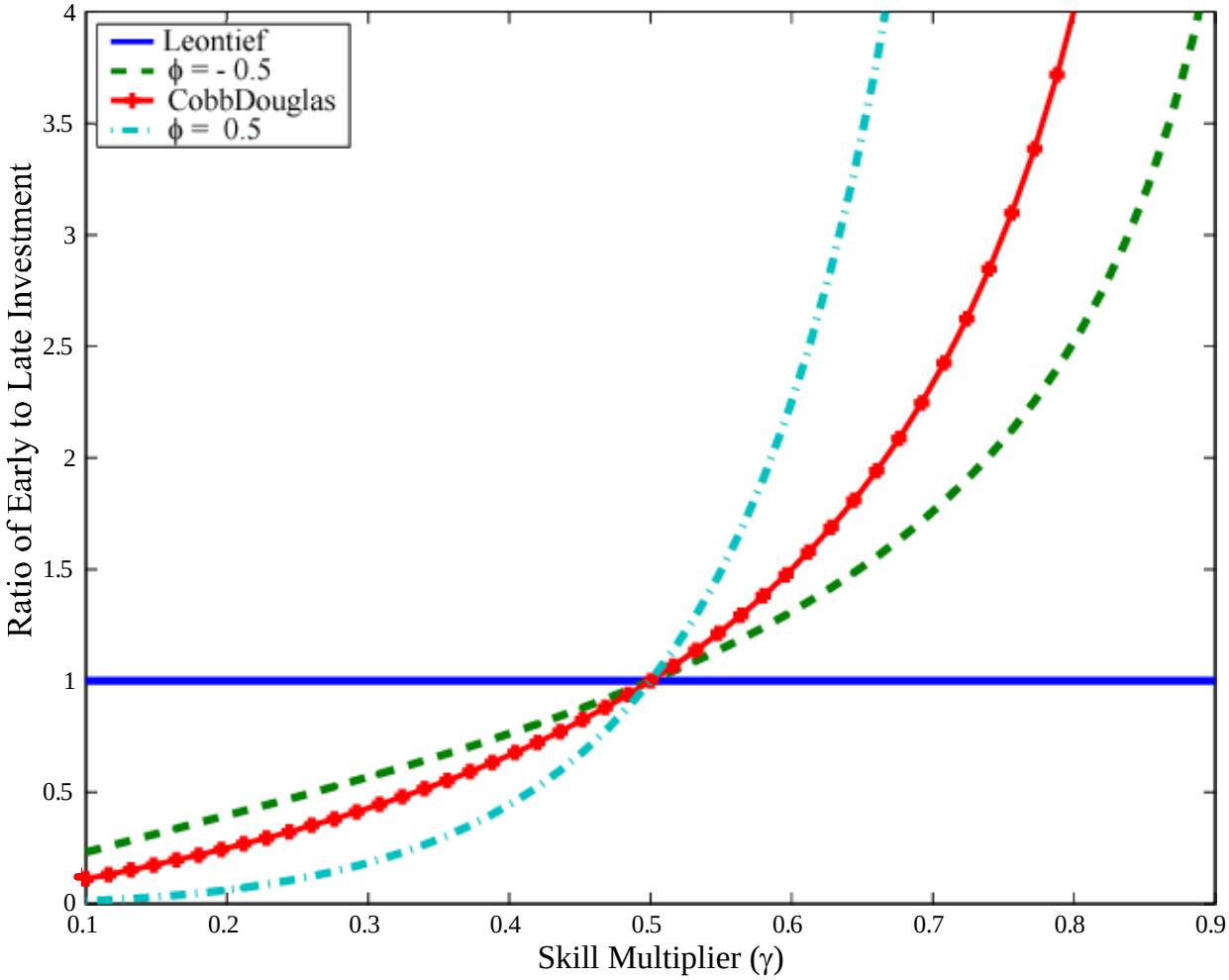
- These children receive higher levels of investment.
- This is a type of market failure due to the "accident of birth" that induces a correlation of human capital and earnings across generations even in the absence of financial market imperfections.
- The initial condition θ_1 also affects investments.
- It creates a second channel of intergenerational dependence due to the "accident of birth" if it is genetically related to parental endowments, as considerable evidence suggests.

- When there are no intra-period constraints,

$$\frac{I_1}{I_2} = \left[\frac{\gamma}{(1-\gamma)(1+r)}\right]^{\frac{1}{1-\phi}}. \tag{5}$$

- $\frac{I_1}{I_2} \uparrow$ as $\gamma \uparrow$, $\phi \uparrow$, and $r \downarrow$.

Figure 1: The Ratio of Early to Late Investment in Human Capital as a Function of the Skill Multiplier for Different Values of Complementarity



(Assumes $r = 0$)

- Imperfect credit markets create another channel of intergenerational dependence.
- One possible constraint is the impossibility of borrowing against the child's future earnings (Becker and Tomes, 1986).

- Because $b' \geq 0$, parental wealth matters in this model when this constraint binds.
- Children coming from constrained families will have lower early and late investments.

- However, even with $b' \geq 0$, the *ratio* of early to late investment is not affected.

- Suppose constraints (3) and (4) bind separately.
- Suppose parental utility is given by $u(c) = \left(c^{\lambda} - 1\right)/\lambda$.
- $\lambda = 1$ corresponds to perfect intertemporal substitutability.

- The ratio of early to late investment is then

$$\frac{I_1}{I_2} = \underbrace{\left(\frac{\gamma}{(1-\gamma)(1+r)}\right)^{\frac{1}{1-\phi}}}_{\substack{\text{unconstrained ratio} \\ \frac{I_1}{I_2}\uparrow \text{ as } \gamma\uparrow,\ \phi\uparrow,\ \text{and } r\downarrow}} \underbrace{[\beta(1+r)]^{\frac{1}{1-\phi}} \left(\frac{c_1}{c_2}\right)^{\frac{1-\lambda}{1-\phi}}}_{\substack{=1 \text{ if unconstrained,} \\ <1 \text{ if constrained } (a \geq \underline{a} \text{ binds})}}. \tag{6}$$

- If early parental income is low compared to later life income, or if λ is small, the level and timing of family resources will influence the parental investment.
- Estimates from Cunha et al. (2010) suggest that $1/(1-\phi) = .\bar{3}$.
- Estimate of $\lambda \in [-3, -1.5]$ (Attanasio and Browning, 1995).
- $(1-\lambda)/(1-\phi) \in [0.8\bar{3}, 1.\bar{3}]$.
- Notice that even if $\lambda = 1$, parents may hit constraints on the level of investment if future resources are of insufficient magnitude.
- This constraint could be very harmful to a child if it binds in a critical period of development and the complementarity parameter ϕ is low so that later life remediation is ineffective.

The Presence of Constraints is Not Synonymous with Low Levels of Investment

- However, the presence of constraints is not necessarily synonymous with a low level of investment.
- For a given family, a binding constraint implies that the investments are lower than the unconstrained optimum.
- Whether a family is constrained, however, is uninformative on how that family compares with others in terms of the effective level of investments provided.

- Caucutt and Lochner (2012) use a variant of the model of Cunha (2007, 2013) to investigate the role of income transfers and credit constraints in the early years.
- They find that a large proportion of young parents are credit constrained (up to 68% among college graduates) but that reducing borrowing constraints is effective in promoting skills only for the children in the generation in which they are relaxed.

Introducing Income Uncertainty

- Cunha (2007, 2013): overlapping generations model with stochastic innovations to parental income.
- If g is stochastic on the interval $[-1, \infty)$, so parents face uncertain income growth, constraints play a dual role.
- First, as before, if the constraints bind, they reduce investments in the constrained periods.
- Second, because future income is uncertain, so is the likelihood of binding future constraints.

- Absent full insurance markets, consumption and investments in children are less than optimal, even if the parent is not currently constrained but expects to be constrained in the future with a probability greater than zero.

See Appendix on **The Problem of the Parent**
(based on Cunha, 2007; revised 2016)

Link

- Under this scenario, young parents who just entered the labor force accumulate more assets than they would in the absence of possible future constraints to ensure against bad future shocks.
- This implies a reduction in household consumption and investments in child human capital.

Recent Extensions of the Basic Model

- The recent literature has moved beyond the simple models just discussed.

- Most assume parental altruism.
- Some are explicitly paternalistic.
- They all feature investment in *goods*.
- Only recently has parental time been analyzed as an explicit input to child quality.
- Most models analyze how child investment depends on parental skills.

- Until recently, most studies considered the self-productivity of skills.
- Some recent papers ignore this feature, despite the empirical evidence that supports it.

- Most analyses assume that parents know the technology of skill formation, as well as the skills of their children, in making investment decisions.
- Some of the recent literature also ignores intergenerational transfers.
- Some papers consider extreme credit constraints that do not permit any borrowing (or lending), even within a lifetime of a generation, much less inter-generational transfers.
- Virtually the entire literature focuses on single-child models, exogenous fertility, and exogenous mating decisions.
- Most models are for single-parent families, for which the characteristics of the spouse are irrelevant.

- First, with the exception of Cunha and Heckman (2008a) and Cunha et al. (2010), as well as very recent papers by Attanasio and Meghir, human capital is treated as a scalar.

- Second, in many, but not all recent models, investments are also treated as scalars.

- Third, families usually have more than one child.

- Fourth, the models in the literature ignore the interaction of parents and children in the process of development.

- Fifth, fertility is taken as exogenous.

- Do not take too literally models of credit constraints interacting with dynamic complementarity that take fertility as exogenously determined.

- Child's development is influenced by the environment outside his family: day care, kindergarten, school, and neighborhood.
- In addition, the effectiveness of policies is determined in part by parental responses to them.
- Policies that complement rather than substitute for family investments will have greater impacts and lower costs.

Credit Constraints and the Effects of Family Income

The Effects of Family Income

- The literature is unanimous in establishing that families with higher levels of long-run (or permanent) income on average invest more in their children and have children with greater skills.
- The literature is much less clear in distinguishing the effect of income by source or in distinguishing pure income effects from substitution effects induced by changing wages and prices (including child-care subsidies or educational incentive payments).

- Levels of permanent income are highly correlated with family background factors such as parental education and maternal ability, which, when statistically controlled for, largely eliminate the gaps across income classes.

- The literature sometimes interprets this conditioning as reflecting parenting and parental investments, but it could arise from any or all of the panoply of correlates of permanent income associated with parental preferences and skills.

- Income operates through altruism or paternalism?

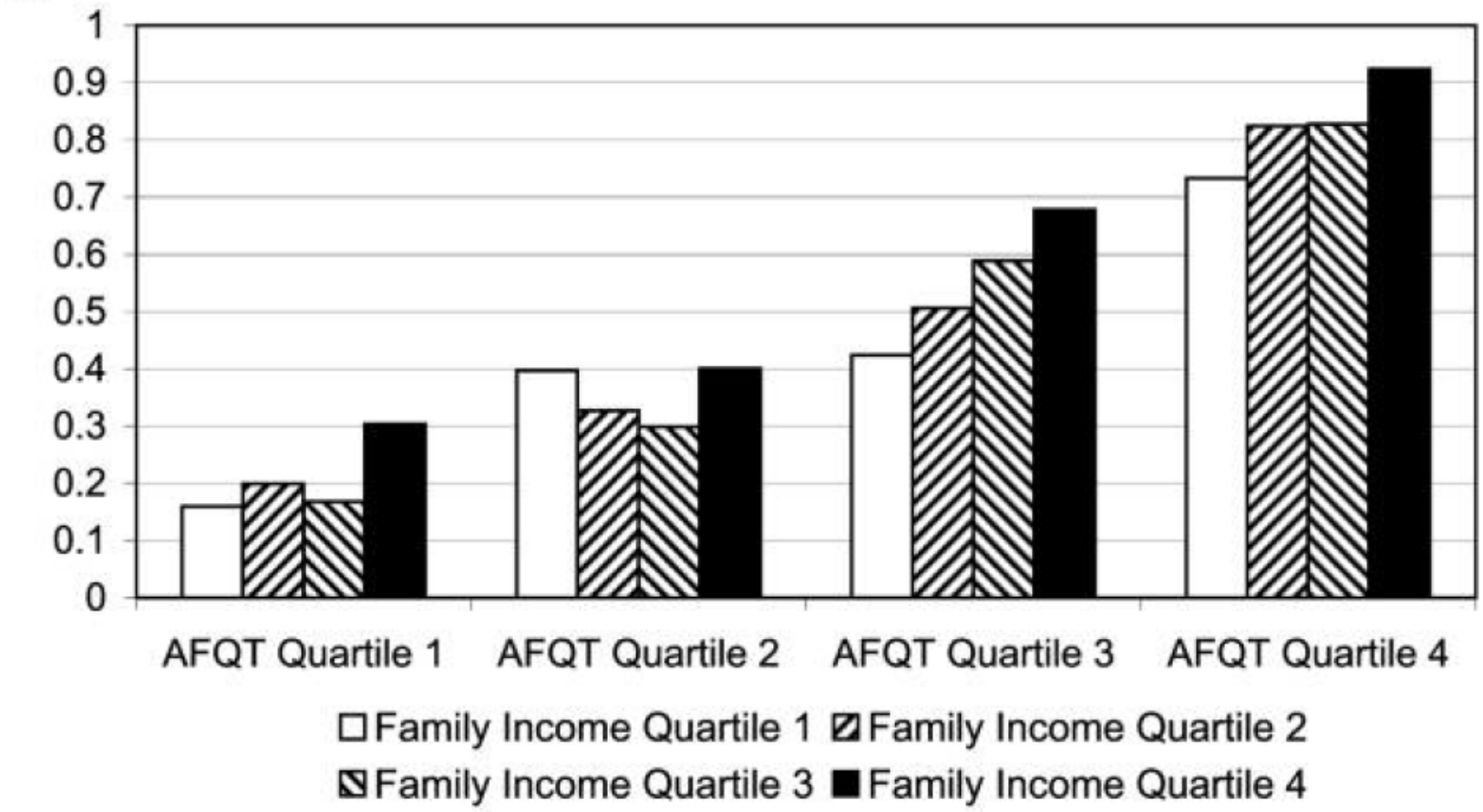
Effects of Borrowing Constraints

- The literature also analyzes the effect of borrowing constraints on child outcomes.

Restrictions in Lending Markets for College Education

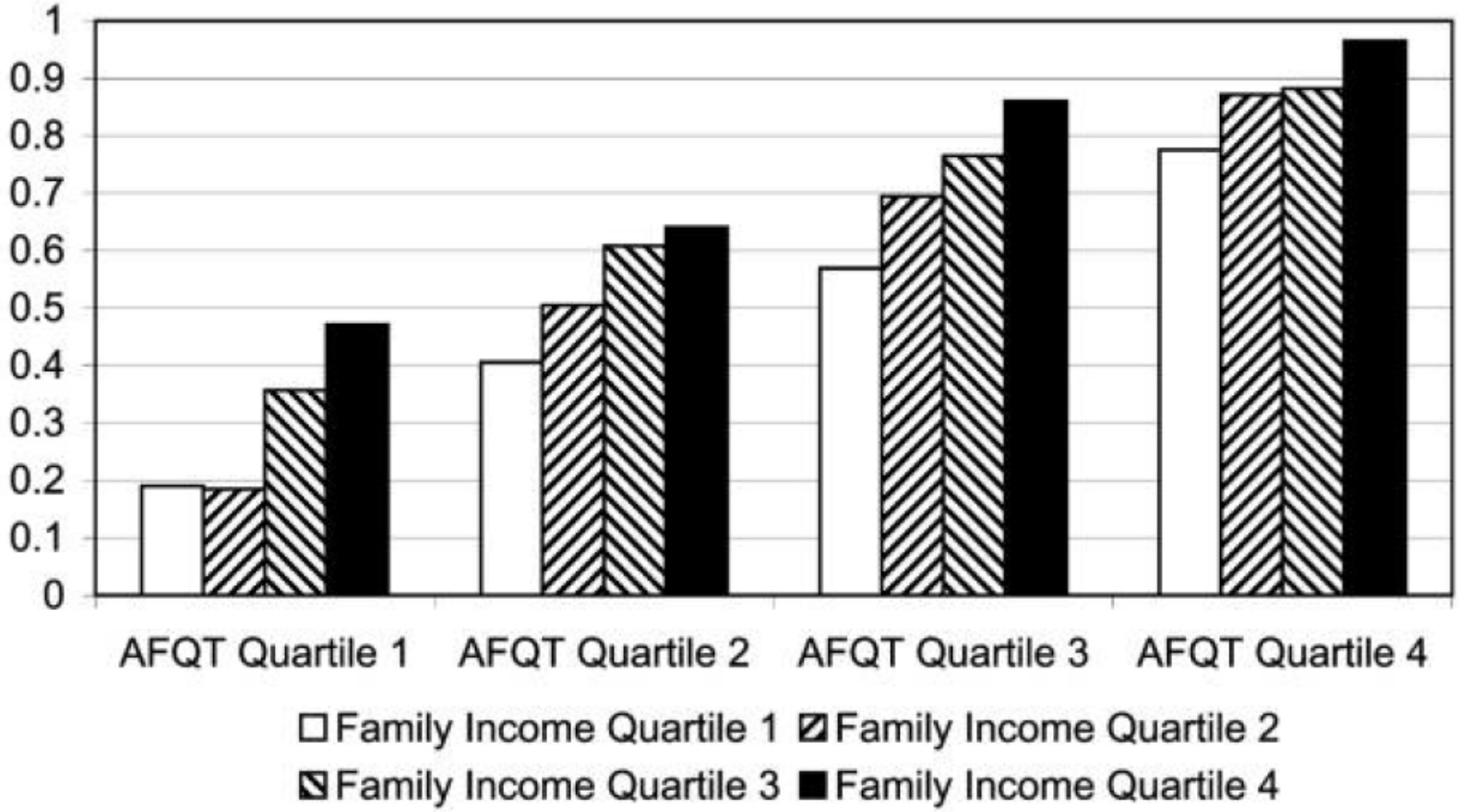
- Belley and Lochner update the NLSY79 analysis of Carneiro and Heckman (2002) using NLSY97 data and claim that credit constraints seem to bind predominantly among less able poor children.
- However, their analysis shows that, across all ability groups, college enrollment increased in 1997 compared to 1979.
- The increases are more substantial for more affluent, low-ability children.

Figure 2: College Attendance by AFQT and Family Income Quartiles (1979)



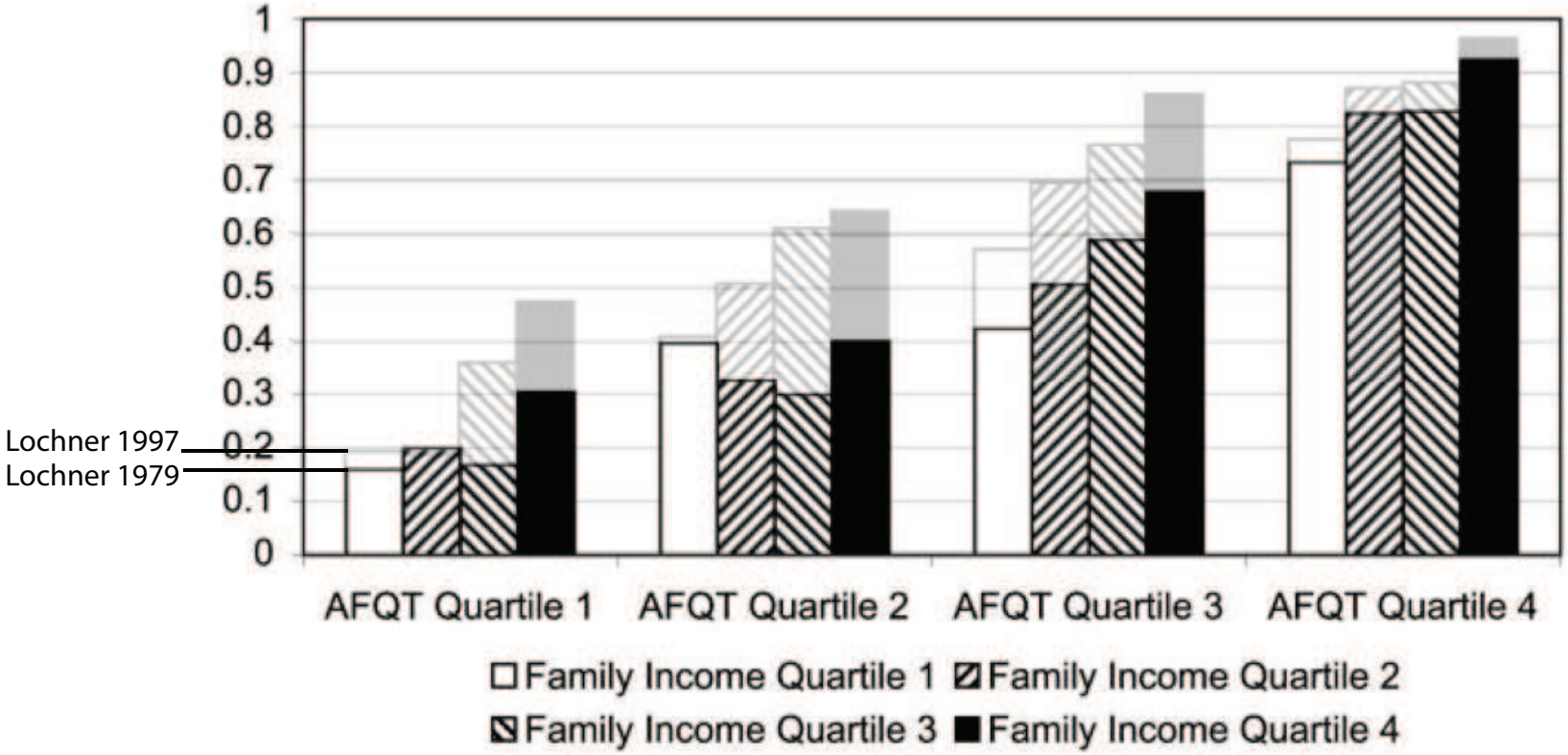
Source: Belley and Lochner (2007).

Figure 3: College Attendance by AFQT and Family Income Quartiles (1997)



Source: Belley and Lochner (2007).

Figure 4: College Attendance by AFQT and Family Income Quartiles (1979 and 1997 on one graph)



Source: Belley and Lochner (2007).

The Timing of Income, Dynamic Complementarity, and Credit Constraints

- The interaction of dynamic complementarity and lifetime liquidity constraints motivates a recent literature.
- Dahl and Lochner (2012) investigate how credit constraints affect test scores of children in early adolescence.
- They exploit the policy variation in the Earned Income Tax Credit (EITC) as an exogenous instrument for the effect of income on child outcomes.

Studies on the Role of Income on Children's Outcomes and on Credit Constraints

Link to Appendix

Lessons from the Literature on Family Income and Credit Constraints

- The literature on credit constraints and family income shows that higher levels of parental resources, broadly defined, promote child outcomes.
- However, a clear separation of parental resources into pure income flows, parental environmental variables, and parental investment has not yet been done.
- **Premature** to advocate income transfer policies alone as effective policies for promoting child development.
- Poverty (measured by income is not (necessarily) a good measure of child disadvantage).

- The literature establishes the first-order importance of child ability for college going, irrespective of family income levels.
- More advantaged families with less able children send their children to college at greater rates than less advantaged families, but the literature does not establish the existence of market imperfections or any basis for intervention in credit markets.
- The observed empirical regularity may result from the exercise of parental preferences.
- Recent work shows that the returns to college for less able children are low, if not negative.

- The literature that presents more formal econometric analyses of the importance of credit market restrictions on educational attainment shows little evidence for them.

- The analysis of Caucutt and Lochner (2012) is an exception.
- They *calibrate* that a substantial fraction of the population is constrained due to the interaction of dynamic complementarity, the receipt of income, and the imperfection of lending markets.
- Much further research is required before definitive policy conclusions can be drawn on the empirical importance of the timing of receipt of income over the life cycle for child outcomes.

Recent work on credit constraints and human capital accumulation

"Inequality in Human Capital and Endogenous Credit Constraints"
by Rong Hai and James J. Heckman (2016)

Link

Structural Models

See Appendix on **Overview of Structural Models of Parental Investments**

Link

- Most studies of the role of income transfer programs discussed earlier do not investigate the interactions of public policy interventions and family investments.
- To do so, some authors have estimated fully specified structural models and use them to study the effect of various types of policy experiments.

Overview of Policy Implications of Structural Models

Four main facts:

- First, subsidies to parental investments are more cost-effective in improving adult outcomes of children such as schooling attainment or earnings, when provided in the early stages of life (Caucutt and Lochner, 2012; Cunha, 2007; Cunha and Heckman, 2007).

- Second, financial investment subsidies have stronger effects for families who are already engaging in complementary investments.
- Targeted public investments and targeted transfers restricted to child-related goods that guarantee minimum investment amounts to every child increase the level of investments received by the children of the least-active parents (Caucutt and Lochner, 2012; Del Boca et al., 2014a; Lochner and Monge-Naranjo, 2016).
- Lee and Seshadri (2016) provide evidence on the importance of targeted education subsidies for increasing the educational expenditures of poor families.

- Third, time-allocation decisions are affected by transfers.
- Del Boca et al. (2014a) show that unrestricted transfers increase the time parents spend with their children through a wealth effect.
- The increase in child quality is minimal.

- Lee and Seshadri (2016) show how this effect is especially strong for parents without college education, whereas, in their model, public transfers negatively affect time spent with children for college-educated parents.
- Fourth, targeted conditional transfers (on a child's ability improvements) are more cost-effective than pure income transfers to achieve any child outcome.

The Implications of Dynamic Complementarity for Investments across Children with Different Initial Endowments

- The average family usually has more than one child, and society allocates public investments across multiple children.
- The problem of intra-child allocations is sometimes formulated as a problem in fairness.
- CES representation of parental utility V is often used:

$$V = \left(\sum_{k=1}^{N} \omega_k V_k^{\sigma} \right)^{\frac{1}{\sigma}} . \tag{7}$$

- Rawlsian version of maximal inequality aversion is obtained when $\sigma \to -\infty$, so utilities are perfect complements, and parents are concerned only with the maximization of the minimum outcome across children.

- In a two-child version of the one-period-of-childhood model analyzed by Becker and Tomes (1979, 1986), under complementarity between initial endowment and investment, the optimal policy when $\sigma = 1$ is to invest less in the initially disadvantaged child.
- Under substitutability, it is optimal to invest more in the disadvantaged child.

- Story richer when we consider a multiperiod model with dynamic complementarity.
- *Investing relatively more in initially disadvantaged young children can be efficient even when the* ω_k *are equal and* $\sigma = 1$.
- This is true even if there is complementarity in each period of the life cycle.

- **Dynamic complementarity** is a force promoting compensating early stage investments.
- In a multiperiod model at stage t

$$\theta_{t+1} = f^{(t)}(\theta_t, I_t), \tag{8}$$

- Even if there is complementarity at all stages, so $f_{12}^{(t)}(\cdot) > 0$ (where $(\cdot)$ denotes the argument of the function), output-maximizing investments can be compensating.

- If $f_{12}^{(1)}(\cdot) < 0$, but $f_{12}^{(2)}(\cdot) > 0$, it is *always* efficient to invest relatively more in the initially disadvantaged child in the first period.

- It can also be productively efficient to invest in the disadvantaged child if $f_{12}^{(1)}(\cdot) > 0$, when initial endowments and investments are complements.

See Appendix on **Targeting Relatively More Investment Toward Disadvantaged Children**

Link

- Intuition.
- Suppose that there is increasing complementarity.
- In this case, the stock of skills in the second period has a greater effect on the productivity of investments than it does in the first period $\left(f_{12}^{(2)}(\cdot) > f_{12}^{(1)}(\cdot)\right)$.
- First-period investments bolster the stock of second-period skills and prepare disadvantaged children to make productive use of them in the second period.
- This effect is stronger when $f_{12}^{(2)}(\cdot)$ is larger.

- Another force promoting greater initial investment in the disadvantaged child is diminishing self-productivity of skills in the first period $\left(f_{11}^{(1)}(\cdot) < 0\right)$.
- The greater the diminishing returns to investment for the better-endowed child, the lower the benefits of early advantage.
- Diminishing productivity of the stock of second-period skills $\left(f_{11}^{(2)}(\cdot) < 0\right)$ operates in the same fashion to limit the effects of any initial advantage.

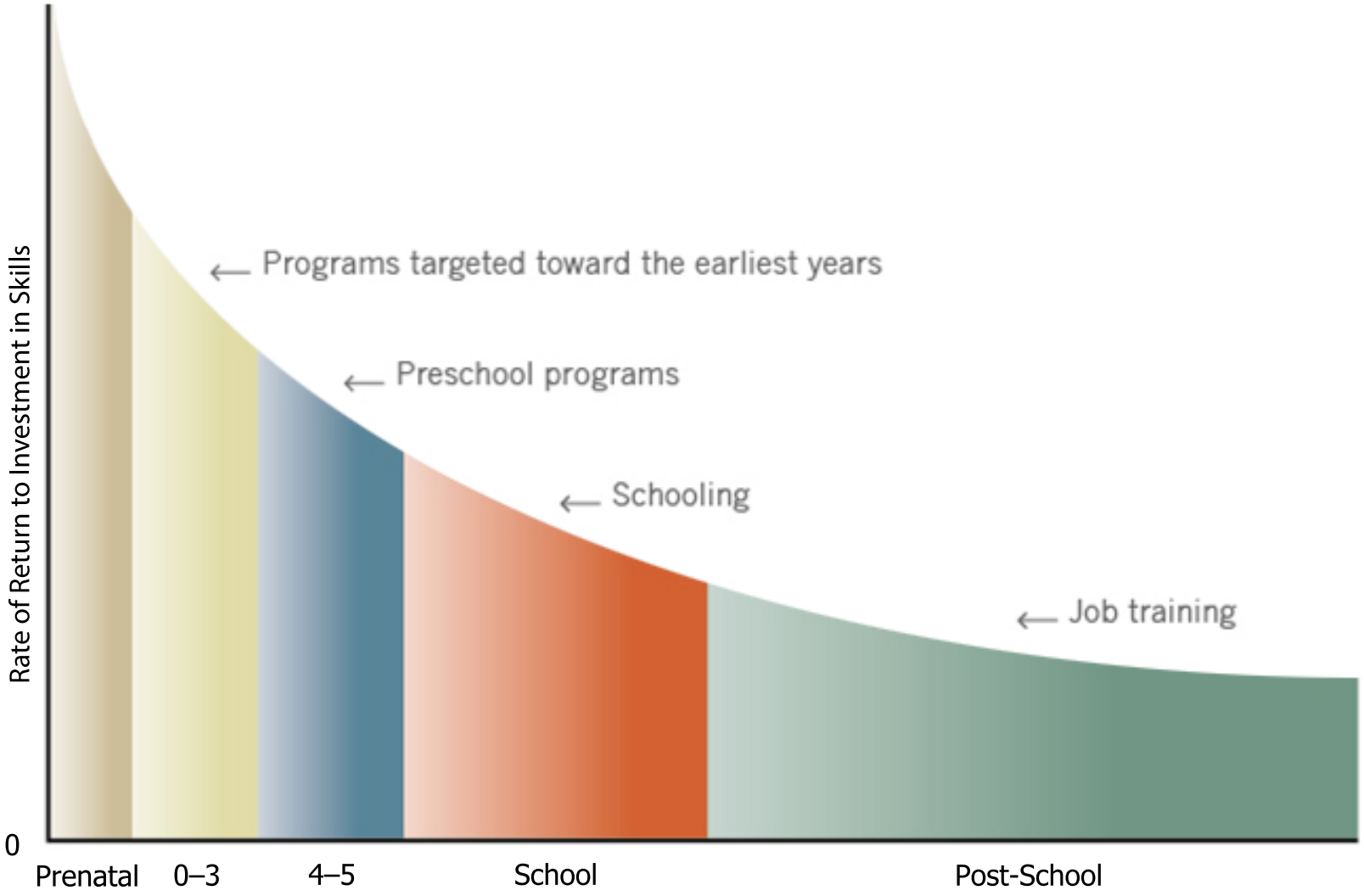
- The smaller the effect of the initial stock of skills on the productivity of investment in the first period $\left(f_{12}^{(1)}(\cdot)\right)$, the weaker is the disequalizing force of complementarity toward promoting investment in the initially advantaged child.

- Summarizing:
  1. The more concave are the technologies in terms of stocks of skills (the more they exhibit decreasing returns in the stocks of skills), the more favorable is the case for investing in more disadvantaged children.
  2. The stronger is second-period complementarity $\left(f_{12}^{(2)}(\cdot)\right)$, the stronger is the case for investing more in the initially advantaged child to build skill stocks to take advantage of this opportunity.
- The weaker is the first-period complementarity $\left(f_{12}^{(1)}(\cdot)\right)$, the less offsetting is the disequalizing effect of complementarity coupled with initial advantage.

- In general, even when investment is greater in the first period for the disadvantaged child, it is optimal for second-period investment to be greater for the initially advantaged child.
- It is generally not efficient to make the disadvantaged child whole in the first period.
- Greater second-period complementarity then kicks in to promote disequalizing second-period investments.

Dynamic complementarity generates the following relationship:

Figure 5: Returns to a Unit Value Invested



Source: Heckman (2008)

See Appendix on **Some Evidence from Simulations on Why Dynamic Complementarity is a Force for Promoting Early Investment in Disadvantaged Children**

Link

- A dynamic state-space model with constraints and family investment decisions is the natural econometric framework for operationalizing the model.
- Discuss in Part IIIB.

Towards a More General Model of Parent-Child Interactions

- The productivity of any investment or parental stimulus is influenced by the child's response to it.
- Parents and children can have different goals.
- For example, the child can be more shortsighted than the parent (Akabayashi, 2006b) or have different values for leisure and future human capital (Cosconati, 2013a).
- The parent may act as a principal whose goal is to maximize the effort from an agent—their child.

Economic Models of Parenting and Scaffolding

García and Heckman

Research Question

- How do child-parent interactions shape the formation of early character and cognitive skills?
- How child matures in this process?

How?

1. Model parent-child interactions
   - Early-childhood, dynamic game
   - Learning and skill development
   - Technology depends on parental and child investment
2. Structurally estimate the model
   - Use data from the Infant Health and Development Program (IHDP)
     - Parent and child efforts
     - Character and cognitive skills
     - Parental expectations

Literature Overview

- Skill formation with passive children:
- Cunha and Heckman (2008b); Cunha and Schennach (2010), Heckman et al. (2010, 2013b)
  - Lots of results on skills and investment
    - Self-productivity (of skills)
    - Dynamic complementarity (of investment)
    - Time without the mother hurts skill development
- Specific features of child-parent interactions:
- Akabayashi (2006a); Cosconati (2013b); Lizzeri and Siniscalchi (2008a)
  - Child-maltreatment (theory only)
  - Optimal parenting styles (theory and estimation)
  - Parental guidance (theory only)

Table 1: Models of Parent-Child Interaction ("✓" means present; "X" means absent)

	Parental Monetary Investments	Discordance in Preferences between Parent and Child	Multiple Children	Parental Incentives for Child Effort
Cosconati (2013a)	X	✓[a,b]	X	✓[c]
Akabayashi (2006b)	✓	✓[a]	X	✓[d,e]
Hao et al. (2008)	✓	✓[b]	✓	✓[e]
Lizzeri and Siniscalchi (2008b)	X	✓[d]	X	X

[a] Difference in discount factors. [b] Differences in utility functions. [c] Restrictions on leisure. [d] Differences in knowledge about proper task execution. [e] Time investments in the child. [f] Monetary investments in the child. [g] The authors analyze the effects of time investments. [h] Implications from the model are empirically tested.

Table 1 (continued): Models of Parent-Child Interaction ("✓" means present; "X" means absent)

	Effort Produces Greater Skill	Parental Learning about Child Quality	Parental Actions Facilitate Acquisition of Information	Parental Beliefs Can Diverge from Truth
Cosconati (2013a)	✓	X	X	X
Akabayashi (2006b)	✓	✓	✓	✓[f]
Hao et al. (2008)	X	X	X	X
Lizzeri and Siniscalchi (2008b)	X	X	✓	X

[a] Difference in discount factors. [b] Differences in utility functions. [c] Restrictions on leisure. [d] Differences in knowledge about proper task execution. [e] Time investments in the child. [f] Monetary investments in the child. [g] The authors analyze the effects of time investments. [h] Implications from the model are empirically tested.

Table 1 (continued): Models of Parent-Child Interaction ("✓" means present; "X" means absent)

	Child's Effort Observable	Child's Human Capital Observable	Model Is Estimated
Cosconati (2013a)	X	✓	✓
Akabayashi (2006b)	✓	X	X
Hao et al. (2008)	X	X	✓[g]
Lizzeri and Siniscalchi (2008b)	✓	X	X

[a] Difference in discount factors. [b] Differences in utility functions. [c] Restrictions on leisure. [d] Differences in knowledge about proper task execution. [e] Time investments in the child. [f] Monetary investments in the child. [g] The authors analyze the effects of time investments. [h] Implications from the model are empirically tested.

Our Contribution to the Literature

- Model and estimate parent-child interactions to understand how:
  - Skill forms
  - Child matures

Parent

- The parent has target skills and target investments
- Target skills evolve according to a predetermined skills production function
- The parent knows the technology of skill formation
- Parent observes a noisy skill realization – function of true skill and child's effort
- Parent chooses investment to minimize deviations from target

Parent's Problem

$$V_t(y_t) = \min_{u_t,\ldots,u_T} \mathbb{E}\left[\sum_{\tau=t}^{T} \beta^{\tau}\left(\widetilde{\theta_\tau}'Q\widetilde{\theta_\tau} + \widetilde{u_\tau}'R\widetilde{u_\tau}\right) + \widetilde{\theta_T}'Q_f\widetilde{\theta_T}|y_t\right] \quad (9)$$

- $Q, Q_f, R \geq 0$ are weighting matrices
- we define

$$\begin{aligned}
\widetilde{\theta_{t+1}} &= A\widetilde{\theta_t} + B\widetilde{u_t} + F\widetilde{a_t} + \omega_t \\
\theta_{t+1} &= \Phi(\theta_t, u_t, a_t); \quad \overline{\theta_{t+1}} = \Phi(\overline{\theta_t}, \overline{u_t}, \overline{a_t}) \\
y_t &= C\widetilde{\theta_t} + B\widetilde{a_t} + v_t \\
\widetilde{x_t} &= x_t - \overline{x_t} \text{ for } x_t = \theta, u_t, a_t
\end{aligned} \quad (10)$$

Child

- Enjoys parental investment and dislikes effort
- Her strategy could be "à la Cournot" or "à la Stackelberg"
- Uncertain on her ability and perfectly observes the rest of the components

Child's Problem

$$
\begin{aligned}
J_t &= \max_{a_t,\ldots,a_T} \lambda_t J_t^1(y_t) + (1-\lambda_t) J_t^2(y_t) \\
J_t^1(y_t) &= \mathbb{E}\left[\sum_{\tau=t}^{T} \beta^\tau \left(-\widetilde{a}_\tau + \widetilde{u}_\tau \widetilde{a}_\tau\right) | y_t\right] \\
J_t^2(y_t) &= \mathbb{E}\left[\sum_{\tau=t}^{T} \beta^\tau \left(-\widetilde{a}_\tau + R_t^P (\widetilde{a}_\tau)^2\right) | y_t\right]
\end{aligned}
$$

Identification

- Extend Cunha and Heckman (2008) to include child inputs
- Linear technology, repeated measures are instrumental variables for measurement error
- Second moments from simultaneous parent and child optimization imply,

$$\lambda_t = \frac{2M_t \mathrm{cov}\left(a_t, u_t\right)}{\mathrm{var}\left(u_t\right) + 2M_t \mathrm{cov}\left(a_t, u_t\right)} \tag{11}$$

- M_t is a function of technology parameters and preference parameters R, β
- Property: $\frac{\partial}{\partial R}\lambda_t < 0$
- Technology identified $\implies$ λ identified up to R, β

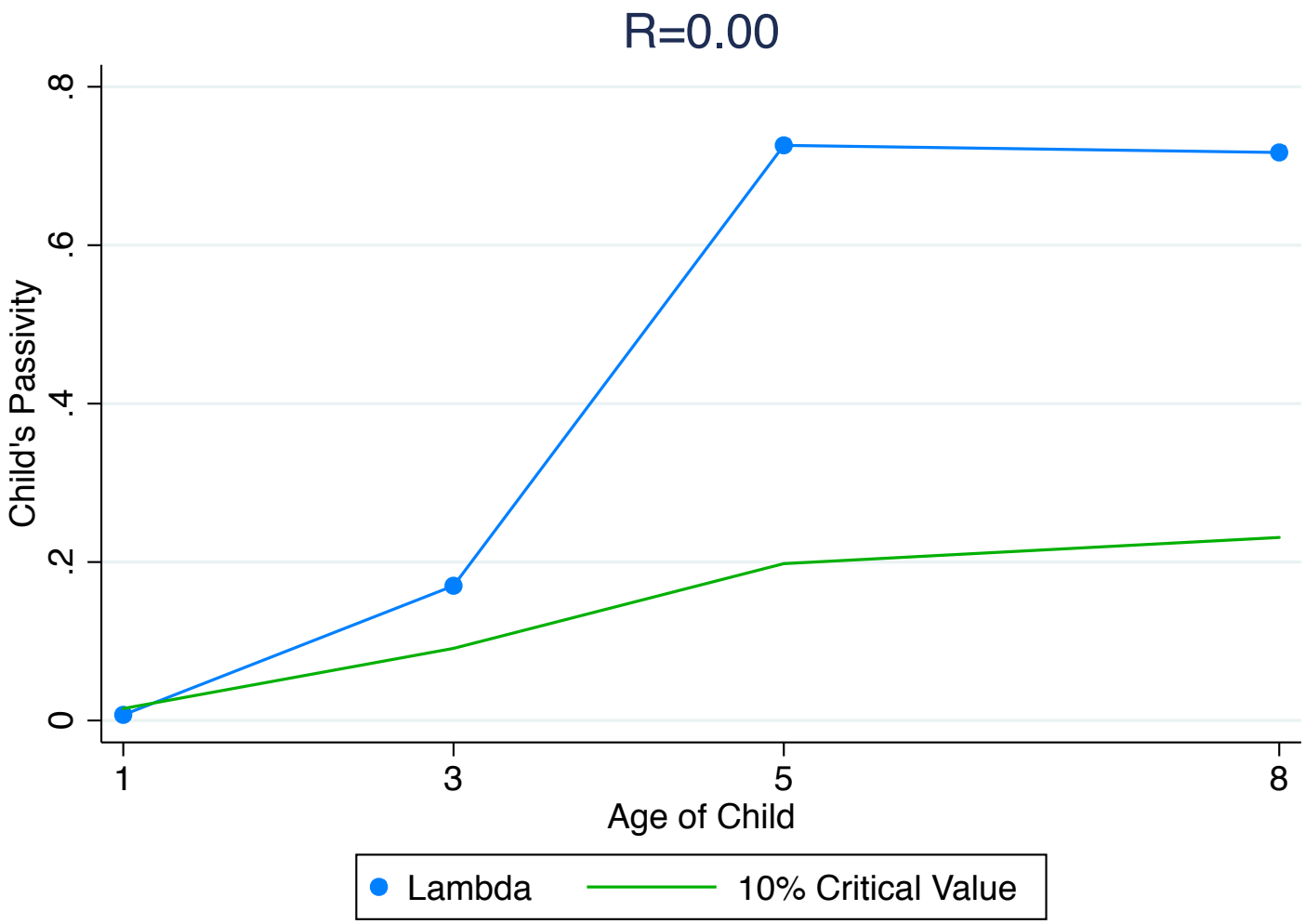
Estimation

- Technology: instrumental variables regression or MLE under distributional assumption
- Requires repeated measures of child's cognitive and non-cognitive skills as well as child and maternal effort over short stages of child development
- λ: Given technology and preferences, plug-in estimator
- Inference: bootstrap the estimators, test for non-zero technology coefficients and positive λ_t

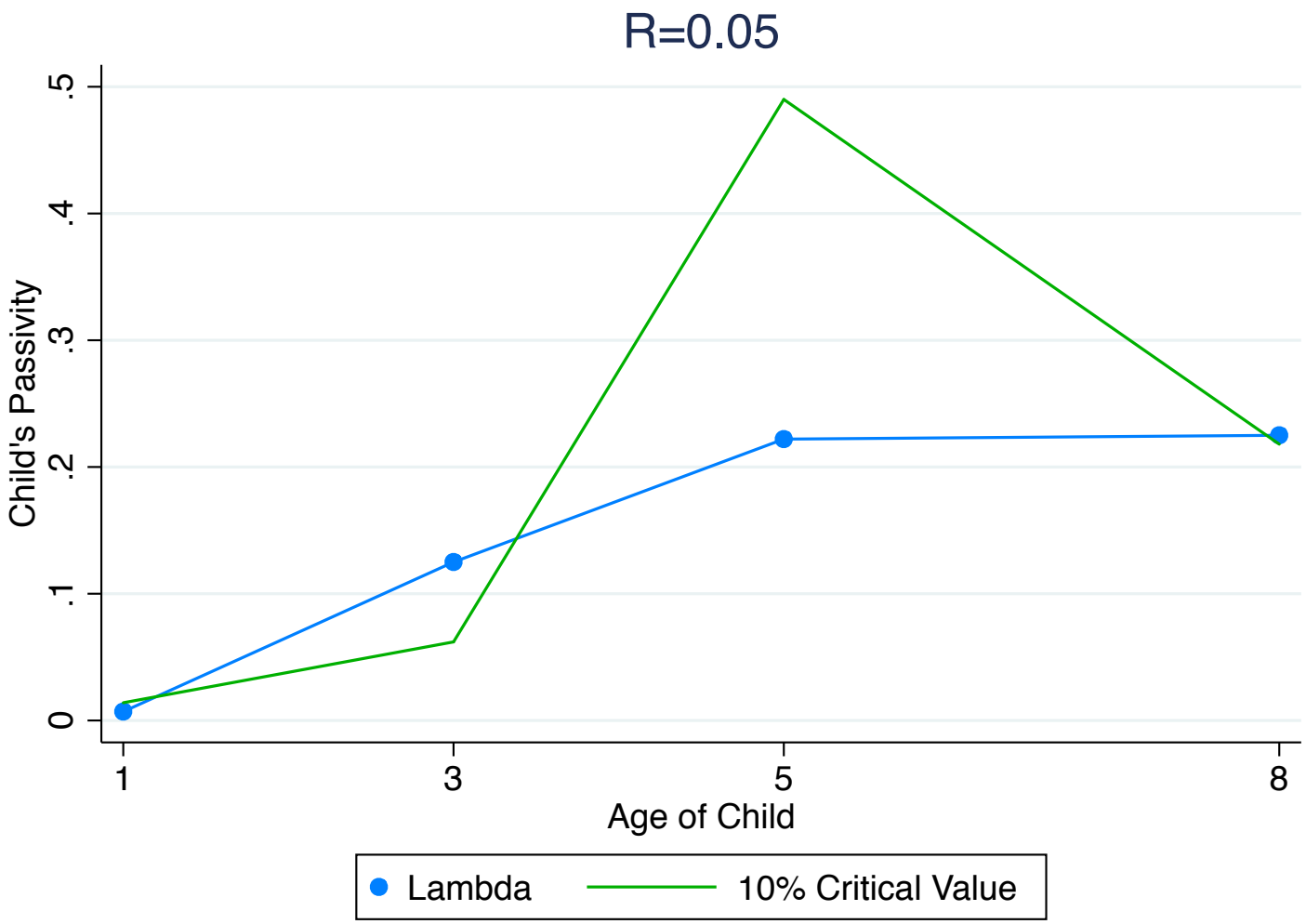
Estimation Details

Link

Child's Maturity over Time



Child's Maturity over Time



A Simulation Exercise Based on Akabayashi (2006)

Link

Link to

"Measuring Early Investments in Children"

by Jennifer Culhane, Flávio Cunha, Irma Elo, and Zoe Pham (2016)

END

Appendix: The Problem of the Parent

The Problem of the Parent

- The parent is assumed to be the decision-maker in the household.
- The child passively accepts investment.
- The consumption of the child is not modeled.
- The problem solved by the parent depends on the age of the child.
- When the child is between ages 1 and $T-1$, he only receives investments and cannot work.

- When the child reaches age T, the parent may invest a minimum level or something beyond that minimum.
- If the parent invests the minimum amount, the child does not attend college but becomes a high school graduate and works full time.
- If the parent invests any amount beyond the minimum, the child attends school (college) full-time.
- At the end of the period, he becomes a college graduate.

The Problem When the Child Is Between 1 and $T-1$ Years Old

- Parental labor supply is assumed to be perfectly inelastic.
- At each age t of the child, the parent is subject to productivity innovations ε_t, corresponding to labor market uncertainty.
- The shocks ε_t are independently and identically distributed across parents.

- The shocks follow a first-order Markov process:

$$\ln \varepsilon_{t+1} = \rho_\varepsilon \ln \varepsilon_t + \sigma_\eta \eta_t^\varepsilon. \tag{12}$$

- Parents are assumed to have positive earnings.
- Productivity innovations are restricted so that there exists ε_{min} with the property that $\varepsilon_t \geq \varepsilon_{\text{min}} > 0$ for any $t = T + 1, \ldots, 2T$.
- The labor income of the parent is $wh\varepsilon_t$, where w is the efficiency wage and r is the risk-free discount rate.
- Innovations in wages and labor market uncertainty are missing in BTS.

- The level of capability of the parent, h, is the outcome of investment decisions made by the grandparent.
- In similar fashion, the level of skill of the child when an adult, h', will also be the consequence of investments made by the parent, and satisfies $h' = \theta_{T+1}$.
- Defining s_t as the stock of savings of the parent at age t, the individual state variables for the parents of children who are between 1 and $T-1$ years old is $(h, \theta_t, s_t, \varepsilon_t, t)$.

- Given the state variables, the parent chooses household consumption c_t, savings s_{t+1}, and investments I_t in the cognitive skill of the child.
- The savings of the parents are in a risk-free asset which pays a rate of interest r.
- p denotes the price of the investment goods in cognitive skill.
- Following Laitner (1992), the parents cannot leave debts to their children and have negative net worth, so savings are subject to the lower bound equal to $\frac{-wh\varepsilon_{\min}}{(1+r)}$ (the "natural" borrowing limit).

- $V(t, h, \theta_t, s_t, \varepsilon_t)$ is the value function of the parent of a child at age t, $1 \leq t \leq T-1$. The problem of the parent is:

$$
\begin{aligned}
& V(t, h, \theta_t, s_t, \varepsilon_t) \\
& = \max_{c_t, l_t, s_{t+1}} \left\{ u(c_t) + \beta \mathbb{E}\left[V(t+1, h, \theta_{t+1}, s_{t+1}, \varepsilon_{t+1}) \middle| \varepsilon_t \right] \right\}.
\end{aligned}
$$

- Subject to:

$$c_t + pl_t + s_{t+1} = wh\varepsilon_t + (1 + r)\, s_t \tag{13}$$

$$s_{t+1} \geq -\left(wh\varepsilon_{\min}\right), \;\; l_t, c_t \geq 0 \tag{14}$$

- And the technology for capability formation

- Associating multiplier μ_t to the borrowing constraint in stage t, the optimal conditions for consumption and investments are given by:

$$u_c(c_t) = \beta(1+r)\mathbb{E}\left[V_s\left(t+1, h, \theta_{t+1}, s_{t+1}, \varepsilon_{t+1}\right)|\varepsilon_t\right] + \mu_t \tag{15}$$

$$\beta\mathbb{E}\left[\frac{\partial\theta_{t+1}}{\partial I_t} V_\theta\left(t+1, h, \theta_{t+1}, s_{t+1}, \varepsilon_{t+1}\right)|\varepsilon_t\right] = \beta(1+r)p\mathbb{E}\left[V_s\left(t+1, h, \theta_{t+1}, s_{t+1}, \varepsilon_{t+1}\right)|\varepsilon_t\right] + \mu_t \tag{16}$$

- This implies that the marginal utility of investments is equated to the marginal utility of consumption and to the marginal utility of future wealth.
- Whenever the constraint binds ($\mu_t \geq 0$), consumption and investment will be reduced as the agent would like to borrow more than ($wh\varepsilon_{min}$), but she is constrained.

- Suppose now that the agent is not constrained in period t.
- Using the envelope condition for assets we can rewrite the optimal condition for investment and consumption making clear the dependence on expected future constraints:

$$\beta \mathbb{E}_t \left[\frac{\partial \theta_{t+1}}{\partial I_t} V_\theta \left(t+1, h, \theta_{t+1}, s_{t+1}, \varepsilon_{t+1}\right) | \varepsilon_t \right] = p u_c(c_t) \tag{17}$$

$$= \beta(1+r) p \mathbb{E}_t \left[V_s \left(t+1, h, \theta_{t+1}, s_{t+1}, \varepsilon_{t+1}\right) | \varepsilon_t \right]$$

$$= [\beta(1+r)]^2 p \mathbb{E}_t \left[\mathbb{E}_{t+1} \left[V_s \left(t+2, h, \theta_{t+2}, s_{t+2}, \varepsilon_{t+2}\right) | \varepsilon_{t+1} \right] + \mu_{t+1} | \varepsilon_t \right] \tag{18}$$

$$= [\beta(1+r)]^2 p \left[\mathbb{E}_t \left[V_s \left(t+2, h, \theta_{t+2}, s_{t+2}, \varepsilon_{t+2}\right) | \varepsilon_{t+1} \right] \right. \tag{19}$$

$$\left. + \mathbb{E}_t [\mu_{t+1} | s_{t+1} = -wh\varepsilon_{min}] \right] P(s^*_{t+1} < -wh\varepsilon_{min})$$

- Where s^*_{t+1} represents the optimal unconstrained amount of savings from stage $t+1$ to stage $t+2$ and

$$P(s^*_{t+1} < -wh\varepsilon_{min}) = P\left(\varepsilon_{t+1}wh - c^*_{t+1}(\varepsilon_{t+1}wh) - pl^*_{t+1}(\varepsilon_{t+1}wh) < -wh\varepsilon_{min} - (1+r)s_t\right) \tag{20}$$

- c^*_{t+1} and I^*_{t+1} represent the optimal unconstrained levels of consumption and investments in period $t+1$ which depend on the realization of income.
- Even when the parent is not constrained in period t, the expectation of future constraints reduces current consumption and investments levels.
- The fear of hitting the constraint in the future induces a precautionary motive for savings which reduces current investments and consumption.

The Problem When the Child Is T Years Old: Go to College or Not?

- Consider the decision to go to college (made by the parent).
- When the child reaches age T, the parent decides to invest the minimum amount, $\underline{I}$, or something beyond that amount.
- The parent uses the relevant information to make that decision, which is contained in the vector of state variables $(h, \theta_t, s_t, \varepsilon_t, n_t)$.

- Let κ be tuition cost. The parent's problem can be stated as:

$$
\begin{aligned}
&V\left(T, h, \theta_T, s_T, \varepsilon_T\right) \\
&= \max_{c_T, I_T, s_1'} \left\{u\left(c_T\right) + \beta \mathbb{E}\left[V\left(1, h', \theta_1', s_1', \varepsilon_1'\right)\right]\right\}.
\end{aligned}
$$

- Subject to:

$$c_T + s_1' + p\underline{I} = wh\varepsilon_T + w\theta_T + (1+r)\, s_T \text{ if } I_T = \underline{I} \tag{21}$$

$$c_T + s_1' + (pI_T + \kappa) = wh\varepsilon_T + (1+r)\, s_T \text{ if } I_T > \underline{I} \tag{22}$$

$$s_T \geq 0 \tag{23}$$

- And the technology for the production of skills

- The budget constraint (21) states that a child who receives the minimum amount of investments $\underline{I}$ works full time.
- Refer to this child as a high school graduate.
- Note that the high-school-graduate child's earnings are pooled with the rest of the parental resources.
- Abstract from productivity shocks for the child before he reaches adulthood.
- If the parent decides to invest any amount above the minimum, so that $I_T > \underline{I}$, then the parent must pay the variable cost of the investment, which is p by unit, plus a fixed cost, φ—college tuition.

- A child who receives more than the minimum amount of investment does not work.
- This is described by the budget constraint (22).
- Note that equation (23) embodies the notion that the parent faces lifetime liquidity constraints.
- The parent dies and cannot leave debts to the child.

- Following Cunha (2007), one can establish a steady state general equilibrium.
- Firms producing final output under constant returns to scale.
- Also a child investment good is produced.
- Cunha (2007) establishes a stochastic general equilibrium for the steady state, extending Laitner to include human capital.

Return to main text

Appendix: Targeting Relatively More Investment Toward Disadvantaged Children Can Be Socially Efficient

Introduction

- Analyze the problem of investing in children with different initial endowments assuming that children are weighted equally ($\omega_k = 1$ for all k). Parents only care about total output.
- No social justice concerns.
- Families are assumed to only care about productivity.
- Consider the following two-stage model of childhood investment:

$$\theta_3 = f^{(2)}(\theta_2, I_2) \tag{24}$$

$$\theta_2 = f^{(1)}(\theta_1, I_1) \tag{25}$$

- θ_3 represents the level of skill at the beginning of adulthood.

- The functions are assumed to be strictly concave in I_2 and I_1, respectively, and twice differentiable.
- Concavity in θ_2 or θ_1 is not required for an optimum, although it plays a role in signing terms in the comparative statics exercise below.
- The assumptions made below imply that all inputs are normal.
- Total resources are E.
- The price of input i is p_i.
- There are two children: A and B.
- Their initial endowments are θ_1^A and θ_1^B, respectively.

- $\theta_1^A = \gamma\theta_1^B$ and consider how, from a position of initial equality ($\theta_1^A = \theta_1^B$ or $\gamma = 1$), raising the initial endowment of A affects Benthamite allocations of investment goods between A and B.
- Denote investment in the first period for child A by I_1^A and in the second period by I_2^A. I_1^B and I_2^B are defined analogously for child B.

One Period of Childhood: Version of the Problem as in Becker and Tomes (1986)

- Parents (or social planners) seek to maximize the aggregate of adult skills (θ_2):

$$\theta_2^A + \theta_2^B$$

$$\text{subject to} \quad E = p_1(I_1^A + I_1^B).$$

- First order condition is:

$$\text{F.O.C.:} \; f_2^{(1)}\left(\gamma\theta_1^B, I_1^A\right) = f_2^{(1)}\left(\theta_1^B, I_1^B\right).$$

$$\text{sign}\left(\frac{\partial I_1^A}{\partial \gamma}\right) = \text{sign}\left(f_{12}^{(1)}(\cdot)\right)_{\gamma=1}.$$

- $f_{12}^{(1)}(\cdot)$ is the value of $f^{(12)}$ in the neighborhood of $(\cdot)$.
- Parents (social planners) invest more in the disadvantaged if inputs are substitutes with initial endowments and they invest less if they are complements.
- These are **direct** complements and substitutes.

Multiperiod Setting

- In the multiperiod setting it is still optimal to invest more in the child with the lower initial endowment if $f_{12}^{(1)}(\cdot) < 0$ even though $f_{12}^{(2)}(\cdot) > 0$.
- Pattern consistent with the evidence on the evolution of complementarity at later stages in the life cycle: $f_{12}^{(1)}(\cdot) < f_{12}^{(2)}(\cdot)$.
- However, targeting relatively more investment to the initially more disadvantaged child can still be efficient if $0 \leq f_{12}^{(1)}(\cdot) \leq f_{12}^{(2)}(\cdot)$.

- To establish this suppose that parents (or social planners) seek to maximize

$$\theta_3^A + \theta_3^B,$$

- Subject to

$$E = p_1(I_1^A + I_1^B) + p_2(I_2^A + I_2^B).$$

First Order Conditions:

$$f_1^{(2)}\left(f^{(1)}\left(\theta_1^A, I_1^A\right), I_1^A\right) f_2^{(1)}\left(\theta_1^A, I_1^A\right) = \lambda p_1$$

$$f_2^{(2)}\left(f^{(1)}\left(\theta_1^A, I_1^A\right), I_2^A\right) = \lambda p_2$$

$$f_1^{(2)}\left(f^{1}\left(\theta_1^B, I_1^B\right), I_2^B\right) f_2^{(1)}\left(\theta_1^B, I_1^B\right) = \lambda p_1$$

$$f_2^{(2)}\left(f^{1}\left(\theta_1^B, I_1^B\right), I_2^B\right) = \lambda p_2$$

$$p_1\left(I_1^A + I_1^B\right) + p_2\left(I_2^A + I_2^B\right) = E.$$

- Consider an enhancement of the endowment of A in the neighborhood of initial equality ($\theta_1^A = \theta_1^B$).
- As before, let $\theta_1^A = \gamma\theta_1^B$.
- Perturb γ from position $\gamma = 1$.
- Take total differentials of the system of first order conditions:

$$
\left.
\begin{aligned}
\left\{ f_{11}^{(2)}(\cdot)\left[f_2^{(1)}(\cdot)\right]^2 + f_1^{(2)}(\cdot)f_{22}^{(1)}(\cdot)\right\} dI_1^A + \left[f_{12}^{(2)}(\cdot)f_2^1(\cdot)\right] dI_2^A + \theta_1^B \underbrace{\left[f_{11}^{(2)}f_1^{(1)}(\cdot)f_2^{(1)}(\cdot) + f_1^{(2)}(\cdot)f_{21}^{(1)}(\cdot)\right]}_{\text{“Term 1”}} d\gamma \\
= (d\lambda)p_1 + \lambda dp_1 \\
\left\{ f_{21}^{(2)}(\cdot)f_2^{(1)}(\cdot)\right\} dI_1^A + \left\{ f_{22}^{(2)}(\cdot)\right\} dI_2^A + \theta_1^B \underbrace{\left\{ f_{21}^{(2)}(\cdot)f_1^{(1)}(\cdot)\right\}}_{\text{“Term 2”}} d\gamma = (d\lambda)p_2 + \lambda dp_2 \\
\left\{ f_{11}^{(2)}(\cdot)\left[f_2^{(1)}(\cdot)\right]^2 + f_1^{(2)}(\cdot)f_{21}^{(1)}(\cdot)\theta_1^B\right\} dI_1^B + \left[f_{12}^{2}(\cdot)f_2^1(\cdot)\right] dI_2^B = (d\lambda)p_1 + \lambda dp_1 \\
\left\{ f_{21}^{(2)}(\cdot)f_2^{(1)}(\cdot)\right\} dI_1^B + \left\{ f_{22}^{(2)}(\cdot)\right\} dI_2^B = (d\lambda)p_2 + p_2 d\lambda \\
-dE + p_1 dI_1^A + p_2 dI_2^A + p_1 dI_1^B + p_2 dI_2^B + I_1^A dp_2 + I_2^A dp_2 + I_1^B dp_1 + I_2^B dp_2 = 0.
\end{aligned}
\right\} \quad (26)
$$

A Three-Stage Analysis

- Fruitful to analyze the problem in three stages.
- In the first stage, we consider, for a single agent, how as $\gamma \uparrow$, the allocation of a fixed bundle of resources between investment in the first period and investment in the second period is affected.
- Then in the second stage consider how, as $\gamma \uparrow$, the productivity of expenditure changes and how resources are allocated across A and B.
- Clearly, resources shift to where they become more productive.

- In the third stage, consider how an increase in resources is allocated between the first and the second periods.
- Sometimes convenient to use fictitious child A specific prices (p_1^A and p_2^A) and child B specific prices p_1^B and p_2^B.

- Expenditures on child A and child B:

$$E_A = p_1 I_1^A + p_2 I_2^A$$
$$E_B = p_1 I_1^B + p_2 I_2^B.$$

- Maximize each of θ_3^A and θ_3^B separately subject to E_A and E_B.
- Then allocate E_A and E_B to equalize marginal productivity of expenditure across A and B.

- Assume concavity of the production functions in terms of θ_1 or θ_2.
- This allows us to use standard results from consumer theory.
- The "$d\gamma$" terms act like income-compensated price changes.
- They do not affect total resources E. Assuming interior solutions, $\gamma \uparrow$ is like an income-compensated change in the (child-specific) input prices p_1 and p_2.

The effect of $\gamma \uparrow$ on the allocation of investments across periods holding E_A fixed

- Consider the effect of an increase in γ on the allocation of period one and period two investment of child A while E_A is fixed.
- We consider the allocation of E_A and E_B across A and B later).
- The displacement system derived from the first order conditions:

$$\underbrace{\begin{bmatrix} c & d & -p_1 \\ d & e & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix}}_{\substack{|M| \\ (+)}} \begin{bmatrix} dI_1^A \\ dI_2^A \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 - \theta_1^B(\text{Term } 1)d\gamma \\ \lambda dp_2 - \theta_1^B(\text{Term } 2)d\gamma \\ 0 \end{bmatrix}. \tag{26}$$

- The income compensated own price changes are negative.
- Cross effects can be shown to be positive under the conditions specified below.
- $|M| > 0$ from the assumption of a regular optimum.
- To simplify the notation, suppress the "$\cdot$" notation. We can sign

$$c = \left[f_{11}^{(2)} [f_2^{(1)}]^2 + f_1^{(2)} f_{22}^{(1)} \right] \leq 0$$

if period 2 production is concave in θ_2 and period 1 production is concave in I_1.

- We assume that all marginal products are strictly positive unless otherwise noted.
- But c might still be negative if period 2 production is convex in θ_2 ($f_{11}^{(2)} > 0$) provided $f_1^{(2)} f_{22}^{(1)}$ is sufficiently negative.

$$d = f_{12}^{(2)} f_2^{(1)} > 0 \quad \text{if there is second period complementarity}$$

and

$$e = f_{22}^{(2)} < 0 \quad \text{from concavity in } l_2.$$

- In displacement system (26),

$$\text{Term 1} \equiv \left[f_{11}^{(2)} f_1^{(1)} f_2^{(1)} + f_1^{(2)} f_{21}^{(1)} \right]$$

may be of either sign.

- The second grouping of terms in Term 1 is positive under first period complementarity.
- It is negative under substitutability.
- The first grouping is negative under concavity of $f^{(2)}$ in θ_2.
- Under second period complementarity ($f_{21}^{(2)} > 0$) and

$$\text{Term 2} = \left[f_{21}^{(2)} f_1^{(1)} \right] \geq 0.$$

- The change associated with Term 1 alone is opposite in sign to the change in the income-constant price of I_1^A which is negative.
- Similarly, a change associated with Term 2 alone is opposite in sign from a change in the price of I_2^A.

- Using Cramer's Rule,

$$\frac{\partial I_1^A}{\partial \gamma} = \frac{\begin{vmatrix} -\theta_1^B(\text{Term } 1) & d & -p_1 \\ -\theta_1^B(\text{Term } 2) & e & -p_2 \\ 0 & -p_2 & 0 \end{vmatrix}}{|M|}$$

$$= \left(\frac{(\text{Term } 1)p_2^2 - (\text{Term } 2)p_1 p_2}{|M|} \right) \theta_1^B.$$

- Focus on the numerator since the denominator is positive.
- Substitute out for p_1 and p_2 using the first order conditions (26).
- The numerator can be written as

$$\underbrace{\theta_1^B \left[\frac{1}{\lambda^2}\right]}_{+} \left\{ \left[f_{11}^{(2)} f_1^{(1)} f_2^{(1)} + f_1^{(2)} f_{21}^{(1)}\right] \left[f_2^{(2)}\right]^2 - \left[f_{21}^{(2)} f_1^{(1)} f_1^{(2)} f_2^{(1)} f_2^{(2)}\right] \right\}.$$

- Focusing further on the term in braces (which is multiplied by a positive term), we obtain

$$\left\{\left[f_{11}^{(2)} f_1^{(1)} f_2^{(1)} \left[f_2^{(2)}\right]^2 + f_1^{(2)} f_{21}^{(1)} \left(f_2^{(2)}\right)^2 - f_{21}^{(2)} \left(f_1^{(2)}\right)^2 f_2^{(2)} f_1^{(1)} f_2^{(1)}\right]\right\}.$$

$$= f_2^{(1)} \left(f_2^{(2)}\right)^2 f_1^{(2)} \left[\frac{f_{11}^{(2)}}{f_1^{(2)}} \left(f_1^{(1)}\right) + \frac{f_{21}^{(1)}}{f_2^{(1)}} - \frac{f_{21}^{(2)}}{f_2^{(2)}} f_1^{(2)} \right]$$

$$= f_2^{(1)} \left(f_2^{(2)}\right)^2 f_1^{(2)} \left[f_1^{(1)} \underbrace{\left[\frac{\partial lnf_1^{(2)}}{\partial \theta_2^A} \right]}_{\substack{(-) \\ \text{Diminishing marginal} \\ \text{productivity of } \theta_2}} + \underbrace{\left(\frac{\partial lnf_2^{(1)}}{\partial \theta_1^A} \right)}_{\substack{\text{Effect of } \theta_1^A \\ \text{on marginal} \\ \text{productivity of } I_1^A \\ (+) \\ \text{(under} \\ \text{complementarity)}}} - \underbrace{\left(\frac{\partial lnf_2^{(2)}}{\partial \theta_2^A} \right)}_{\substack{\text{Effect of } \theta_2^A \\ \text{on marginal} \\ \text{productivity of } I_2^A \\ (+) \\ \text{(under} \\ \text{complementarity)}}} f_1^{(1)} \right]$$

- Note that $f_1^{(1)} = \frac{\partial \theta_2^A}{\partial \theta_1^A}$.
- This is the marginal self productivity of θ_1.

- Thus the term in brackets is:

$$\left[\underbrace{\frac{\partial lnf_1^{(2)}}{\partial \theta_1^A}}_{\substack{\text{The effect of } \theta_1^A \\ \text{on the marginal} \\ \text{productivity of } \theta_2^A}} + \underbrace{\frac{\partial lnf_2^{(1)}}{\partial \theta_1^A}}_{\substack{\text{The effect of } \theta_1^A \\ \text{on the marginal} \\ \text{productivity of } I_1^A}} - \underbrace{\frac{\partial lnf_2^{(2)}}{\partial \theta_1^A}}_{\substack{\text{The effect of } \theta_1^A \\ \text{on the marginal} \\ \text{productivity of } I_2^A}}\right]$$

$$= \frac{\partial}{\partial \theta_1^A}\left[lnf_1^{(2)} + lnf_2^{(1)} - lnf_2^{(2)}\right] \tag{27}$$

- Consider the three effects inside the bracket going from left to right.
- The first term is the effect of θ_1^A on the marginal product of θ_2^A in period 2 production.
- From concavity (in terms of θ_2^A), this term is negative.
- Diminishing returns is a force toward investing less in the first period.
- This term reflects how first period stocks of skills augment second period stocks of skills.

- If, example, $f_1^{(1)} = 0$ (so $\frac{\partial \theta_2^A}{\partial \theta_1^A} = 0$), this term is zero.
- This could occur if there is 100% depreciation of skills or if there is a threshold value of θ_1 beyond which increases in θ_1 do not affect θ_2 and the agent is at or beyond the threshold.
- If θ_2^A has a low or zero productivity in second period production, this term is small or zero.

- The second term is the effect of increasing θ_1^A on augmenting the productivity of first period investment in producing θ_2^A.
- This is the term that drives the analysis in a one period model of childhood.

- The third term is the effect of increasing θ_1^A on augmenting the productivity of second period investment.
- Again, if there is no self-productivity ($\frac{\partial \theta_2^A}{\partial \theta_1^A} = 0$), this term is zero.
- Greater complementarity with later stages in the life cycle is a force toward investing less in the first period.

- Thus, in the absence of self-productivity ($f_1^1 = \frac{\partial \theta_2^A}{\partial \theta_1^A} = 0$), the effect is driven solely by the second term.
- Under complementarity, the sign of the effect is positive.

- Thus,

$$\frac{\partial I_1^A}{\partial \gamma} < 0$$

- if (a) $f^{(2)}$ concave in θ_2^A, $f_{21}^{(1)} < 0$, $f_{21}^{(2)} > 0$
- and/or (b) $f^{(2)}$ is concave in θ_2 and $\frac{\partial lnf_2^{(1)}}{\partial \theta_1} < f_1^{(1)} \frac{\partial lnf_2^{(2)}}{\partial \theta_2}$, or
- if there are other configurations so that the term in brackets in (27) is positive.

- Because of the budget constraint it follows that

$$\frac{\partial I_2^A}{\partial \gamma} > 0 \quad \text{if} \quad \frac{\partial I_1^A}{\partial \gamma} < 0.$$

- The effects are offsetting.
- This is an analysis for allocation of investment *within* the life cycle of child A.

Stage 2
The Effects on Productivity: Allocation Between A and B

- Let λ_A be the productivity of expenditure on A.
- λ_B is defined analogously for B.
- If, as $\gamma \uparrow$, $\lambda_A \uparrow$, it is optimal to allocate to $A(E_A \uparrow)$.
- If $\lambda_A \downarrow$ it is optimal to allocate less to A $(E_A \downarrow)$.
- The sign of this relationship hinges on the sign of Term 1 as we now show.

$$\frac{\partial \lambda_A}{\partial \gamma} = \frac{\begin{vmatrix} c & d & (-\text{Term } 1) \\ d & e & (-\text{Term } 2) \\ -p_1 & -p_2 & 0 \end{vmatrix}}{|M|} \theta_1^B$$

- Collecting terms and using the first order conditions (26), using

$$p_1 = \frac{1}{\lambda} f_1^{(2)} f_2^{(1)} \quad \text{and} \quad p_2 = \frac{1}{\lambda} f_2^{(2)}$$

$$\frac{\partial \lambda_A}{\partial \gamma} = \underbrace{\frac{\theta_1^B}{\lambda}\frac{1}{|M|}}_{(+)} \left[\underbrace{(\underbrace{\text{Term 1}}_{(?)})\underbrace{[f_2^{(2)} d - f_1^{(2)} f_2^{(1)} e]}_{\substack{Q_1 \\ (+)}}}_{} - \underbrace{\underbrace{(\text{Term 2})}_{(+)}\underbrace{[f_2^{(2)} c - d f_1^{(2)} f_2^{(1)}]}_{\substack{Q_2 \\ (-)}}}_{+} \right],$$

- Remember: $d > 0$; $e < 0$; $c < 0$.

$$Q_1 = f_2^{(1)} \left[f_2^{(2)} f_{12}^{(2)} - f_1^{(2)} f_{22}^{(2)} \right] > 0$$

- and

$$Q_2 = f_2^{(2)} f_{11}^{(2)} [f_2^{(1)}]^2 + f_2^{(2)} f_1^{(2)} f_{22}^{(1)} - f_{12}^{(2)} f_2^{(1)} f_1^{(2)} f_2^{(1)} < 0.$$

- Thus

$$\frac{\partial \lambda}{\partial \gamma} = \frac{\theta_1^B}{\lambda} \frac{1}{|M|} \left[\underset{?}{(\text{Term } 1)} \underset{(+)}{(Q_1)} + \underset{(+)}{(\text{Term } 2)} \underset{(+)}{Q_2} \right].$$

- So if Term 1 $(+)$, then $\frac{\partial \lambda_A}{\partial \gamma} > 0$.
- This is a sufficient condition.
- In this case, as $\gamma \uparrow$ it is efficient to allocate *more* to $A(E_A \uparrow)$.

- If Term 1 is sufficiently negative, it is optimal to allocate *less* to $A(E_A) \downarrow$.
- Recall that a sufficient condition for Term 1 to be negative is that $f_{21}^{(1)} < 0$.
- But even if $f_{21}^{(1)} > 0$, if there is sufficiently strong diminishing returns in $\theta_1(f_{11}^2 < 0)$, the optimal response of an increase in γ is to reduce I_1^A (i.e. to favor the disadvantaged child).

Stage 3:
Allocation of Changes in Endowments over Periods

- From standard results in consumer theory,

$$\frac{\partial I_1^A}{\partial E_A} = \frac{(-1)}{|M|}\begin{vmatrix} d & -p_1 \\ e & -p_2 \end{vmatrix} = \frac{dp_2 - p_1 e}{|M|} = \frac{(f_2^{(1)})}{\lambda |M|}\left[f_{12}^{(2)} f_2^{(2)} - f_1^{(2)} f_{22}^{(2)}\right] \geq 0.$$

- $d > 0$, $e < 0$.
- Recall we assume $f_{12}^{(2)} > 0$.
- From concavity it follows that $f_{22}^{(2)} < 0$.
- Thus $\frac{\partial I_1^A}{\partial E_A} > 0$.

$$\frac{\partial I_2^A}{\partial E_A} = \frac{\begin{vmatrix} c & -p_1 \\ d & -p_2 \end{vmatrix}}{|M|} = \underbrace{\frac{1}{\lambda |M|}}_{(+)} \left\{ \left(f_2^{(1)}\right)^{(2)} f_1^{(2)} f_{12}^{(2)} - \left(f_2^{(2)}\right)\left[f_{11}^{(2)} \left(f_2^{(1)}\right)^{(2)} + f_1^{(2)} f_{22}^{(1)}\right] \right\}$$

- This expression is also positive.
- Thus inputs are normal under our assumptions.
- For the case $p_1 = p_2 = 1$ (which we can assume with no loss of generality).

$$\frac{\partial I_1^A}{\partial E_A} = \frac{f_{12}^{(2)} f_2^{(1)} - f_{22}^{(2)}}{|M|}$$

$$\frac{\partial I_2^A}{\partial E_A} = \frac{f_{11}^{(2)} \left(f_2^{(1)}\right)^2 - f_1^{(2)} f_{22}^{(1)} + f_{12}^{(2)} f_2^{(1)}}{|M|}.$$

- Observe that $\frac{\partial I_1^A}{\partial E_A}$ is larger
  (a) the greater the second period complementarity ($f_{12}^{(2)}$) (so that I_1^A has greater productivity in producing final output θ_3^A),
  (b) the larger $f_2^{(1)}(= \frac{\partial \theta_2^A}{\partial I_1^A})$ (so that I_1^A is more productive in producing the intermediate product θ_2^A);
  (c) the more rapidly the decline in the productivity of I_2^A.
- Intuitively, relatively more is allocated to first period investment the more productive is the first period investment.

Putting It All Together

- The second step is the key one.
- It determines the allocation of expenditure across children in response to an increase in endowment ($\gamma \uparrow$).
- The greater the decline in self productivity with increases in θ_1 (the more negative $f_{11}^{(2)}$), the more likely it is that more resources are devoted to the less advantaged child.
- This negative effect is amplified by greater productivity of θ_1 in period 1 ($f_1^{(1)}$) and greater productivity of I_1 in period 1.

- These effects are reinforced if there is substitutability between θ_1 and $I_1(f_{21}^{(1)} < 0)$.
- If $f_{21}^{(1)}$ is positive, the redistributive effect is attenuated.
- This offsetting effect is weaker the smaller the productivity of θ_2 in period 2 production.

- The first step explores substitution effects arising from the change in γ.
- The third step explores income effects across periods arising from transfers across children.
- The other steps determine the allocation of investment across periods for each child.
- The analysis of the third step for each child informs us that resources are differentially allocated to the more productive period.
- The analysis of the first step makes a similar claim but investigates how changes in γ affect the relative productivity of investment in each period.

- In Section 208 below, we establish that if first period investment (I_1) and initial endowment (θ_1) are substitutes, ($f_{12}^{(1)} < 0$), but θ_2 is complementary with second period investments ($f_{12}^{(2)} > 0$), first period investments are greater for the more disadvantaged child.

- But even if ($f_{12}^{(1)} > 0$), greater first period investment in the initially disadvantaged child may be optimal.
- This is more likely (ceteris paribus).

(a) the more steeply diminishing is the productivity of second period skills ($f_{22}^{(2)}$);

(b) the greater the self productivity of the stock of skills in the first period ($f_1^{(1)} = \frac{\partial \theta^2}{\partial \theta_1}$);

(c) the smaller first period complementarity ($f_{21}^{(1)}$) relative to second period complementarity and absolutely

(d) the more rapidly diminishing the marginal productivity of θ_1($f_{11}^{(1)}$);

(e) the greater the second period complementarity ($f_{12}^{(2)}$);

(f) the greater the first period productivity of investment ($f_2^{(1)}$) and

(g) the more rapidly diminishing the productivity of second period investment ($f_{22}^{(2)}$).

- Roughly speaking, the more concave are the technologies in terms of stocks of skills, the more favorable is the case for investing relatively more in the disadvantaged child.
- The greater the second period complementarity ($f_{12}^{(2)}$), the greater the case for investing more in the initially disadvantaged child to allow the child to benefit from greater second period complementarity of the stock of skills with second period investment.

- In general, even when investment is greater in the first period for the disadvantaged child, second period investment is greater for the initially advantaged child.
- It is generally not efficient to make the initially disadvantaged child whole as it enters the second period when the effect of greater second period complementarity kicks in.

Appendix

- Direct proofs of some additional propositions

Proof that $f_{12}^{(1)} < 0$ is sufficient for $\frac{\partial I_A^1}{\partial \gamma} < 0$.

- Consider the bordered Hessian displacement system associated for the problem for both children treated together:

$$\begin{bmatrix} c & d & 0 & 0 & -p_1 \\ d & e & 0 & 0 & -p_2 \\ 0 & 0 & c & d & -p_1 \\ 0 & 0 & d & e & -p_2 \\ -p_1 & -p_2 & -p_1 & -p_2 & 0 \end{bmatrix} \begin{bmatrix} dl_1^A \\ dl_2^A \\ dl_1^B \\ dl_2^B \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 - \theta_A^1(\text{Term 1})d\gamma \\ \lambda dp_2 - \theta_1^B(\text{Term 2})d\gamma \\ \lambda dp_1 \\ \lambda dp_2 \\ -dE + \sum\limits_{\substack{j\in\{A,B\} \\ l\in\{1,2\}}} l_l^j dp_l \end{bmatrix} \tag{28}$$

- where as before

$$c = \left[f_{11}^{(2)} [f_2^{(1)}]^2 + f_1^{(2)} f_{22}^{(1)} \right] \leq 0$$

- if period 2 production is concave in θ_2 and period 1 production is concave in I_1.
- But it might also arise if period 1 production is convex in θ_2.

$$d = f_{12}^{(2)} f_2^{(1)} > 0 \quad \text{if there is second period complementarity}$$

$$e = f_{22}^{(2)} < 0 \quad \text{from concavity in } I_2.$$

- Recall that

$$T_1 \equiv \text{Term 1} \equiv \left[f_{11}^{(2)} f_1^{(1)} f_2^{(1)} + f_1^{(2)} f_{21}^{(1)} \right]$$

- may be of either sign.

- The second grouping of terms in Term 1 is positive under complementarity in the first period; negative under substitutability.
- The first grouping is negative under concavity of $f^{(2)}$ in θ_2 (but it might be positive if there are increasing returns).
- Under second period complementarity ($f_{21}^{(2)} > 0$)

$$T_2 \equiv \text{Term 2} = \left[f_{21}^{(2)} f_1^{(1)} \right] \geq 0.$$

- Let H be the bordered Hessian associated with displacement system (28) and let $|H|$ be the determinant of the Hessian.
- $|H| > 0$ under the assumption of a regular optimum.

- Then the income-compensated effect of a change in p_2^A on I_1^A is

$$\frac{\partial I_1^A}{\partial p_2^A} = \lambda \begin{vmatrix} d & 0 & 0 & -p_1 \\ 0 & c & d & -p_1 \\ 0 & d & e & -p_2 \\ 0 & -p_1 & -p_2 & 0 \end{vmatrix} / |H|$$

$$= \lambda d \underbrace{\begin{vmatrix} c & d & -p_1 \\ d & e & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix}}_{<0} / |H|. \tag{29}$$

- The numerator of (29) is negative from the sufficiency conditions for an optimum for the two stage budgeting problem for A and from second period dynamic complementarity ($d > 0$).

- Hence both inputs are Hicks-compensated cross substitutes:

$$\frac{\partial I_1^A}{\partial p_2^A} < 0.$$

- and from symmetry

$$\frac{\partial I_1^A}{\partial p_2^A} = \frac{\partial I_2^A}{\partial p_1^A} < 0.$$

- Collecting results,

$$\text{let} \quad S_{ij} = \frac{\partial I_i^A}{\partial p_j^A} \qquad i, j \in \{1, 2\}$$

$$\frac{\partial I_1^A}{\partial \gamma} = -\left\{ \underset{(-)}{[S_{11}]} \underset{(?)}{[\text{Term } 1]} + \underset{(-)}{[S_{12}]} \underset{(+)}{[\text{Term } 2]} \right\} d\gamma \tag{30}$$

$$\frac{\partial I_2^A}{\partial \gamma} = -\left\{ \underset{(-)}{[S_{12}]} \underset{(?)}{[\text{Term } 1]} + \underset{(-)}{[S_{22}]} \underset{(+)}{[\text{Term } 2]} \right\} d\gamma. \tag{31}$$

- If Term 1 is sufficiently negative, which could happen even if $f_{21}^{(1)}(\cdot) > 0$, then

$$\frac{\partial I_1^A}{\partial \gamma} < 0.$$

- (Term 1 would be negative if $f_{21}^{(1)} < 0$) and possibly even

$$\frac{\partial I_2^A}{\partial \gamma} < 0.$$

- Term 1 positive $\Rightarrow \frac{\partial I_1^A}{\partial \gamma} > 0$ and $\frac{\partial I_1^B}{\partial \gamma} < 0$.
- Thus it may be efficient to allocate more to the less endowed, even in both periods.

- We can say something stronger.
- If $f_{12}^{(1)} < 0$, but $f_{12}^{(2)} > 0$, then as $\gamma \uparrow, I_A \downarrow$ and the term in braces in (30) is positive.
- To prove this define $T_1 =$ Term 1 and $T_2 =$ Term 2 and notice that

$$\frac{\partial I_1^A}{\partial \gamma} = \begin{vmatrix} -T_1\theta_1^B & d & 0 & 0 & -p_1 \\ -T_2\theta_1^B & e & 0 & 0 & -p_2 \\ 0 & 0 & c & d & -p_1 \\ 0 & 0 & d & e & -p_2 \\ 0 & -p_2 & -p_1 & -p_2 & 0 \end{vmatrix} = \frac{|N|}{|H|}\theta_1^B,$$

$$\underbrace{|H|}_{(+)}$$

where

$$|N| = \left\{ -T_1 e \underbrace{\begin{vmatrix} c & d & -p_1 \\ d & e & -p_2 \\ -p_2 & -p_2 & 0 \end{vmatrix}}_{|M|>0} - T_1 p_2 \begin{vmatrix} 0 & 0 & -p_2 \\ c & d & -p_1 \\ d & e & -p_2 \end{vmatrix} \right.$$

$$\left. + \; T_2 d \underbrace{\begin{vmatrix} c & d & -p_1 \\ d & e & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix}}_{|M|>0} + T_2 p_2 \begin{vmatrix} 0 & 0 & -p_1 \\ c & d & -p_1 \\ d & e & -p_2 \end{vmatrix} \right\} \theta_1^B$$

$$|N| = \left[(-T_1 e + T_2 d)|m| - \underbrace{\underset{(?)}{(T_1)}\,\underset{(+)}{p_2}\,\underset{(-)}{(-p_2)}\underset{(+)}{\begin{vmatrix} c & d \\ d & e \end{vmatrix}}}_{-\text{if } (T_1)<0} + \underbrace{\underset{(+)}{T_2}\,\underset{(+)}{p_2}\,\underset{(-)}{(-p_1)}\underset{(+)}{\begin{vmatrix} c & d \\ d & e \end{vmatrix}}}_{(-)} \right] \theta_1^B$$

- Thus it follows as a sufficient condition that

$$|N| < 0 \text{ if } \Big[(-T_1 e + T_2 d) < 0\Big].$$

- Writing out $(-T_1 e + T_2 d)$,

$$(-T_1 e + T_2 d) = -f_{11}^{(2)} f_1^{(1)} f_2^{(1)} f_{22}^{(2)} - f_1^{(2)} f_{21}^{(1)} f_{22}^{(2)} + f_{21}^{(2)} f_1^{(1)} f_{12}^{(2)} f_2^{(1)},$$

- and collecting the first and the last terms:

$$\underbrace{-\underbrace{f_1^{(1)} f_2^{(1)}}_{(+)} \underbrace{\left[f_{11}^{(2)} f_{22}^{(2)} - [f_{12}^{(2)}] \right]}_{(+) \text{ by concavity}}}_{(-)} - \underbrace{\underset{(+)}{f_1^{(2)}} \underset{(?)}{f_{21}^{(1)}} \underset{(-)}{f_{22}^{(2)}}}_{(-) \text{ if } f_{21}^{(1)} < 0} \tag{32}$$

- so

$$(-T_1 e + T_2 d) < 0 \quad \text{if} \quad f_{21}^{(1)} < 0,$$

- and hence

$$|N| < 0 \quad \text{if} \quad f_{21}^{(1)} < 0,$$

- so

$$\frac{\partial I_1^A}{\partial \gamma} < 0 \quad \text{if} \quad f_{21}^{(1)} < 0.$$

- Notice, however, that even if $f_{21}^{(1)}(\cdot) > 0$, it is possible that

$$\frac{\partial I_1^A}{\partial \gamma} < 0.$$

- (See the second term in equation (32).)
- Notice that the more negative $f_{22}^{(2)}$ (i.e., the more sharply are the diminishing returns to I_2^A in period 2), the more negative is $\frac{\partial I_1^A}{\partial \gamma}$.

- The intuition for this offsetting effect is that as second period investments become less effective, then it is more productive to invest relatively more in the first period.
- Concavity in terms of θ_2 is not strictly required.

- Next consider

$$\frac{\partial I_2^A}{\partial \gamma} =$$

$$\frac{\begin{vmatrix} c & -T_1\theta_1^B & 0 & 0 & -p_1 \\ d & -T_2\theta_1^B & 0 & 0 & -p_2 \\ 0 & 0 & c & d & -p_1 \\ 0 & 0 & d & e & -p_2 \\ -p_1 & 0 & -p_1 & -p_2 & 0 \end{vmatrix}}{|H|}$$

$$= \frac{\tilde{N}}{|H|}\theta_1^B$$

$$\tilde{N} = T_1 \begin{vmatrix} d & 0 & 0 & -p_2 \\ 0 & c & d & -p_1 \\ 0 & d & e & -p_2 \\ -p_1 & -p_1 & -p_2 & 0 \end{vmatrix}$$

$$-T_2 \begin{vmatrix} c & 0 & 0 & -p_1 \\ 0 & c & d & -p_1 \\ 0 & d & e & -p_2 \\ -p_1 & -p_1 & -p_2 & 0 \end{vmatrix}$$

$$= T_1 \left[d \begin{vmatrix} c & d & -p_1 \\ d & e & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} + p_1 \begin{vmatrix} 0 & 0 & -p_2 \\ c & d & -p_1 \\ d & e & -p_2 \end{vmatrix} \right]$$

$$-T_2\left[c\begin{vmatrix} c & d & -p_1 \\ d & e & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} + p_1\begin{vmatrix} 0 & 0 & -p_1 \\ c & d & -p_1 \\ d & e & -p_2 \end{vmatrix}\right]$$

$$= (T_1 d - T_2 c) \underset{(+)}{\begin{vmatrix} c & d & -p_1 \\ d & e & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix}}$$

$$- \underbrace{T_1 p_1 p_2}_{(-) \text{ if } T_1 < 0} \underset{(+)}{\begin{vmatrix} c & d \\ d & e \end{vmatrix}} + \underset{(+)}{T_2 (p_1)^2} \underset{(+)}{\begin{vmatrix} c & d \\ d & e \end{vmatrix}}$$

- Focus on the term $(T_1 d - T_2 c)$

$$= \underset{(-)}{(}\underset{(+)}{T_1}\, d \underset{)}{)} - \underset{(+)}{(}\underset{(-)}{T_2}\, c \underset{)}{)} \underset{(+)}{\begin{vmatrix} c & d & -p_1 \\ d & e & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix}} - \underset{(-)}{(}\underset{(+)}{T_1 p_1 p_2} - \underset{+}{T_2} \underset{+}{p_1^2}\underset{)}{)} \underset{(+)}{\begin{vmatrix} c & d \\ d & e \end{vmatrix}}$$

- Observe that

$$
\begin{aligned}
(T_1 d - T_2 c) = & \left[f_{11}^{(2)} f_1^{(1)} f_2^{(1)} + f_{12}^{(1)} f_1^2 \right] f_{12}^{(2)} f_2^{(1)} \\
& - f_{21}^{(2)} f_1^{(1)} \left[f_{11}^{(2)} \left(f_2^{(1)} \right)^2 + f_1^{(2)} f_{22}^{(1)} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \cancel{f_{11}^{(2)} f_1^{(1)} f_2^{(1)} f_{12}^{(2)} f_2^{(1)}} \\
&\quad + f_{12}^{(1)} f_1^{(2)} f_{12}^{(2)} f_2^{(1)} \\
&\quad \cancel{- f_{21}^{(2)} f_1^{(1)} f_{11}^{(2)} \left(f_2^{(1)} \right)^2} \\
&\quad - f_{21}^{(2)} f_1^{(1)} f_1^{(2)} f_{12}^{(1)} \\
&= f_{12}^{(2)} f_1^{(2)} \underset{(+)}{} \underbrace{\left[f_{12}^{(1)} f_2^{(1)} - f_1^{(1)} f_{22}^{(1)} \right]}_{T_3}
\end{aligned}
$$

- and the last term is positive ($T_3 > 0$), if in the period 1 production function $f_{12}^{(1)} > 0$ (first period complementarity).

- This is a sufficient condition for

$$\frac{\partial I_2^A}{\partial \gamma} > 0.$$

- Notice that when Term 1 (T_1) is negative, then T_3 can be negative.[1]
- Thus, it is possible that the efficient policy redistributes to the less endowed in period 1 but to the more endowed in period 2.

- It is also possible that as $\gamma \uparrow$, it is socially efficient to invest in the disadvantaged child in both periods, although this seems unlikely.
- In general, it is not efficient to make the initially disadvantaged child whole by the start of the second period, and second period complementarity reinforces starting of second period discrepancies.

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Appendix: Some Evidence from Simulations on Why Dynamic Complementarity is a Force Toward Targeting Disadvantaged Children in the Early Years

- Dynamic complementarity is a force toward equalization of early stage investments even in the absence of family inequality aversion.
- To illustrate the mechanism underlying this claim, suppose that, for each child k, the outcome of interest for parents are children's earnings E_k and that they are a function of children's adult human capital determined by "genes" $(\theta_{1,k})$ and early $(I_{1,k})$ and late $(I_{2,k})$ parental investments.

$$E_k = wf^2(\theta_{2,k}, I_{2,k}) = f^2\left(\gamma_2\theta_{2,k}^{\phi_2} + (1-\gamma_2)I_{2,k}^{\phi_2}\right)^{\frac{\rho_2}{\phi_2}} \tag{33}$$

with

$$\theta_{2,k} = f^1(\theta_{1,k}, I_{1,k}) = f^1\left(\gamma_1\theta_{1,k}^{\phi_1} + (1-\gamma_1)I_{1,k}^{\phi_1}\right)^{\frac{\rho_1}{\phi_1}} \tag{34}$$

- Where w is the payment to skill corresponding to one unit of human capital which is determined by equilibrium in the factor markets.
- Since w is common across families and siblings we assume that the measurement of human capital is chosen so that $w = 1$.

- The budget constraint faced by the parents with total resources R^e is:

$$p_1 \sum_{k=1}^{n} I_1 + p_2 \sum_{k=1}^{n} I_2 = R^e. \tag{35}$$

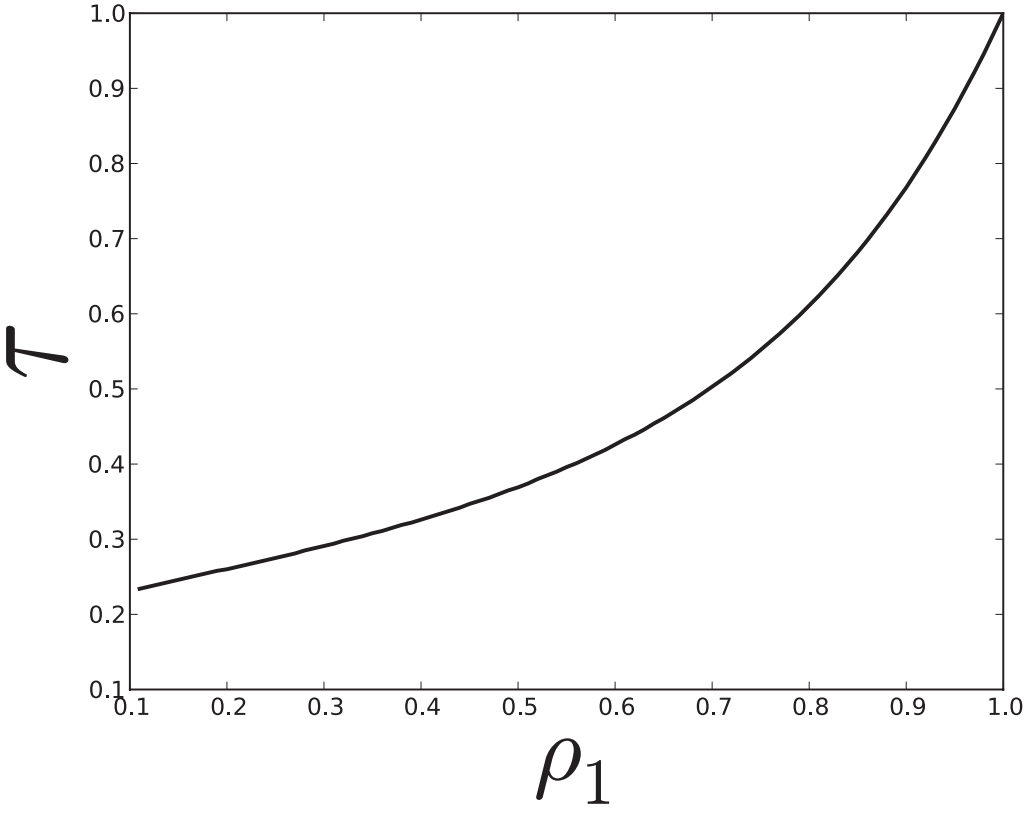
- Consider the case of a parent with two children i and j.
- We show that even in the absence of inequality aversion, the shape of the technology, and in particular the presence of decreasing returns in at least one of the two periods, might induce parents to follow a compensating strategy devoting more resources to the less endowed child, say j ($\theta_{1,i} > \theta_{1,j}$).

- As a measure of parental compensation with respect to initial inequality we define the parameter τ as:

$$\tau \equiv \left(\frac{E_i}{E_j}\right) \Big/ \left(\frac{\theta_i}{\theta_j}\right), \tag{36}$$

- Which captures how much earnings differences are inflated compared to initial endowment differences.
- If $\tau = 1$, the parents perfectly translate "genetic" differences into earnings.
- In results from a simulation exercise, Figure 6 shows that earnings differences are dampened compared to differences in initial endowments whenever $\rho_1 < 1$.

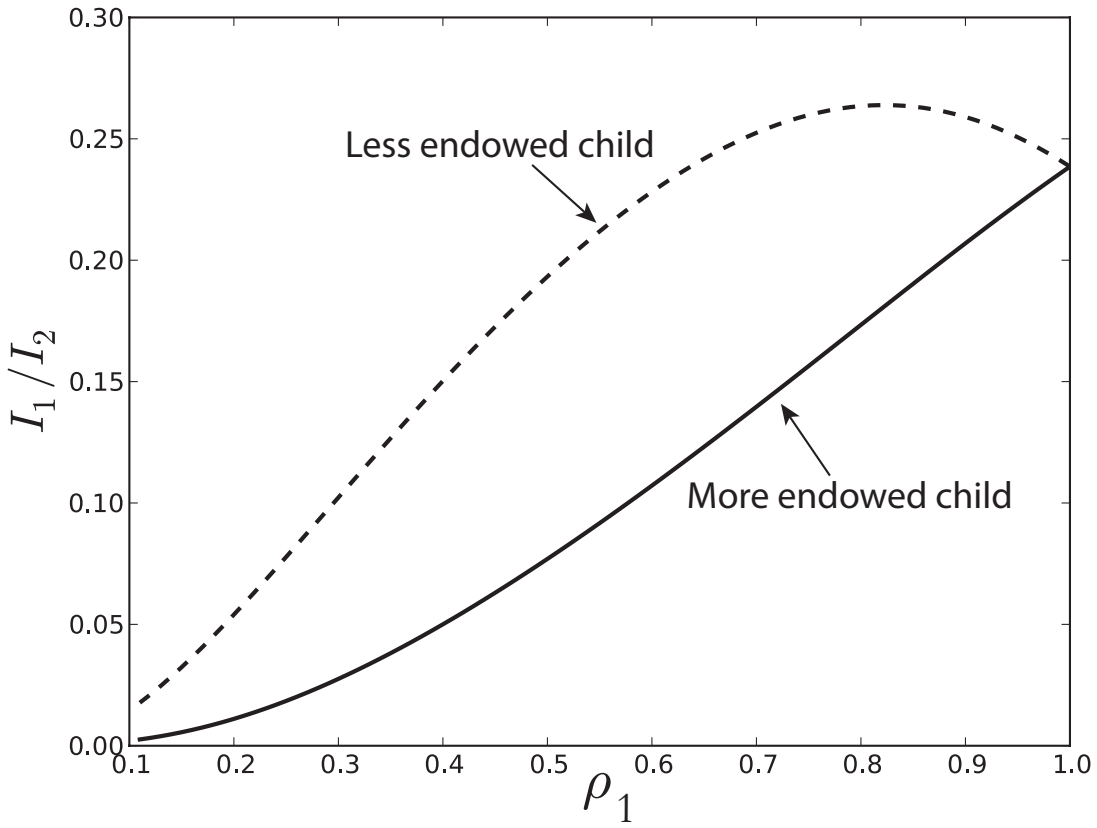
Figure 6: Earnings Equalization



Notes: The parental preference parameters used in the simulation are $\sigma = 1$ and $\omega_i = \omega_j = 0.5$. Total resources are $R^e = 4$. The technology of skill formation parameters, capturing increasing complementarity between skills and investments over time, are: $\gamma_1 = \gamma_2 = 0.5$, $\phi_1 = 0.6$, $\phi_2 = -0.5$, $\rho_2 = 1$. The parameter ρ_1 defines the degree of homogeneity of the first period technology. We vary the value of ρ_1 over the range $[0.1, 1]$. Child i has a skill endowment of 5 while child j of 1.

- We also consider the how changes in ρ_1 affect parental behavior in Figures 7, 8, and 9.
- Figure 7 shows the ratio of early (I_1) to late (I_2) investments.

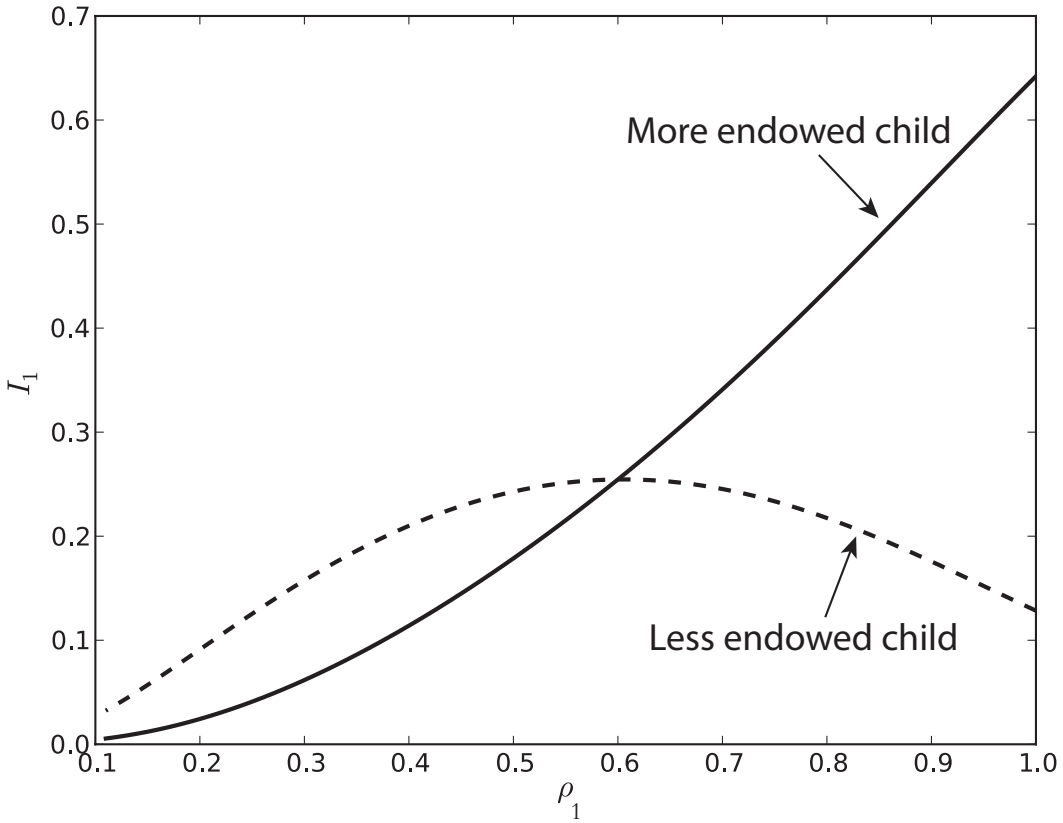
Figure 7: Ratio Early to Late Investments



Notes: The solid line refers to the most endowed child, the dashed line to the least endowed chid. The parameters used are as in Figure 6.

- This ratio is always higher for the less endowed child j whenever ρ_1 is smaller than one.
- Figure 8 shows that the *less* endowed child receives a *higher* amount of early investment whenever the period 1 technology exhibits substitutability between skills (initial endowments) and investments (i.e. when $\rho_1 < \phi_1$).

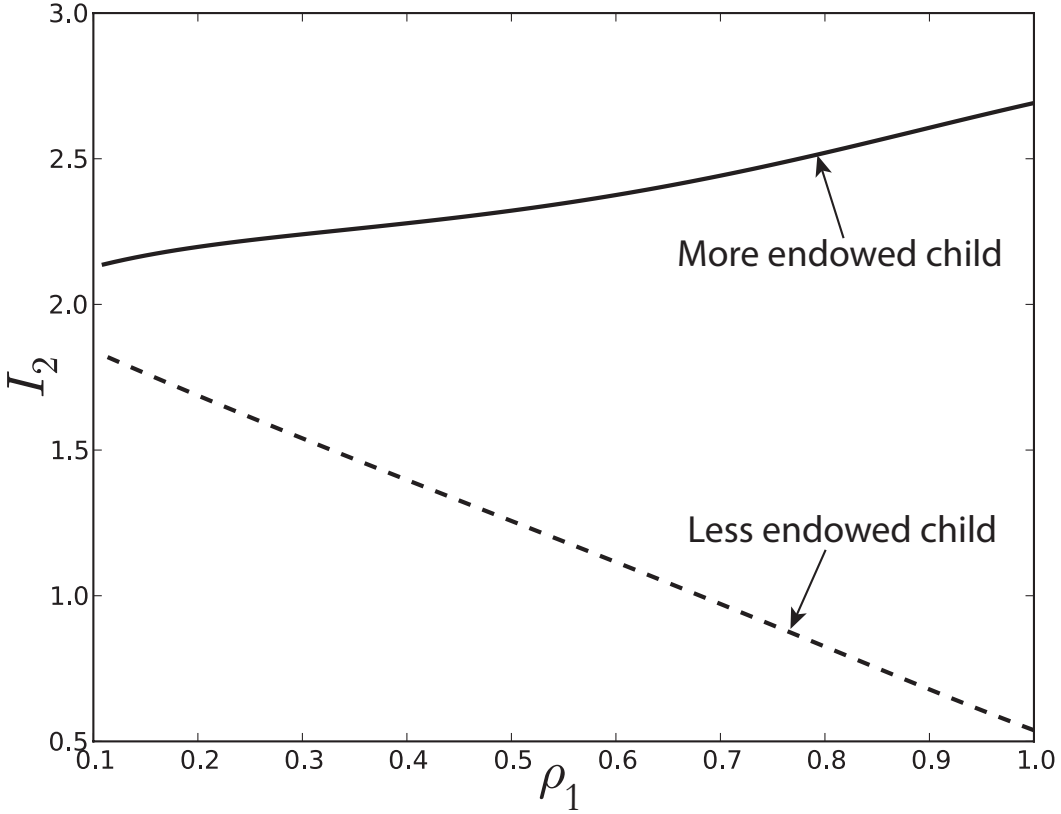
Figure 8: Levels of Early Investments



Notes: The solid line refers to the most endowed child, the dashed line to the least endowed chid. The parameters used are as in Figure 6.

- Figure 9 shows that the most endowed child always receives a higher level of late investment.

Figure 9: Levels of Late Investments



Notes: The solid line refers to the most endowed child, the dashed line to the least endowed chid. The parameters used are as in Figure 6.

- Late investments are an increasing function of ρ_1 for the more endowed child while they are decreasing in ρ_1 for the less endowed child.
- As ρ_1 decreases the less endowed child receives a higher level of early investments and a level of late investments which is increasingly closer to the one of his more endowed brother.
- This explains why earnings tend to be equalized as ρ_1 decreases.

- We conclude that if the technology of skill formation is defined over more than one period, parents might exhibit compensating behavior in investments in children's human capital even in absence of inequality aversion.
- In particular, less endowed children receive a higher level of early investment than their more endowed siblings if the technology of skill formation exhibits substitutability between initial (genetic) endowments and the level of early investments.

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Appendix:
Overview of Structural Models of Parental Investments

Table 2: Structural Models of Parental Investments ("✓" means present; "X" means absent)

	OLG Model	Dynastic Links	Explicit Models of Parental Preferences, Altruism (a) or Paternalism (p)	Model Estimated
Cunha and Heckman (2007)	✓	A, B, C,	✓(a)	X
Cunha (2007)	✓	A, B, C	✓(a)	✓
Caucutt and Lochner (2012)	✓	B, C	✓(a)	✓
Del Boca et al. (2014b)	X	X	✓(p)	✓
Gayle et al. (2015)	✓	A, C	✓(p)	✓
Cunha et al. (2013)	X	X	✓(p)	✓
Bernal (2008)	X	X	✓(p)	✓
Lee and Seshadri (2016)	✓	A, B, C	✓(a)	✓
Restuccia and Urrutia (2004)	✓	A, C	✓(a)	X
Attanasio et al. (2015)	X	A, B, C	✓(a)	✓
Meghir et al. (2015)	X	A, B, C	✓(a)	✓

A: Through parental skills, B: Through asset transfers, C: Through genes (initial conditions), [D]Natural borrowing limit, [E]Limits can be more stringent than natural limit.

Table 2: Structural Models of Parental Investments ("✓" means present; "X" means absent)

	Parental Goods Investment	Parental Time Investment	Technology Depends on Parental Skill	Self-productivity
Cunha and Heckman (2007)	✓	X	✓	✓
Cunha (2007)	✓	X	✓	✓
Caucutt and Lochner (2012)	✓	X	X	✓
Del Boca et al. (2014b)	✓	✓	X	✓
Gayle et al. (2015)	✓	✓	✓	X
Cunha et al. (2013)	✓	X	✓	✓
Bernal (2008)	✓	✓	✓	✓
Lee and Seshadri (2016)	✓	✓	✓	✓
Restuccia and Urrutia (2004)	✓	X	X	✓
Attanasio et al. (2015)	✓	✓	✓	✓
Meghir et al. (2015)	✓	X	✓	✓

A: Through parental skills, B: Through asset transfers, C: Through genes (initial conditions), [D]Natural borrowing limit, [E]Limits can be more stringent than natural limit.

Table 2: Structural Models of Parental Investments ("✓" means present; "X" means absent)

	Parental Learning About Technology	Bequests	Intragenerational Borrowing	Multiple Skills of Children
Cunha and Heckman (2007)	X	✓	✓[E]	X
Cunha (2007)	X	✓	✓[D]	X
Caucutt and Lochner (2012)	X	X	✓[E]	X
Del Boca et al. (2014b)	X	X	X	X
Gayle et al. (2015)	X	X	X	X
Cunha et al. (2013)	✓	X	X	X
Bernal (2008)	X	X	X	X
Lee and Seshadri (2016)	X	✓	✓[D]	X
Restuccia and Urrutia (2004)	X	X	X	X
Attanasio et al. (2015)	X	X	X	✓
Meghir et al. (2015)	X	X	X	X

A: Through parental skills, B: Through asset transfers, C: Through genes (initial conditions), [D]Natural borrowing limit, [E]Limits can be more stringent than natural limit.

Table 2: Structural Models of Parental Investment ("✓" means present; "X" means absent)

	Multichild Families (Preferences for Equity vs. Efficiency)	Endogenous Fertility Decisions	Multiple Parents	Endogenous Mating Decisions
Cunha and Heckman (2007)	X	X	X	X
Cunha (2007)	X	X	X	X
Caucutt and Lochner (2012)	X	X	X	X
Del Boca et al. (2014b)	✓	X	✓	X
Gayle et al. (2015)	✓	✓	✓	✓
Cunha et al. (2013)	X	X	X	X
Bernal (2008)	X	X	X	X
Lee and Seshadri (2016)	X	X	X	X
Restuccia and Urrutia (2004)	X	X	X	X
Attanasio et al. (2015)	X	X	X	X
Meghir et al. (2015)	X	X	X	X

A: Through parental skills, B: Through asset transfers, C: Through genes (initial conditions), [D]Natural borrowing limit, [E]Limits can be more stringent than natural limit.

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Inequality in Human Capital and Endogenous Credit Constraints (2016)

Rong Hai and James J. Heckman

- A dynamic model of schooling and working
- Agents subject to uninsured human capital risks, and face restrictions on their borrowing possibilities

Table 3: Structural Models on Educational Choices and Credit Constraints

	Human Capital Investment	Labor Supply	GSL	Private Loan Limit	CRRA Risk Aversion	Parental Influence	Data
Keane and Wolpin (2001)	Education and work experience	Yes	No	Borrowing limits not observed, proxied by a function of age and human capital; the parameters of the borrowing limit are estimated	Estimate $\gamma = 0.4826$	Parental transfers is a function of parental education and individuals choices	NLSY79 (1979-1992)
Navarro (2011)	Education	No	No	Models an endogenous natural borrowing limit based on education and inelastic labor supply, due to borrowers' limited repayment ability in the presence of uninsurable wage risk; however, in estimation, the borrowing limit is set equal to the lowest level of assets observed in samples in each period independently of the model	Estimate $\gamma = 0.82$	No	NLSY79 and PSID
Lochner and Monge-Naranjo (2011)	Education and work experience	No	Yes	Endogenous credit limit based on borrowers' cost of default (including temporary exclusion from credit market and wage garnishments), due to private lenders' limited ability to punish default; parameters on the cost of default are calibrated outside the model	Set $\gamma = 2$	No	NLSY79 (1979-2006)

Source: NLSY: National Longitudinal Survey of Youth. BHPS: British Household Panel Survey.

Table 3: Structural Models on Educational Choices and Credit Constraints Cont'd

	Human Capital Investment	Labor Supply	GSL	Private Loan Limit	CRRA Risk Aversion	Parental Influence	Data
Johnson (2013)	Education and work experience	Yes	Yes	Borrowing limits not observed; proxied by a function of age and human capital; the parameters of the borrowing limit proxy equation are estimated	Set $\gamma = 2$	Parental transfers is a function of parental income and child choices	NLSY97 (1997 to 2007)
Abbott, Gallipoli, Meghir, Violante (2016)	Education	Yes	Yes	No borrowing for high-school students; exogenous fixed debt limits for workers in the work-stage and college students whose parents are wealthy, respectively; borrowing limits are calibrated to match the fraction of households with zero or negative net worth and the aggregate private loans/GSL ratio, respectively	Set $\gamma = 2$	Parental transfers explicitly modeled	Multiple data, including NLSY79, NLSY97
Blundell, Costas Dias, Meghir, Shaw (2016)	Education and work experience	Yes	Yes	No borrowing permitted	Set $\gamma = 1.56$	Parental income and background factors affect youth's psychic cost of schooling	BHPS (1991 to 2008)
This paper	Education and work experience	Yes	Yes	Model-determined natural borrowing limit based on education and labor supply decisions, due to borrowers' limited repayment ability in the presence of uninsurable wage risk; no new auxiliary parameters for borrowing limit is added in estimation, unlike many previous papers	Set $\gamma = 2$	Parental transfer is a function of parental education and net worth, and individuals choices; parental education affects youth's psychic cost of schooling	NLSY97 (1997-2013)

Source: NLSY: National Longitudinal Survey of Youth. BHPS: British Household Panel Survey.

- Build model with natural borrowing limit
- Extend the existing literature by analyzing how cognitive and noncognitive ability affect choices through:
  (i) Psychic costs of working and schooling
  (ii) The technology of human capital production
  (iii) The discount factor

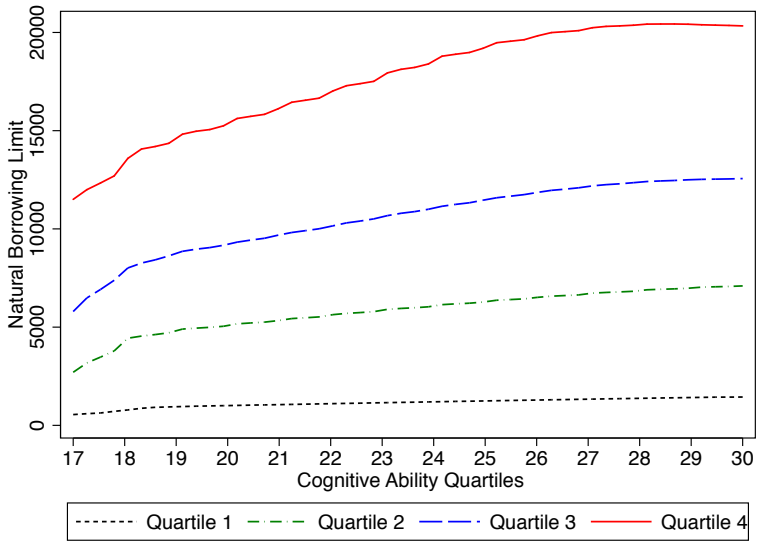
- Strong effects of adolescent endowments of cognitive and noncognitive ability on human capital development.
- Tuition costs and family transfers to children play important roles in explaining differences in life outcomes due to human capital investments.

Who Are the Credit Constrained?

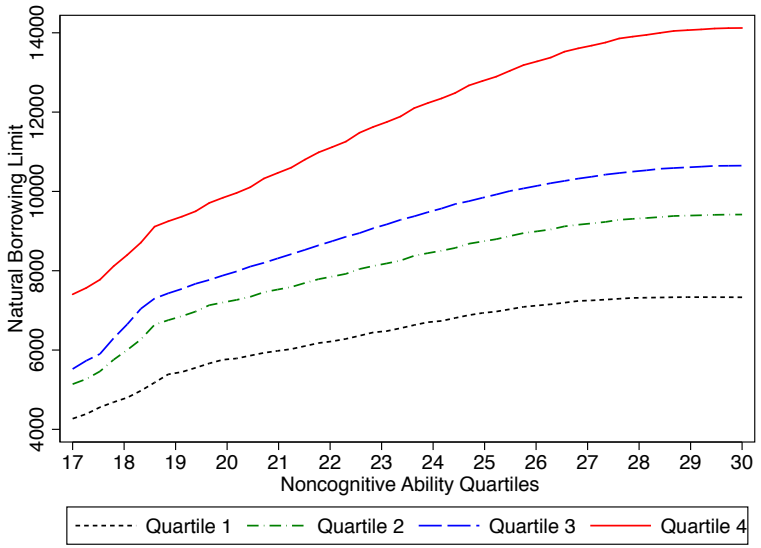
- Credit constrained agents fall into two groups:
  (a) Those with poor initial endowments and family background who acquire little human capital and have low wage levels and low life cycle wage growth
  (b) The very able and those from good family backgrounds who have high levels of human capital, high wage levels, and high life cycle wage growth

Figure 10: Evolution of Average Natural Borrowing Limit by Ability Endowments

(a) Natural Borrowing Limit *vs* Cognitive Ability over Ages 17 to 50



(b) Natural Borrowing Limit *vs* Noncog. Ability over Ages 17 to 50



Kuhn Tucker Multiplier on the Borrowing Constraint
(This is constraint on intertemporal lending and borrowing)

$$\lambda_{s,t}(c_t, s_{t+1}; \mathbf{\Omega}_t) = \frac{\partial u_c(c_t; \mathbf{\Omega}_t)}{\partial c_t} - \exp(-\rho(\theta_c, \theta_n)) \left(\frac{\partial \mathbb{E} V_{t+1}}{\partial s_{t+1}} \right) \tag{37}$$

Figure 11: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 21

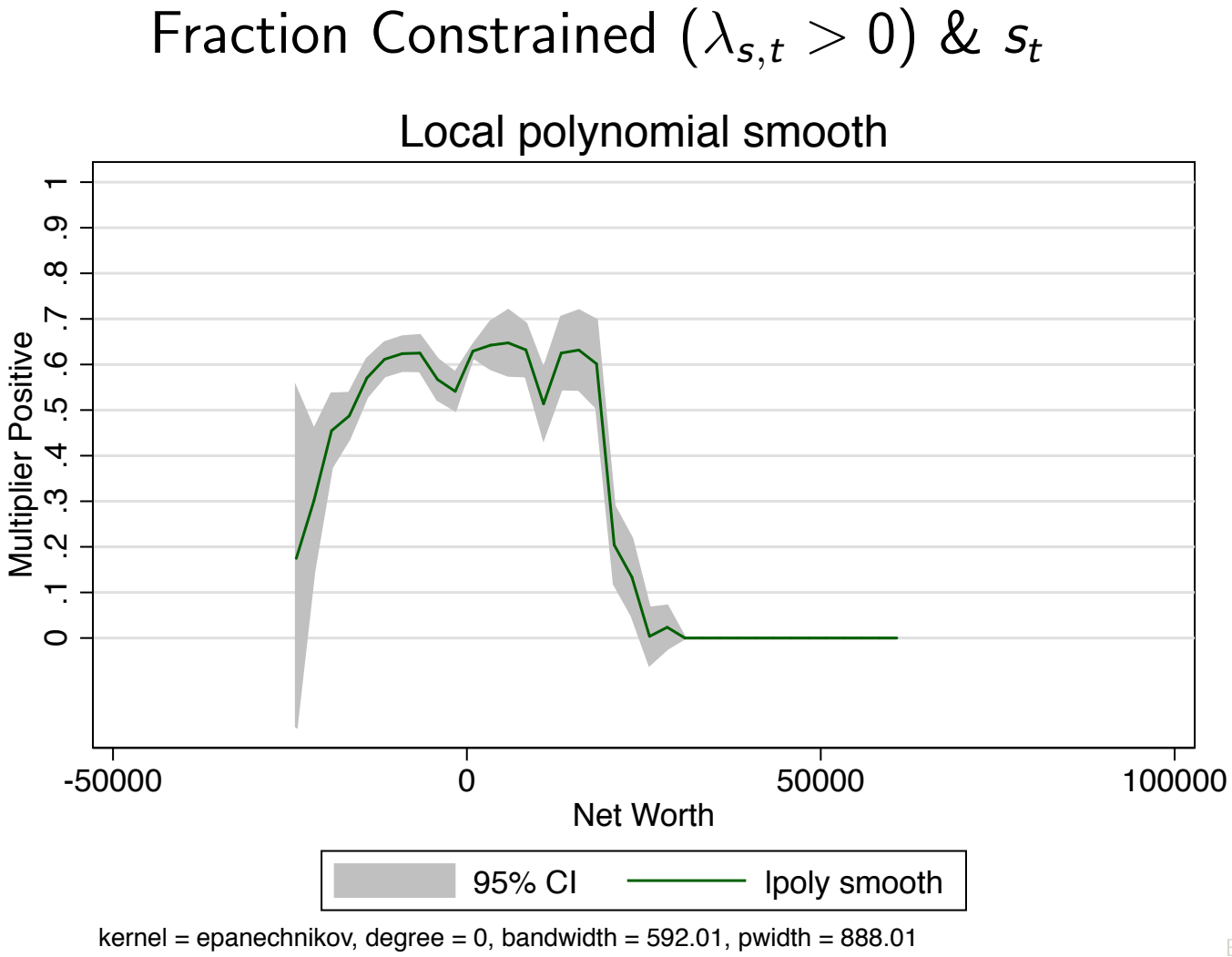


Figure 11: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 21 Cont'd

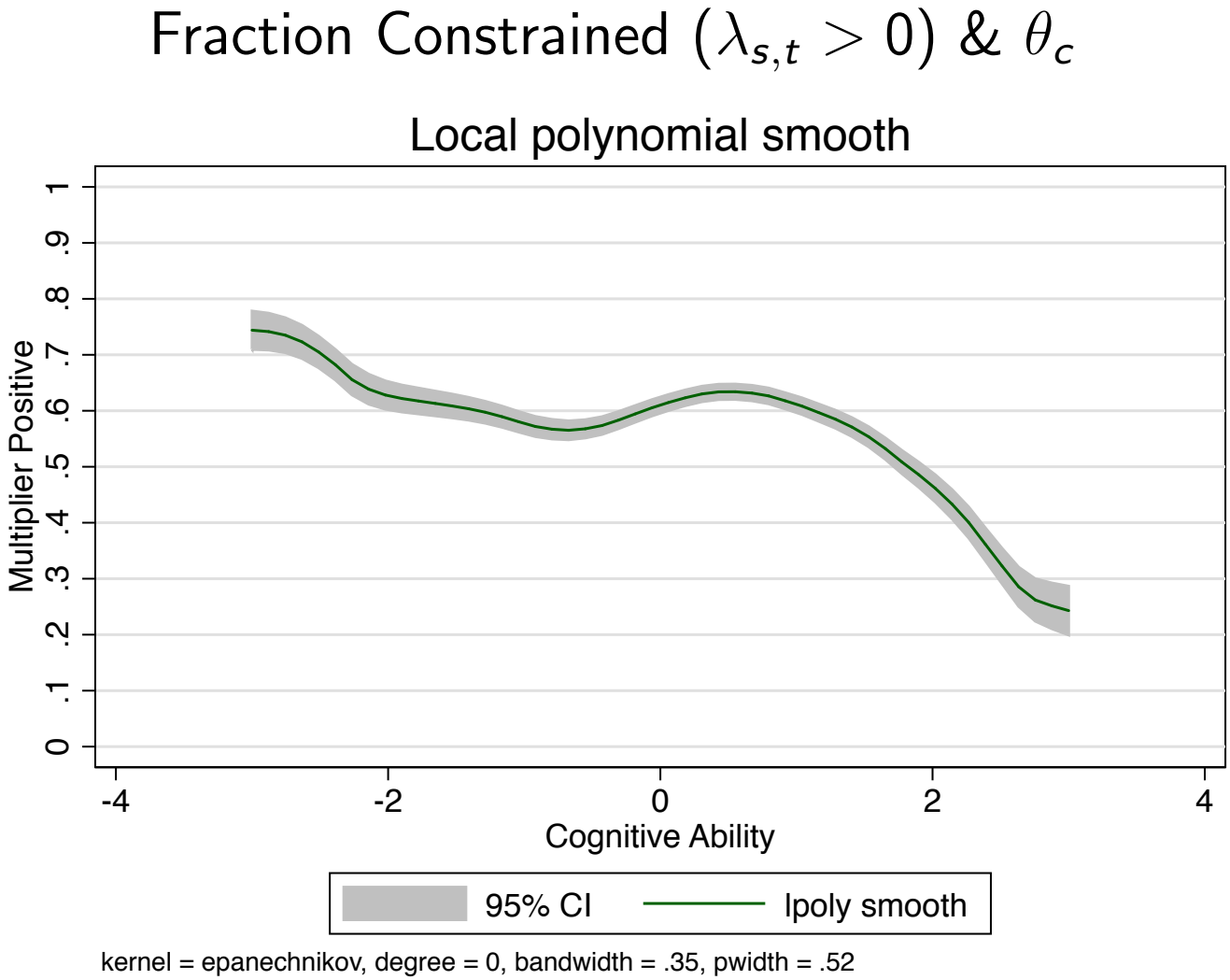


Figure 11: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 21 Cont'd

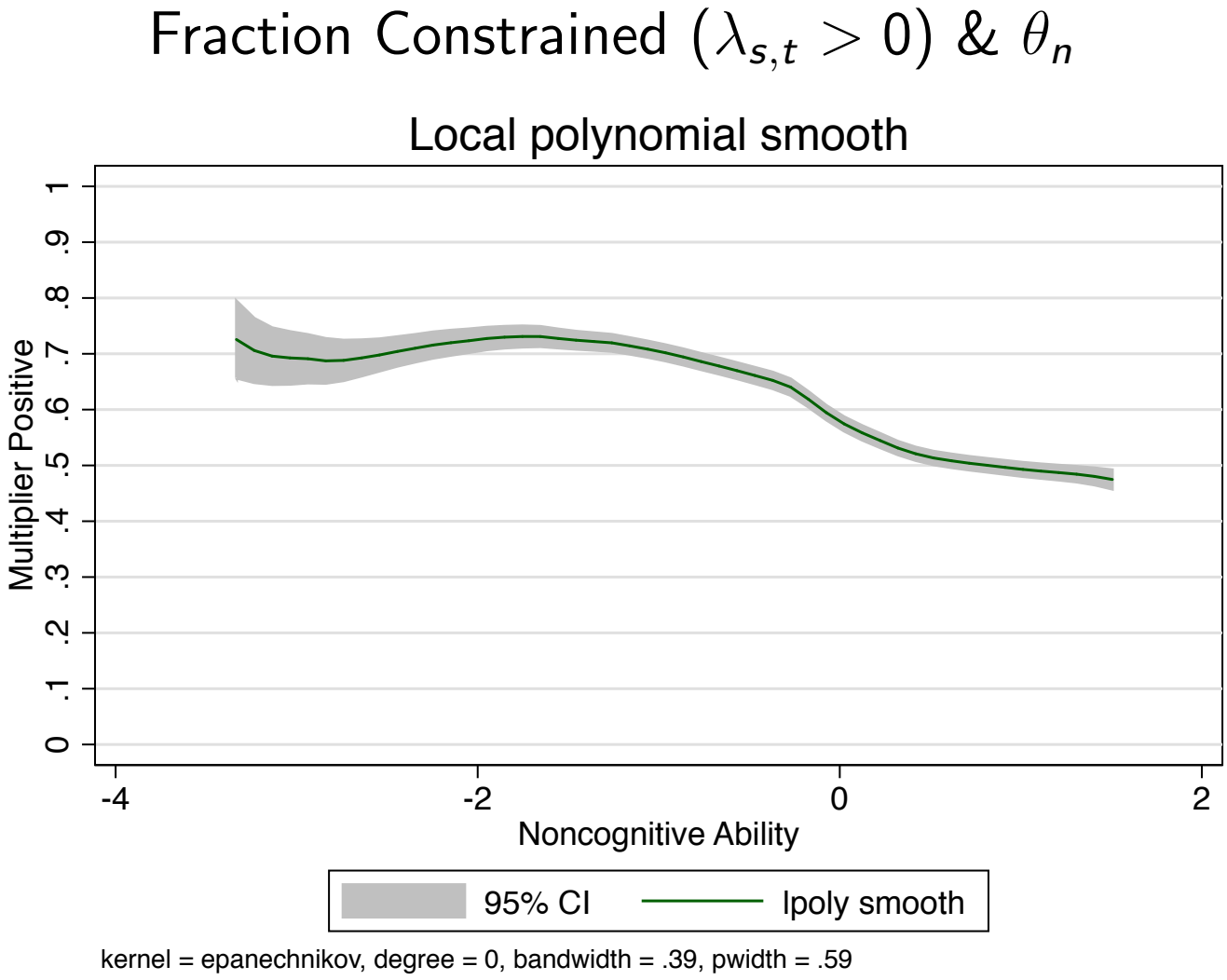


Figure 11: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 21 Cont'd

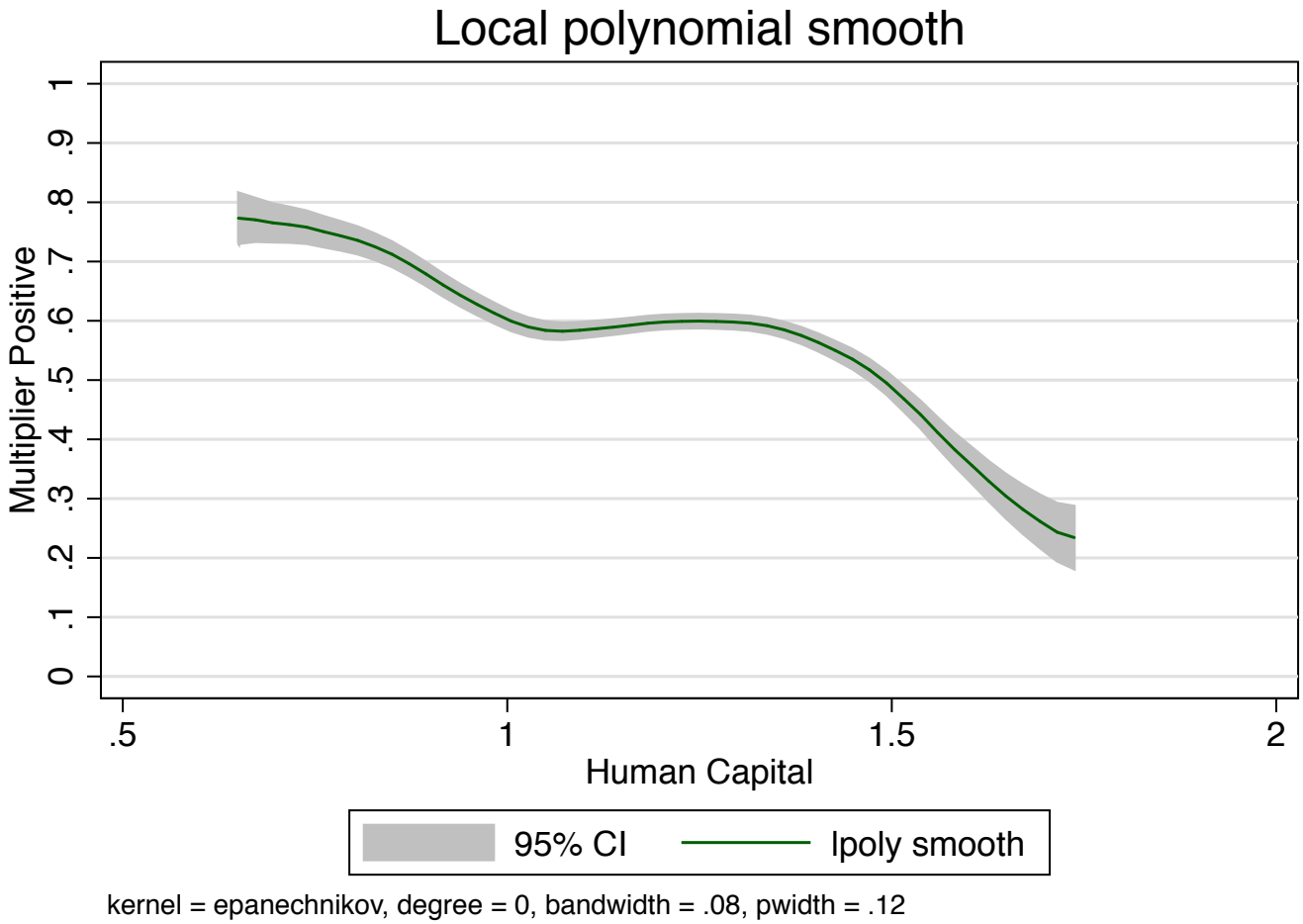


Figure 12: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 30

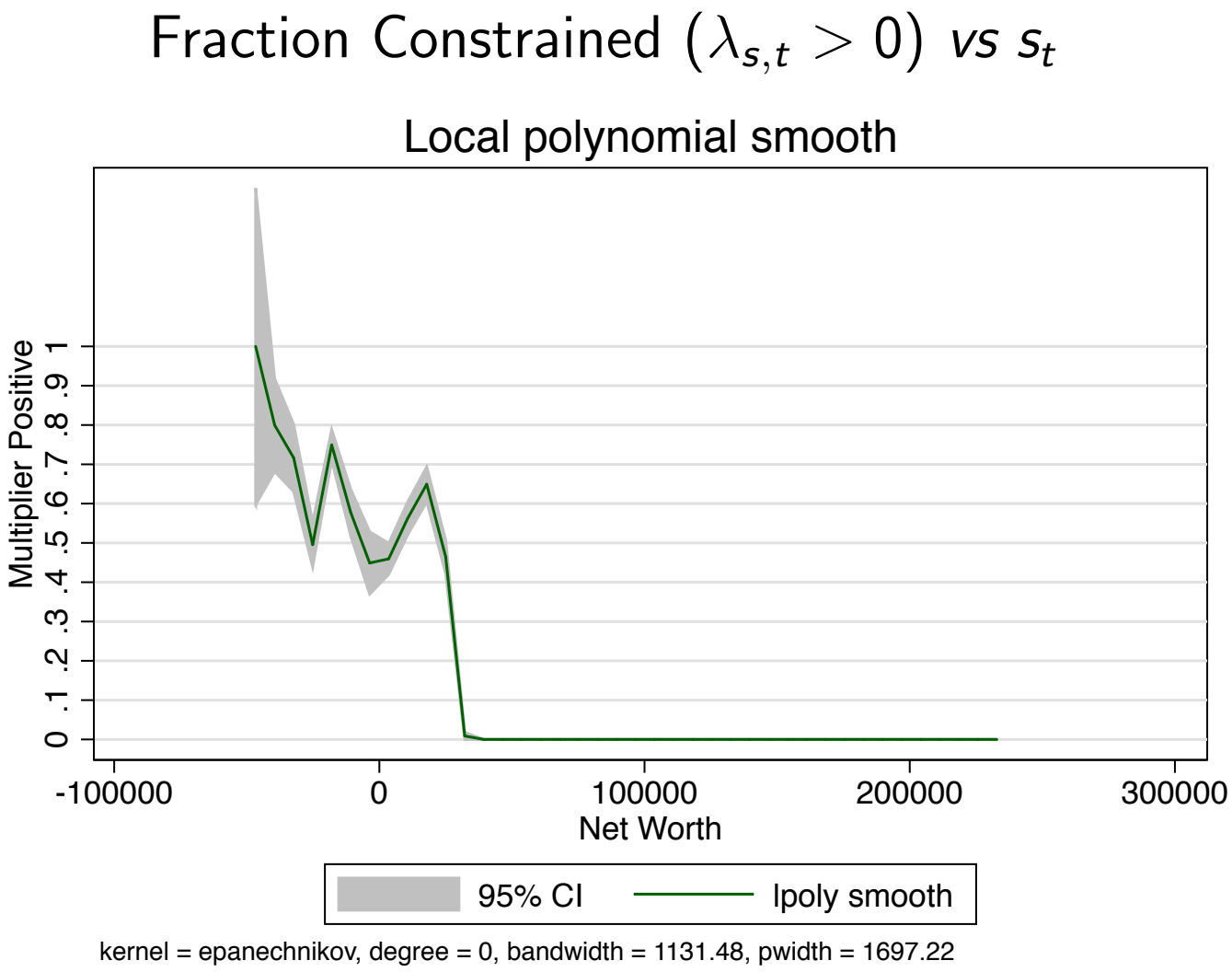


Figure 12: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 30 Cont'd

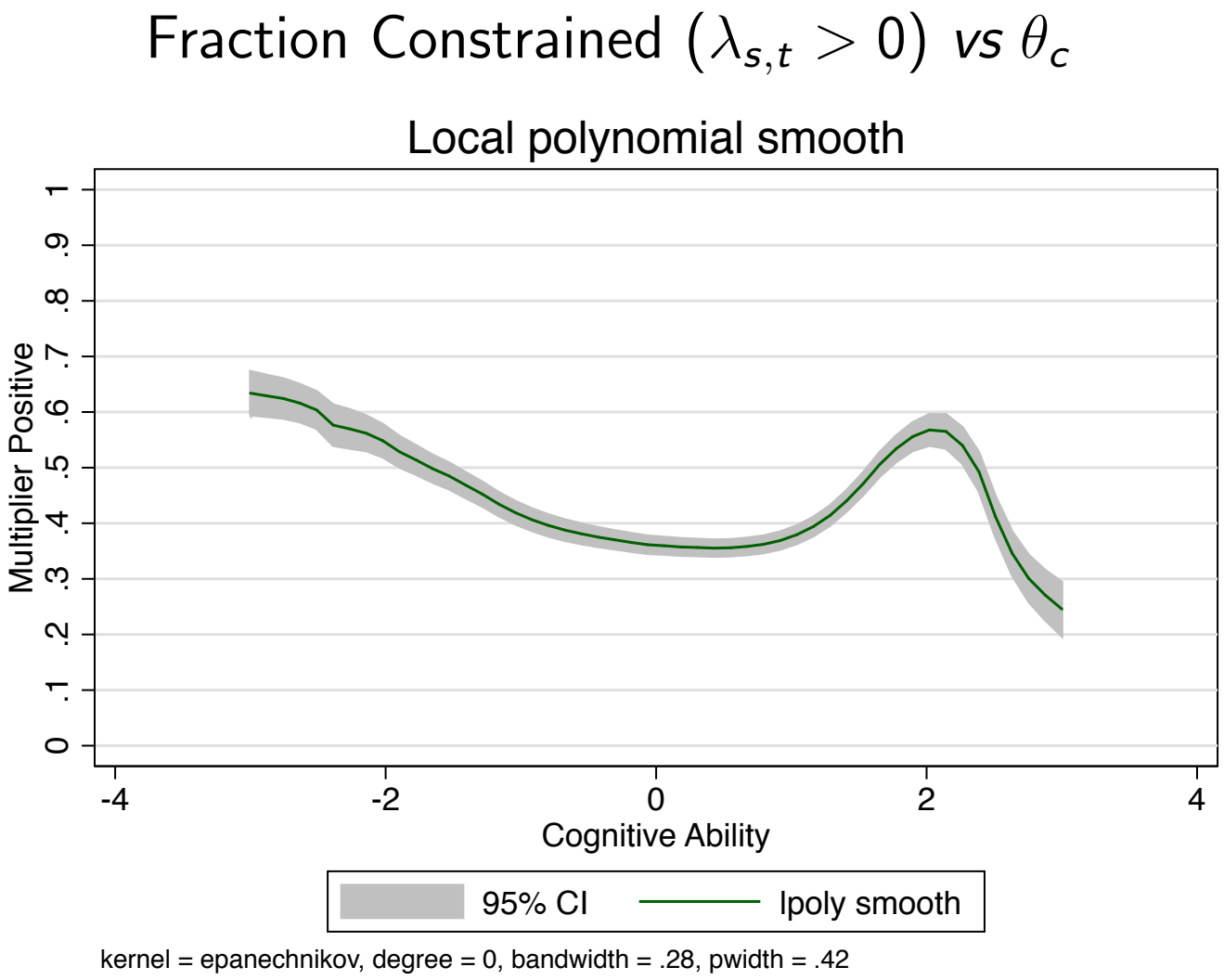


Figure 12: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 30 Cont'd

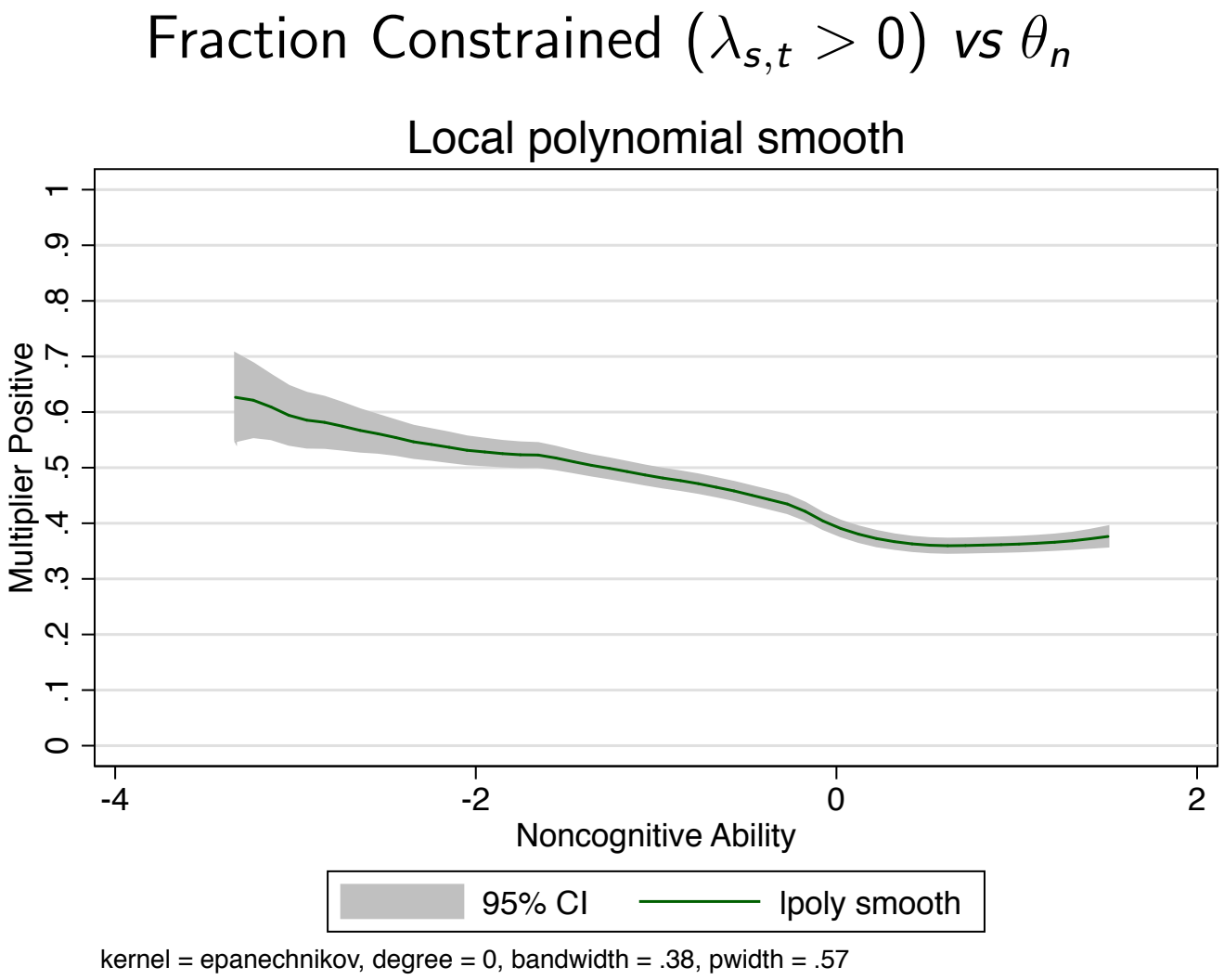
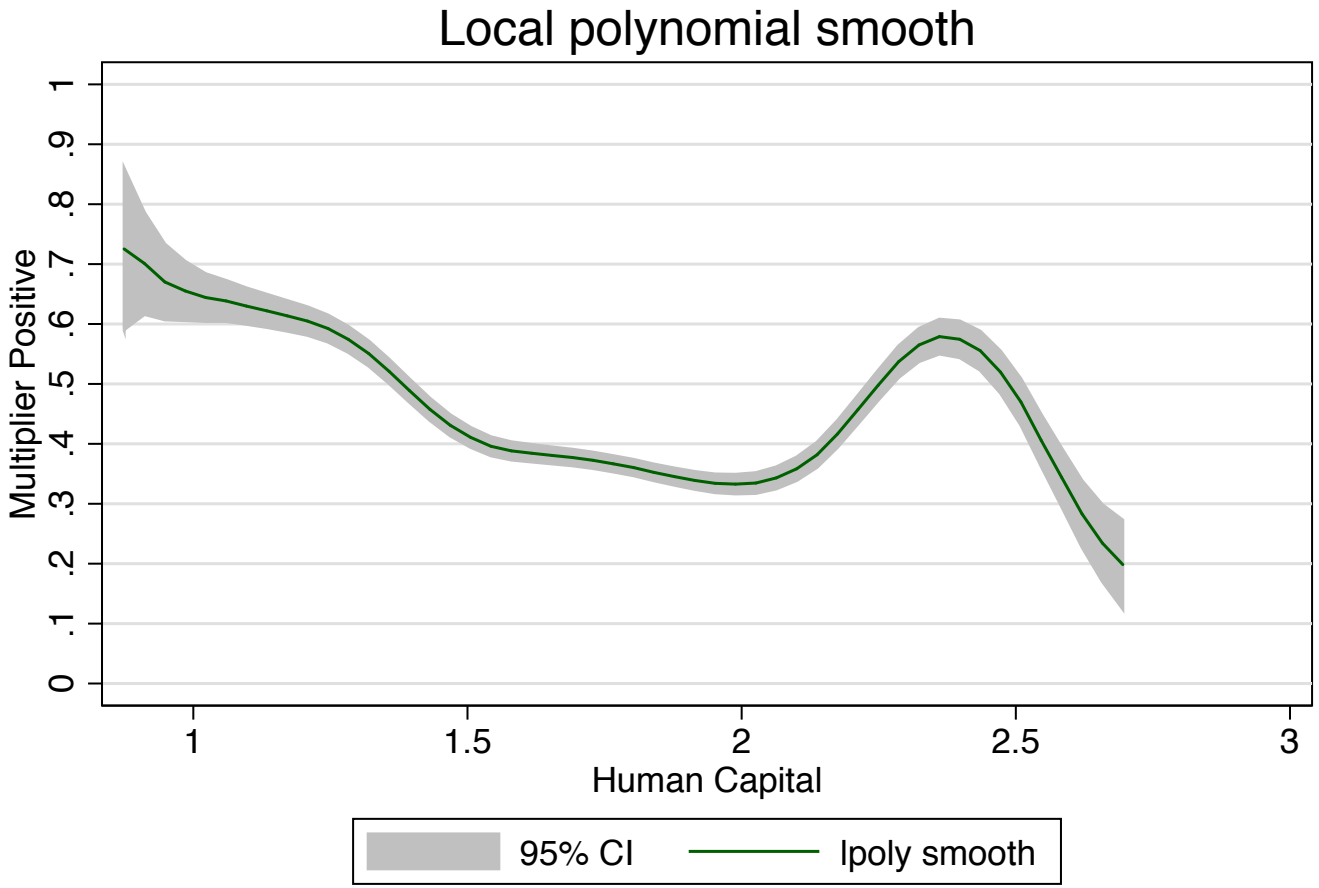


Figure 12: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 30 Cont'd

Fraction Constrained $(\lambda_{s,t} > 0)$ *vs* $F^{\psi}(e_t, k_t, \boldsymbol{\theta}, \epsilon_{w,t})$



kernel = epanechnikov, degree = 0, bandwidth = .09, pwidth = .13

Figure 13: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 40

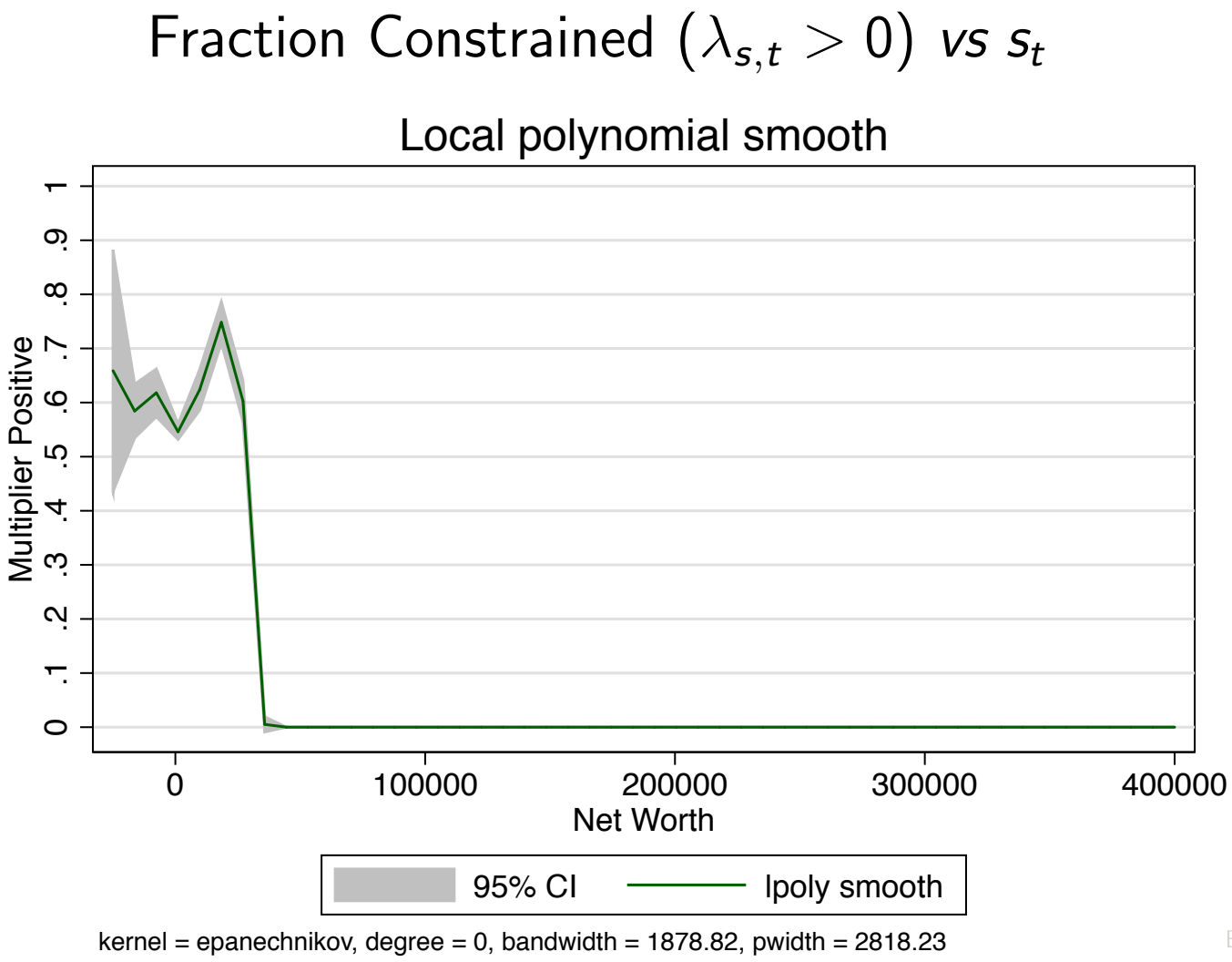


Figure 13: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 40 Cont'd

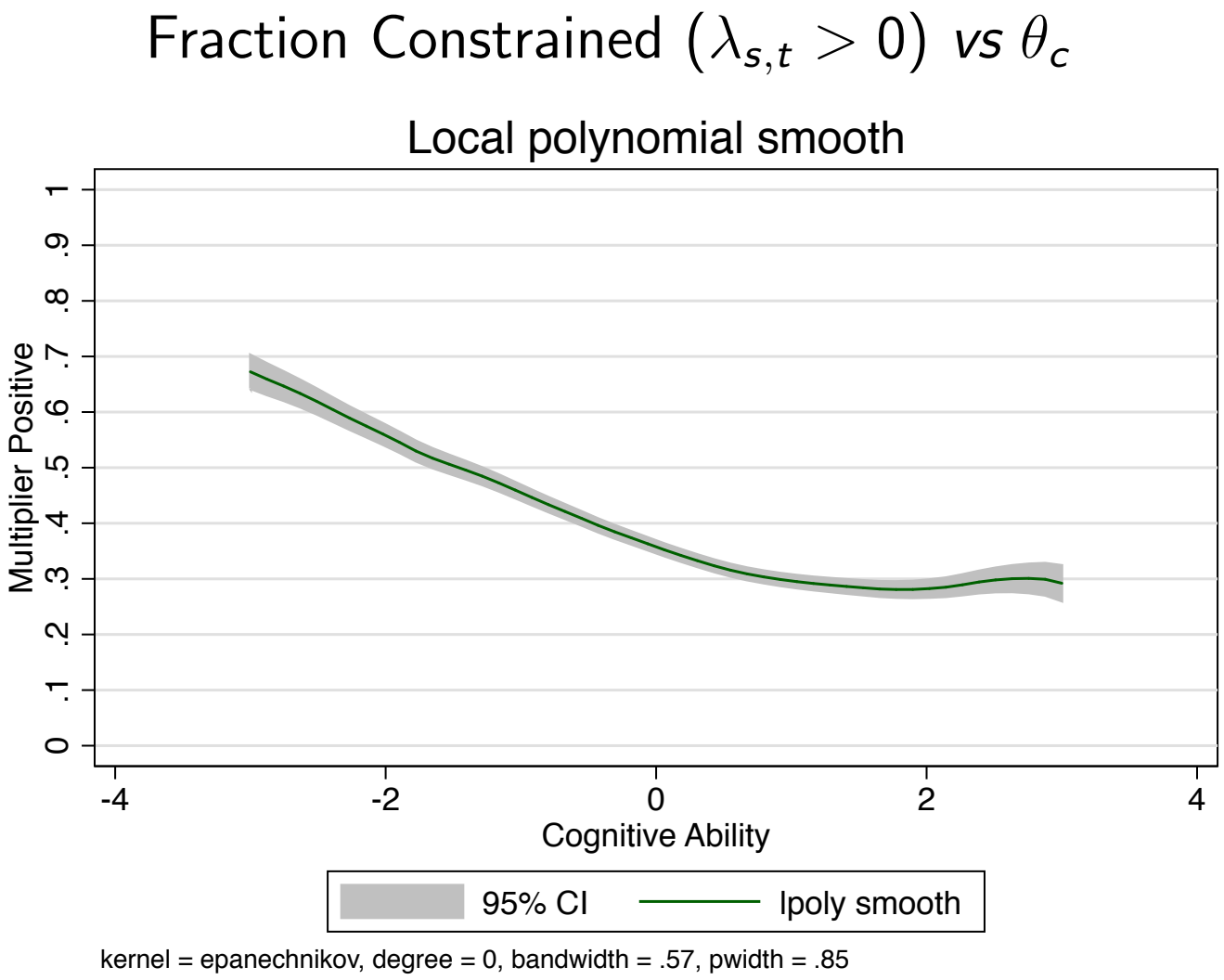


Figure 13: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 40 Cont'd

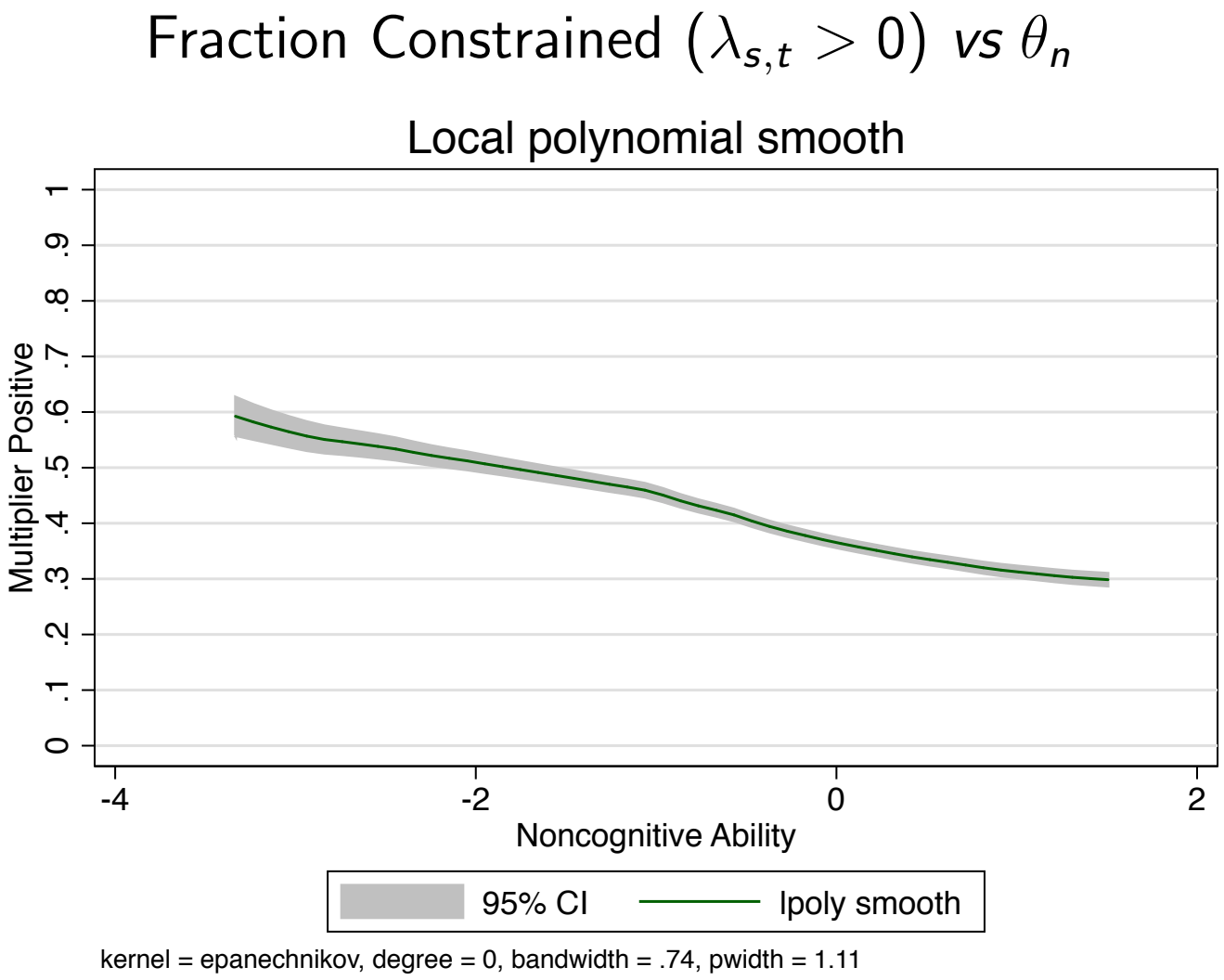


Figure 13: Multiplier $\lambda_{s,t}$ and Borrowing Constrained Youths $\lambda_{s,t} > 0$ at Age 40 Cont'd

Fraction Constrained $(\lambda_{s,t} > 0)$ *vs* $F^{\psi}(e_t, k_t, \boldsymbol{\theta}, \epsilon_{w,t})$

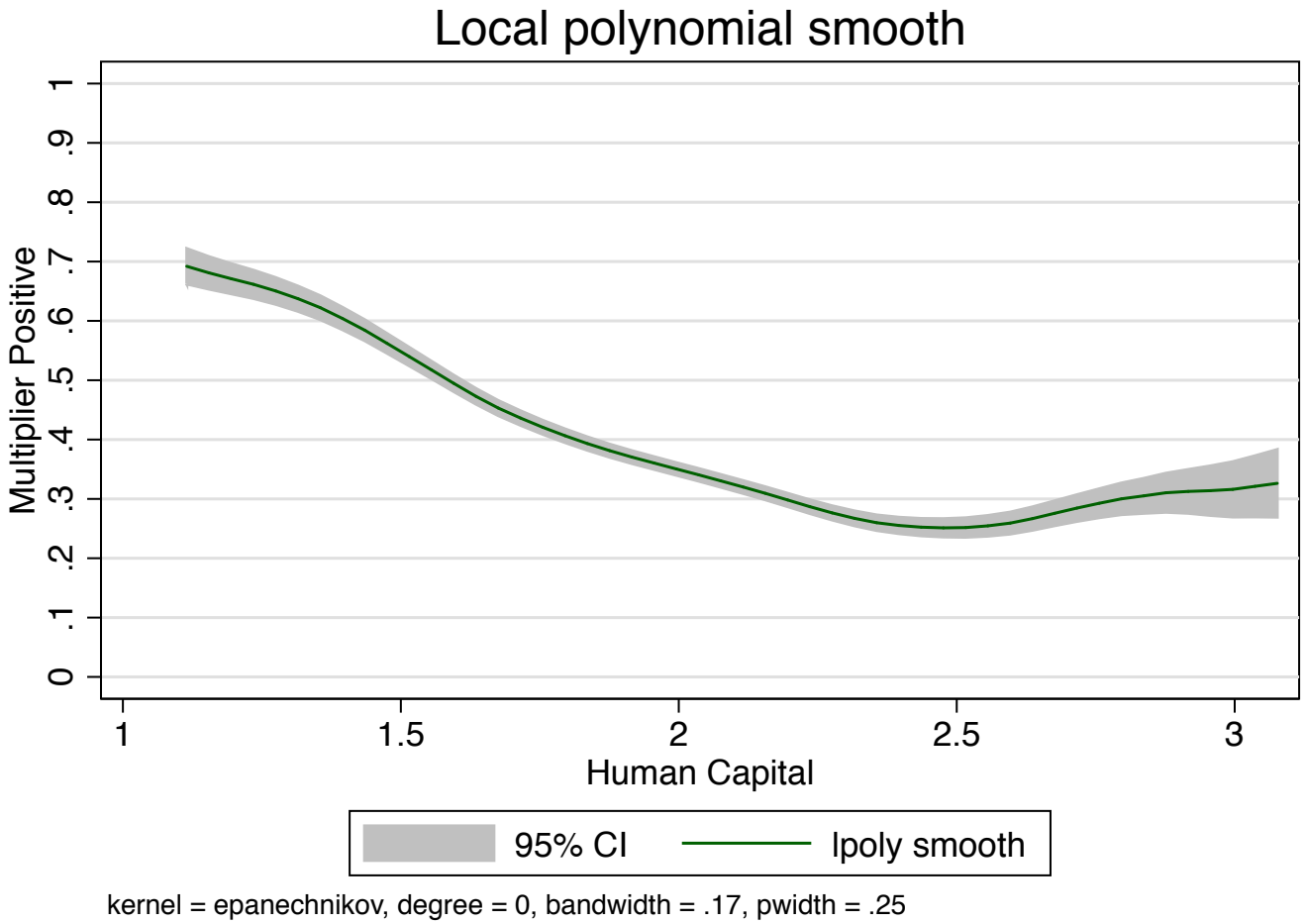
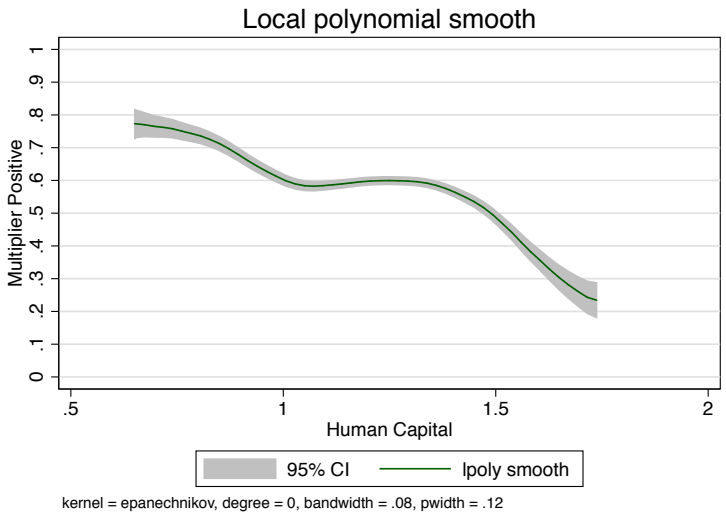
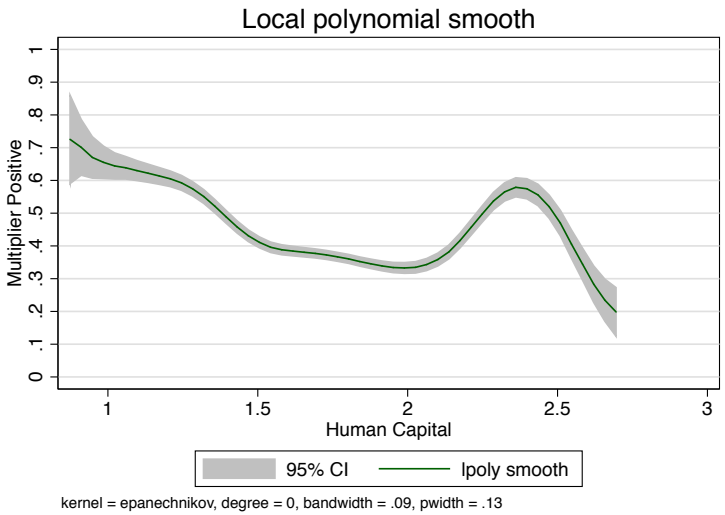


Figure 14: Borrowing Constrained ($\lambda_{s,t} > 0$) & Human Capital $F^{\psi}(e_t, k_t, \boldsymbol{\theta}, \epsilon_{w,t})$

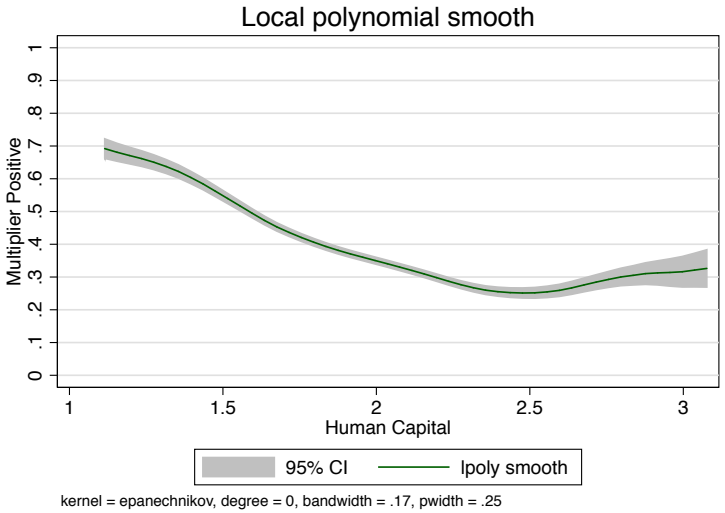
Age 21



Age 30



Age 40



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Estimation Details

Data Source: IHDP

- Mid to late 1980s
- Eight sites across the US
- Site-specific, stratified randomization
- 985 families
- Low birth weight, premature infants
- Randomized intervention lasted first 3 years of child's life
- Data collected frequently from 0-8 years, also 18 years
- We do not make use of randomized treatment

Preparation of IHDP Data

- We use data from 5 ages: 1,3,5,8,18 years
- After attrition and non-response, sample size 833 of 985
- Partially-missing response ($<$20%): nonparametric imputation
- For each type of factor, measures chosen so that units are consistent over time
- Measures preserved by monotonic transforms, we use ranks
- Inference: 10,000 bootstrap samples

IHDP Factor Measures

- Cognitive measures: IQ, PPVT, and math examinations
- Character Skills: Maternally-reported antisocial attitudes, social withdrawal, and depressive behavior indices
  - Example: Doesn't get along with other children? Not true, sometimes, often
- Child Effort/Compliance: Maternally-reported indices of rule-breaking behavior, aggression, and destruction indices
  - Example: Runs away from home? Not true, sometimes, often
- Maternal Effort/Investment: Learning material provision, time spent helping with reading/homework, and activity frequency/quality indices
  - Example: Read books, magazines together? Daily, weekly, etc.

Cognitive Skills

Parameter	$t = 1$	$t = 3$	$t = 5$	$t = 8$
Self-Productivity	0.826	0.855	0.980	0.911
(SD)	(0.022)	(0.014)	(0.010)	(0.010)
Cross-Productivity	0.323	0.077	-0.006	-0.036
(SD)	(0.041)	(0.012)	(0.014)	(0.017)
Child's Effort Productivity	-0.045	0.000	0.000	0.096
(SD)	(0.027)	(0.011)	(0.011)	(0.012)
Mother's Effort Productivity	0.477	0.365	0.103	-0.071
(SD)	(0.012)	(0.015)	(0.018)	(0.020)

Non-cognitive Skills

Parameter	$t = 1$	$t = 3$	$t = 5$	$t = 8$
Self-Productivity	0.929	0.417	0.904	0.199
(SD)	(0.022)	(0.016)	(0.011)	(0.012)
Cross-Productivity	0.275	-0.014	-0.010	0.163
(SD)	(0.054)	(0.017)	(0.026)	(0.022)
Child's Effort Productivity	0.125	0.066	0.160	0.487
(SD)	(0.030)	(0.014)	(0.017)	(0.017)
Mother's Effort Productivity	0.118	0.022	-0.033	0.043
(SD)	(0.014)	(0.019)	(0.026)	(0.027)

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A Simulation Exercise Based on Akabayashi (2006)

- In this document I simulate the model that Akabayashi (2006) develops in his paper *An Equilibrium Model of Child Maltreatment*.
- The layout of the model is the following.
- The child's human capital development in each period follows a linear low of motion:

$$h_{\tau+1} = (1-\delta)h_\tau + \varphi s_\tau H^\gamma + \phi a_\tau \tag{38}$$

for $\tau = 1, \ldots, T$.

- δ is a human capital depreciation parameter, φ, ϕ, γ are technology parameters, and H is the given and fixed parent's human capital.
- $s_\tau \in [0, 1]$ and a_τ are endogenous variables.

- The parents are not able to observe either the true level of human capital of the children or the effort they make.
- Instead, at each time τ, they observe an outcome variable, y_τ, which evolves according to the following linear rule:

$$y_\tau = h_\tau + a_\tau + \nu_\tau \tag{39}$$

  where $\nu_\tau \sim N(0, \sigma^2_{\nu\tau})$ for $\tau = 1, \ldots, T$.
- The author lets $\sigma^2_{\nu\tau} \equiv \frac{K}{s_\tau}$ because more time spent with the child reduces the uncertainty of the observation error.
- (38) is interpreted as the state equation and (39) as the observation equation.
- The author postulates a linear incentive schedule.

- The service, d_τ, is as follows:

$$d_\tau = (s_\tau + b_\tau \mathbb{E}[a_\tau | I_\tau])\, H^\gamma \tag{40}$$

  for $\tau = 1, \ldots, T$ where b_τ is defined as the slope of the incentive schedule and is an endogenous decision of the parent.
- $I_t \equiv \{y_t, \ldots y_1\}$ is the information set at t.
- Suppose that a parent picks a relatively high b_τ and that the observation of his daughter's performance, y_τ, deviates from his human capital forecast, $\hat{h}_\tau$, by a lot such that $y_\tau - \hat{h}_\tau$ is very negative.
- Then, d_τ is relatively low and the child suffers from a low service, which the author interprets as abuse.

- The child's utility function is

$$\max_{\{a_\tau\}_{\tau=1}^{T}} \mathbb{E}\left[u\left(\sum_{\tau=t}^{T} \frac{1}{1+\rho_{ct}}^{\tau-t}(d_\tau - v(a_\tau)) + \frac{1}{1+\rho_{ct}}^{T-t+1} Bh_{T+1}\right) | I_{t-1}\right] \tag{41}$$

subject to (38) and (40) and given $\{s_\tau, b_\tau\}$.

- The parent's utility function is

$$\max_{\{b_\tau, s_\tau\}_{\tau=1}^{T}} \mathbb{E}\left[U\left(\sum_{\tau=t}^{T} \frac{1}{1+\rho_{pt}}^{\tau-t}[c_\tau + \alpha u(\cdot, a_\tau))]\right) | I_{t-1}\right] \tag{42}$$

subject to $c_\tau = \pi(1 - s_\tau)H^\gamma$, (38), (39), (40) and to the child's optimal decision rule, and where c_τ is consumption at τ, and π efficiency unit wage.

- The Nash Equilibrium of the model can be solved easily.

- The child decides the optimal level of effort to solve her utility maximization problem by taking parent's choices, $\{b_\tau, s_\tau\}_{\tau=1}^{T}$, as given.
- The parent solves his utility maximization problem to choose the optimal level of time and incentive slope by taking the optimal decision rule of the kid as given.

- First, when the parent receives a new observation, y, he updates his contemporaneous belief about the child's current level of human capital by taking an average of the previous belief and the new observation weighted by the degree of uncertainty:

$$\hat{h}^u = \hat{h} + \frac{A'(\hat{h})\sigma_h^2}{A'(\hat{h})^2\sigma_h^2 + \sigma_v^2}(y - A(\hat{h}) - C(\sigma_h^2)) \tag{43}$$

- Then, the parent uses the updated belief, $\hat{h}^u$, and the human capital formation rule to forecast the child's human capital level in next period:

$$\hat{h}' = F(\hat{h}^u) + G(\sigma_h^2) \tag{44}$$

- Finally, the parent updates the uncertainty regarding the child's human capital:

$$(\sigma_h^2)' = \Phi\sigma_h^2 \tag{45}$$

where

$$A(\hat{h}) = \hat{h} + \phi\psi D_t(\hat{h}) \tag{46}$$

$$A'(\hat{h}) = 1 + \phi\psi D_t'(\hat{h}) \tag{47}$$

$$C(\sigma_h^2) = \underline{a} + \frac{\psi}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t) \tag{48}$$

$$F(\hat{h}^u) = (1-\delta)\hat{h}^u + \phi^2\psi D_t(\hat{h}^u) \tag{49}$$

$$G(\sigma_h^2) = \phi[\underline{a} + \frac{\psi}{R\alpha\sigma_h^2 - \psi}(\underline{a} + \psi\phi D_t(H) - KR\alpha Q_t)] + \psi s^* H^\gamma \tag{50}$$

$$\Phi = \frac{F'(\hat{h}^u)^2\sigma_v^2}{A'(\hat{h}^u)^2\sigma_h^2 + \sigma_v^2} \tag{51}$$

$$F'(\hat{h}^u) = (1-\delta) + \phi^2\psi D_t'(\hat{h}^u) \tag{52}$$

Simulation Parameterization

- I use the following parameters and functional forms to simulate the model:
  - Time Horizon: $T = 80$.
  - Technology parameters: $\phi = .7, \varphi = .01, \delta = .001, \gamma = 0.5$
  - Preference parameters:
    $\psi =, 7, \underline{a} = 7, K = 1.5, R = 2, \alpha = .9, B = 50$
  - Human Capital Parameters: $h_1 = 100, H = 40000$.
  - Filter Initial Values: $\hat{h}_1 = 200, \sigma^2_{h1} = 5000$.
  - Other parameters: $K = 1.5, \pi = 2$.
  - Discount Rates: $\rho_i(h) = \exp(-.02 * h)$ for $i = p, c$.

Simulation Results

Figure 15: log Incentive Slope and log Time Spent with the Child

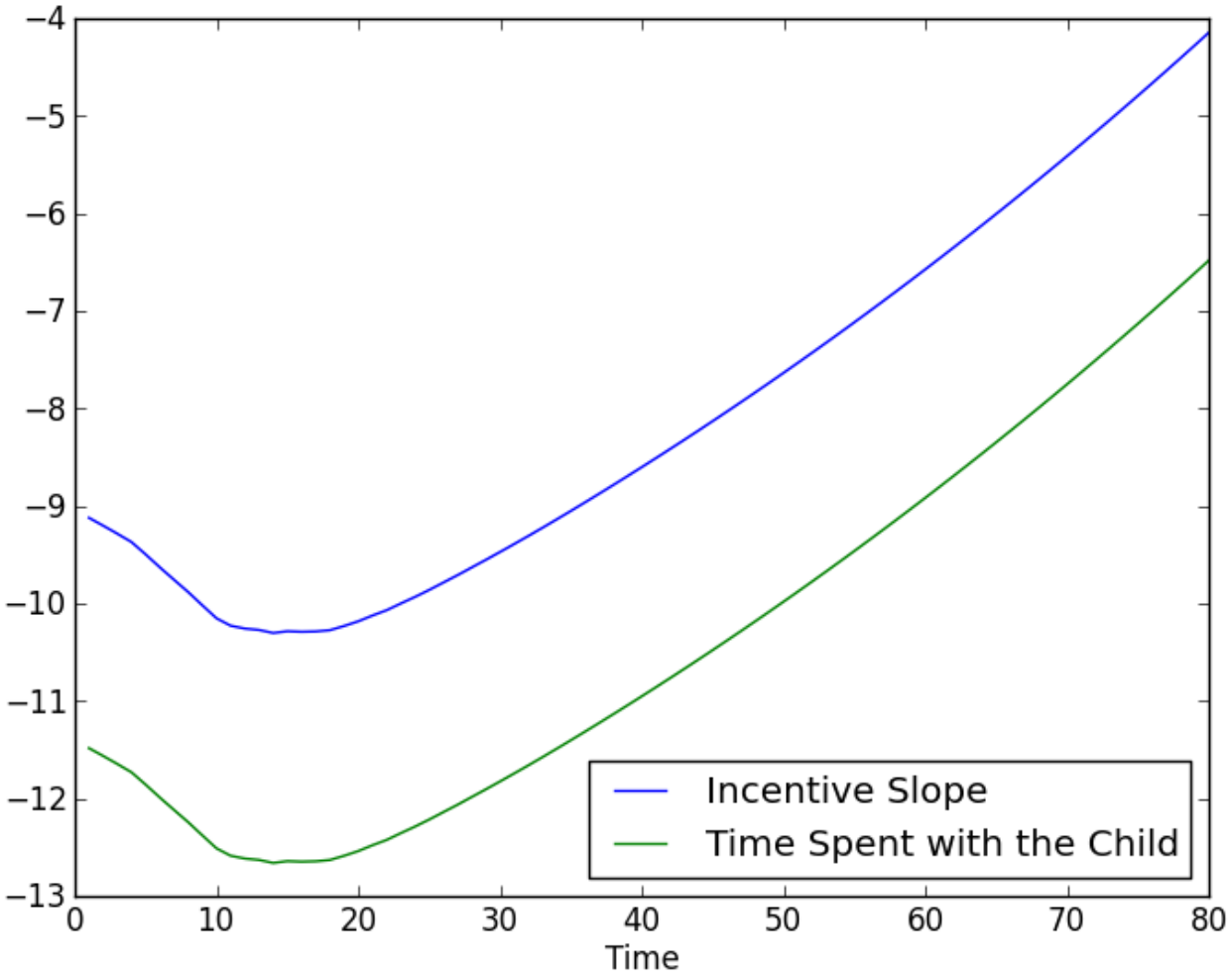


Figure 16: Child's Effort

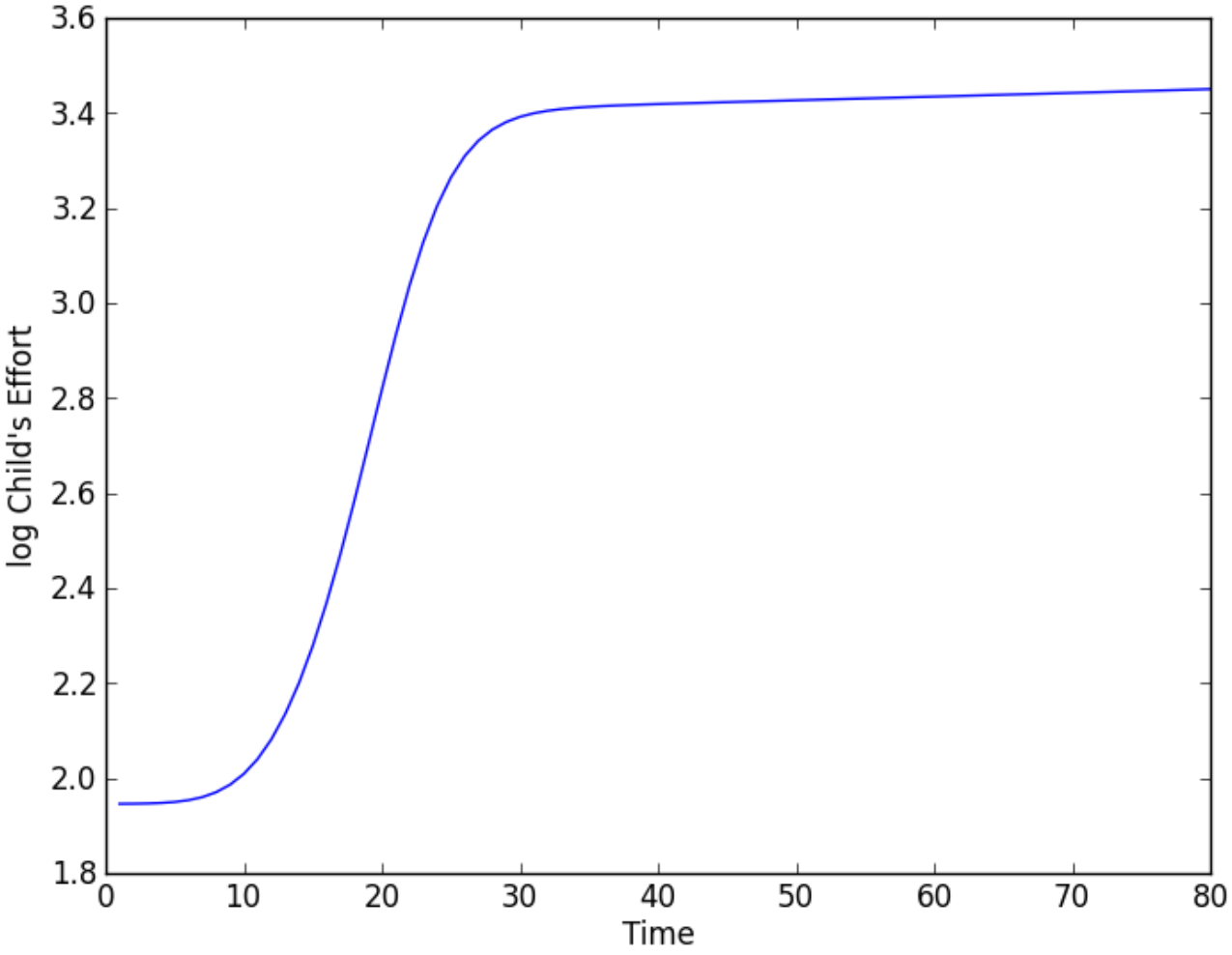


Figure 17: Human Capital

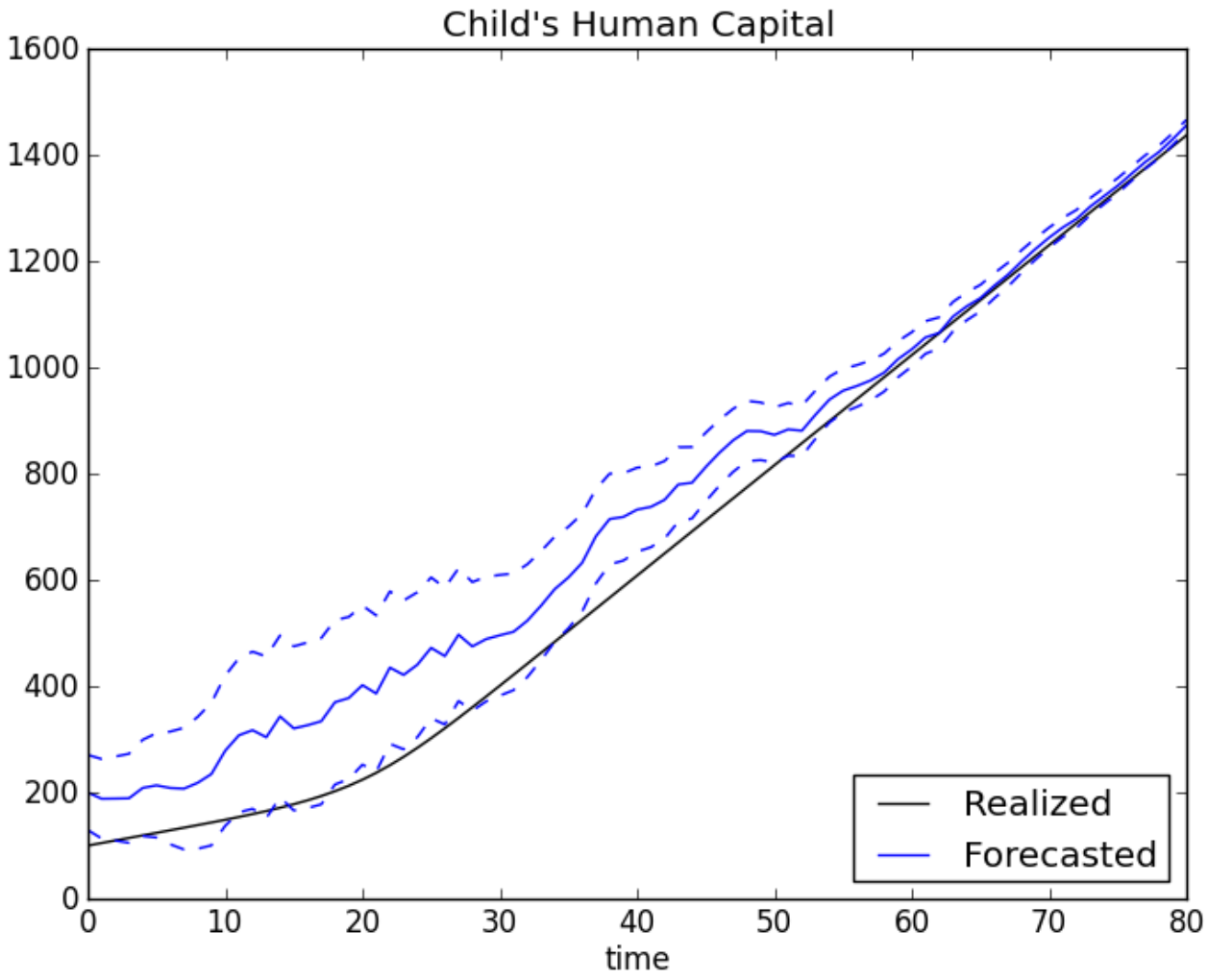
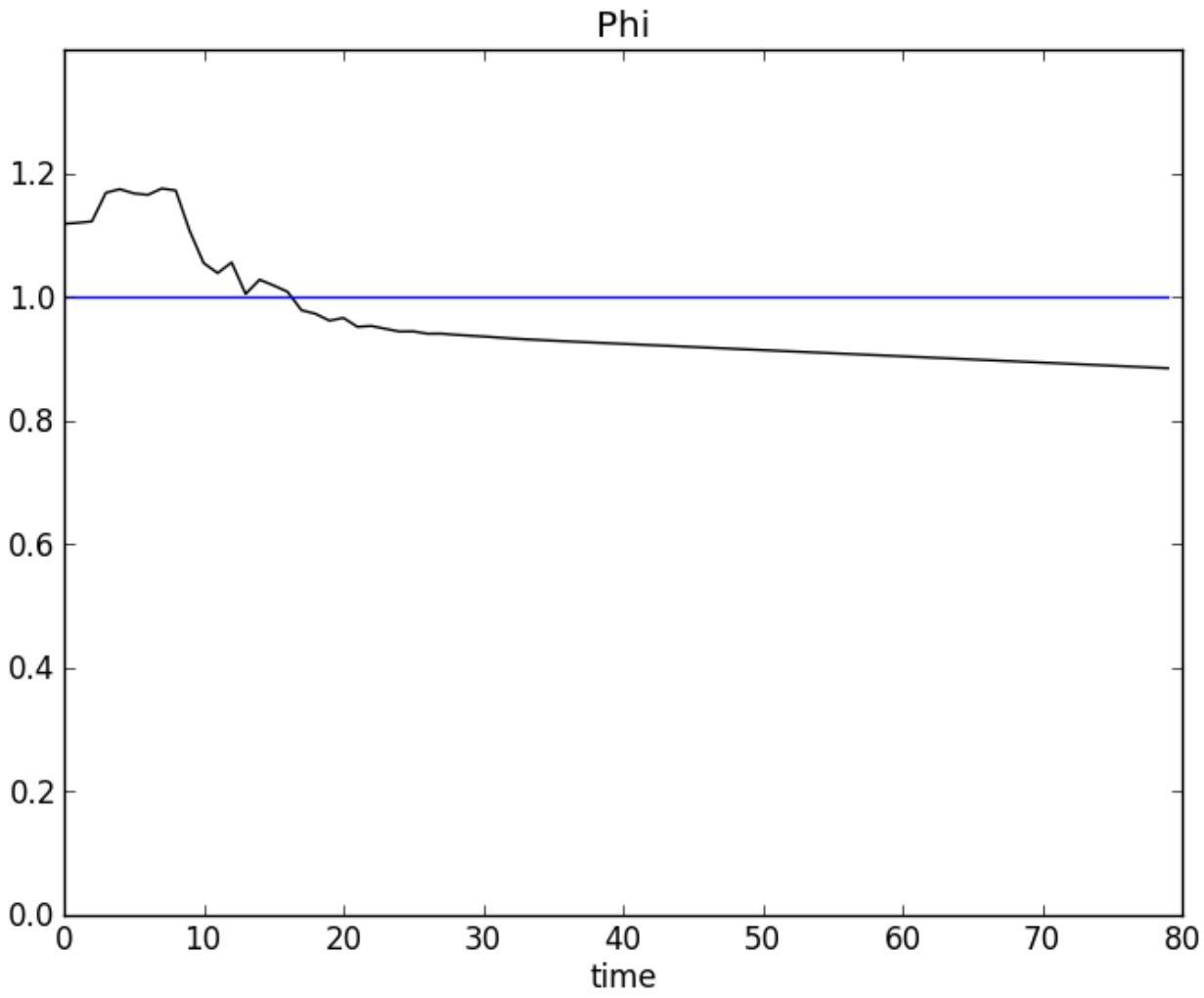


Figure 18: Phi



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Studies on the Role of Income on Children's Outcomes and Credit Constraints

Table 4a: Studies on the Role of Income on Children's Outcomes

	Dataset	Outcome Studied: Test Scores (T), Schooling (S)	Timing of Income (Developmental Stage of the Child at Which Income Effects are Studied)	Separate the Effect of Income from Changes in Labor Supply or Family Environment
Carneiro and Heckman (2002)	NLSY79*	S	Early (E) and Late (L) College Enrollment	X
Belley and Lochner (2007)	NLSY79*, NLSY97*	S	L High school completion and College Enrollment	X
Dahl and Lochner (2012)	NLSY79*, C-NLSY79*	T	E Preadolescence (ages 8 to 14)	X[a]
Duncan et al. (1998)	PSID*	S[d]	E and L Childhood and Preadolescence (ages 0 to 15)	X

Table 4a (cont.): Studies on the Role of Income on Children's Outcomes

	Dataset	Outcome Studied: Test Scores (T), Schooling (S)	Timing of Income (Developmental Stage of the Child at Which Income Effects are Studied)	Separate the Effect of Income from Changes in Labor Supply or Family Environment
Duncan et al. (2011)	Randomized Interventions on Welfare Support	T	E Early Childhood (ages 2 to 5)	X
Loken (2010)	Norwegian Administrative Data	S	E Childhood (ages 1 to 11)	✓[c]
Loken et al. (2012)	Norwegian Administrative Data	S	E Childhood (ages 1 to 11)	X
Milligan and Stabile (2011)	CCTB**, NCBS***	T	E Childhood (ages 0 to 10)	X
Carneiro et al. (2013)	Norwegian Registry	S[e]	E and L Childhood to Adolescence (ages 0 to 17)	X

Table 4b: Studies on the Role of Income on Children's Outcomes

	Distinguishes the Effects of Contemporaneous vs. Permanent Income	Sources of Income Whose Effects are Studied	Instrument Used
Carneiro and Heckman (2002)	✓	Total family income	None
Belley and Lochner (2007)	X	Total family income	None
Dahl and Lochner (2012)	X[b]	Total family income	Policy variation in EITC eligibility
Duncan et al. (1998)	✓	Total family income	None

Table 4b (continued): Studies on the Role of Income on Children's Outcomes

	Distinguishes the Effects of Contemporaneous vs. Permanent Income	Sources of Income Whose Effects are Studied	Instrument Used
Duncan et al. (2011)	X	Total family income	Random assignment to programs offering welfare transfers conditional on employment or education related activities, or full time work
Loken (2010)	X	Total family income	Oil discovery (inducing regional increase in wages)
Loken et al. (2012)	X	Total family income	Oil discovery (inducing regional increase in wages)
Milligan and Stabile (2011)	X	Child related tax benefits and income transfers	Variation in benefits eligibility
Carneiro et al. (2013)	✓	Total family income	None

Table 4c: Studies on the Role of Income on Children's Outcomes

	Effect of Income on Human Capital Investments
Carneiro and Heckman (2002)	Percentage of people constrained = weighted gap in educational outcome to highest income group: 5.1% are constrained in college enrollment (1.2% among low income, low ability, 0.2% low income high ability), 9% in completion of 2-year college (5.3% among low income, low ability, 0.3% low income high ability). No effect of timing of receipt of family income on child outcomes.
Belley and Lochner (2007)	High school completion: +8.4% for highest income quartile compared to lowest in 79, +6.7 in 97 cannot reject equal effect of income; college enrollment: +9.3% for highest income quartile compared to lowest in 79, +16 in 97, cannot reject equal effect of income.
Dahl and Lochner (2012)	$1,000 extra per year for 2 years: +6% of a standard deviation in math and reading combined PIAT score.
Duncan et al. (1998)	$10,000 increase in average (age 0-15) family income: +1.3 years of schooling in low income (¡$20,000) families, +0.13 in high income ones. Relevance of income is stronger in the early years (age 0-5): $10,000 increase in average (age 0-5) family income leads to extra 0.8 years of schooling in low income families, 0.1 in high income ones. Income at age 6-10 and 11-15: no significant effect. Similar results in a sibling differences model.

Table 4c (continued): Studies on the Role of Income on Children's Outcomes

	Effect of Income on Human Capital Investments
Duncan et al. (2011)	$1,000 extra per year for 2 to 5 years: +6% of a standard deviation in child's achievement score.
Loken (2010)	OLS: positive relationship of average (age 1-13) family income on children's education, IV: no causal effect. Results are robust to different specification and splitting the sample by parental education.
Loken et al. (2012)	Non-linear IV (quadratic model): increase of $17,414, +0.74 years of education for children in poor families, +0.05 for children in rich families.
Milligan and Stabile (2011)	Low education mothers: positive effects of child benefits on cognitive outcomes for boys, on emotional outcomes for girls, weak on health. Results are non robust to the exclusion of Quebec.
Carneiro et al. (2013)	All outcomes: monotone and concave relationship with permanent income. £100,000 increase in permanent father's earnings: +0.5 years of schooling. Timing of income: a balanced profile between early (age 0-5) and late childhood (age 6-11) is associated with the best outcomes; shifting income to adolescence is associated with better outcomes in dropping out of school, college attendance, earnings, IQ and teen pregnancy. Early and late childhood income are complements in determining schooling attainment, early and adolescent income are substitutes.

Table 5a: Studies on Tests of Credit Constraints

	Dataset	Outcome Studied: Schooling (S)	Timing of Income (Developmental Stage of the Child at Which Constraints are Studied)	Explicit Dynamic Model	Who is Affected by Constraints: Parent of the Agent (P), Agent / Child (C)
Keane and Wolpin (2001)	NLSY79*	S	L College Enrollment	✓	C
Carneiro and Heckman (2002)	NLSY79*, C-NLSY79*	S	E and L College Enrollment, Completion, Delayed Entry	X	P

Table 5a (continued): Studies on Tests of Credit Constraints

	Dataset	Outcome Studied: Schooling (S)	Timing of Income (Developmental Stage of the Child at Which Constraints are Studied)	Explicit Dynamic Model	Who is Affected by Constraints: Parent of the Agent (P), Agent / Child (C)
Cameron and Taber (2004)	NLSY79*	S	L Adolescence and College Enrollment	X	C
Caucutt and Lochner (2012)	NLSY79*, C-NLSY79*	T[a]	E and L Childhood and Adolescence	✓	P

Table 5b: Studies on Tests of Credit Constraints

	Method to Test for Credit Constraints	Find Presence of Credit Constraints	Effect of Income or Constraints on Human Capital Investments
Keane and Wolpin (2001)	Structural estimation of the lower bound on asset level	YES But irrelevant for schooling decisions	Increase borrowing limit to \$3,000 (3× max estimated): no change in mean highest grade completed; +0.2% in college enrollment; -0.2\$ on mean hourly wage rate; increase in consumption and reduction in market hours; moderate reduction in parental transfers especially for the least educated parents.
Carneiro and Heckman (2002)	(1) Gap to students from highest income quartile. (2) Timing of income. (3) Difference in IV and OLS estimates of Mincer coefficient	YES But affecting at most 8% of students	(1) Conditioning on ability and family background factors, the role of income in determining schooling decisions is minimal. The strongest evidence is in the low ability group. The test is not robust to accounting for parental preferences and paternalism. Observed differences in attendance might be due to a consumption value of child's schooling for parents. (2) There is no evidence of a independent effect on college enrollment of early or late income once permanent income is accounted for. (3) The claim that higher IV than OLS estimates of the Mincer coefficient implies credit constraints are incorrect: instruments used are invalid, the quality margin is ignored and self selection and comparative advantage can produce the result also in absence of financial constraints.

Table 5b (continued): Studies on Tests of Credit Constraints

	Method to Test for Credit Constraints	Find Presence of Credit Constraints	Effect of Income or Constraints on Human Capital Investments
Cameron and Taber (2004)	IV estimation of "returns" to schooling using costs of schooling or foregone earnings as instruments	NO	Theoretical prediction: if borrowing constraints, IV estimates using direct costs of schooling higher than using opportunity costs. Data: IV estimates using the presence of a local college are smaller than the ones using foregone earnings. Regressions which interact college costs and characteristics potentially related to credit availability: no evidence of excess sensitivity to costs for potentially constrained sample. Structural model: almost 0% of the population is found to borrow at a rate higher than the market one.
Caucutt and Lochner (2012)	Structural estimation of the lower bound on asset level	YES Stronger effect on high skilled parents	50% of young parents are constrained: high school dropouts (50%), high school graduates (38%), college dropouts (60%), college graduates (68%); and 12% of old parents are constrained. Families with college graduate parents benefit the most from a reduction in credit constraints.

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"Measuring Early Investments in Children"

by Jennifer Culhane, Flávio Cunha, Irma Elo, and Zoe Pham (2016)

Measuring Early Investments in Children

Jennifer Culhane, Flávio Cunha, Irma Elo, and Zoe Pham

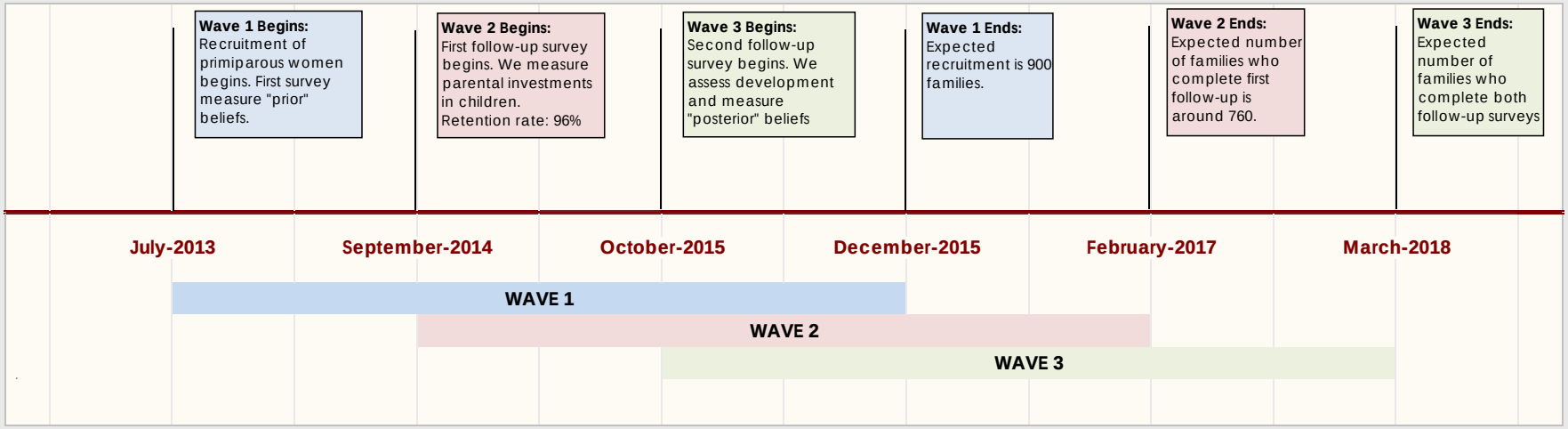
Children's Hospital of Philadelphia, Rice University, University of Pennsylvania, Rice University

June 5, 2016

Early Investments and Human Capital Formation

- A large body of research has demonstrated that investments in early childhood.
- In this paper, we consider two forms of measurement of investments:
  - Home Observation for the Measurement of the Environment (HOME).
  - Language Environment Analysis (LENA).
- HOME (or one of its many adaptations/versions) is widely used in many studies; LENA increasingly so.

Figure: PHD Study Data Collection Timeline



Home Observation for the Measurement of the Environment

- Created by Bettye Caldwell and Robert Bradley in late 1960s, early 1970s (first published in 1980s)
- Evaluates a child's home environment as well as parent-child interaction.
- Administered by trained professional at the child's home with both child and primary caregiver present.
- Semi-structured interview and observation period: 45-60 minutes.

HOME: Strenghts and Weakenesses

- Strenghts
  - Easy to administer and score.
  - Reliability and validity.
  - Easy to adapt for specific purposes.
  - Provides objective information on home, child, and parent-child interaction.
- Weakenesses:
  - Training of administrators to follow standardized measurement.
  - Only Yes/No questions.
  - Score: Simple summation gives "too much" weight to items that do not vary a lot across households.

HOME: IRT Analysis

- Let θ_i denote the latent quality of the environment experienced by child i.
- Let $d^*_{i,j} = a_j(\theta_i - b_j) + \epsilon_{i,j}$ and define $d_{i,j} = 0$ if $d^*_{i,j} \leq 0$ and $d_{i,j} = 1$, otherwise.
- Assume $\epsilon_{i,j}$ has logistic distribution and let θ_i be normally distributed with mean zero and variance σ^2.
- Parameter a_j is item discrimination while b_j is item difficulty.

Interpretation of IRT Parameters

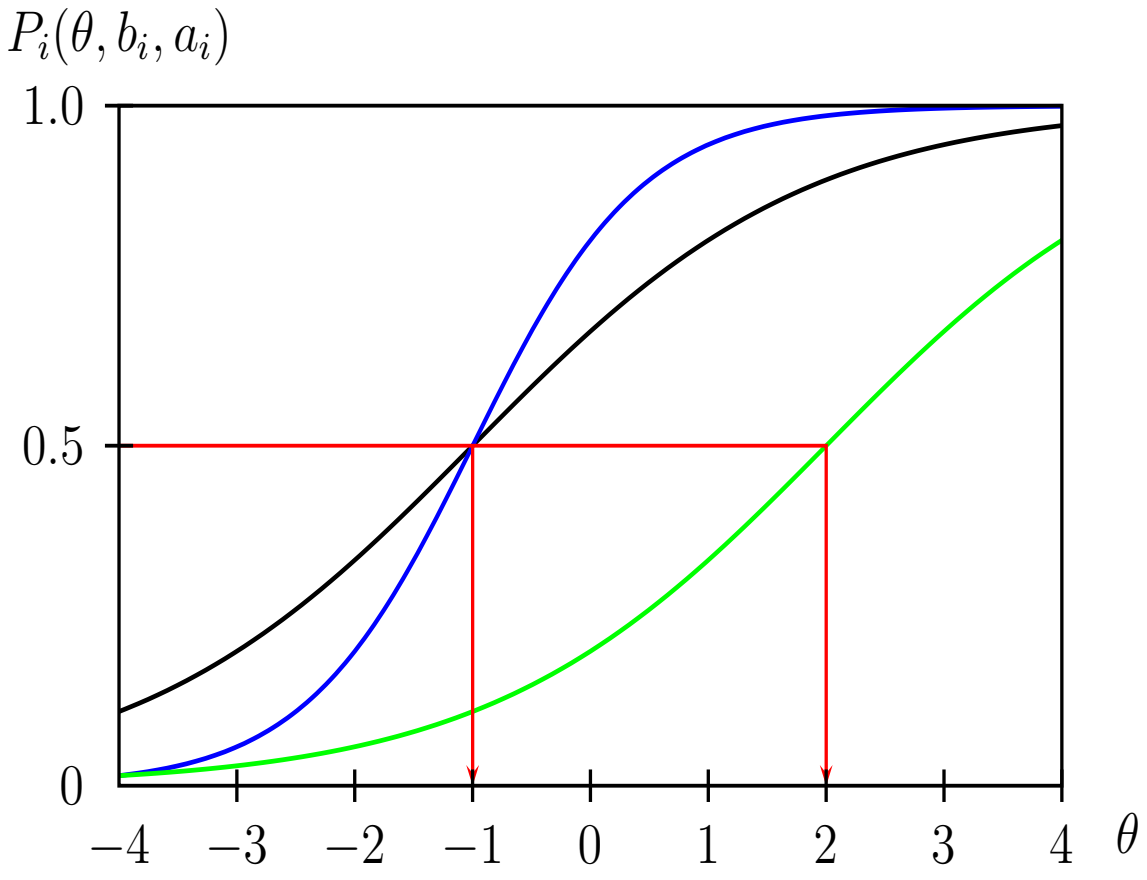
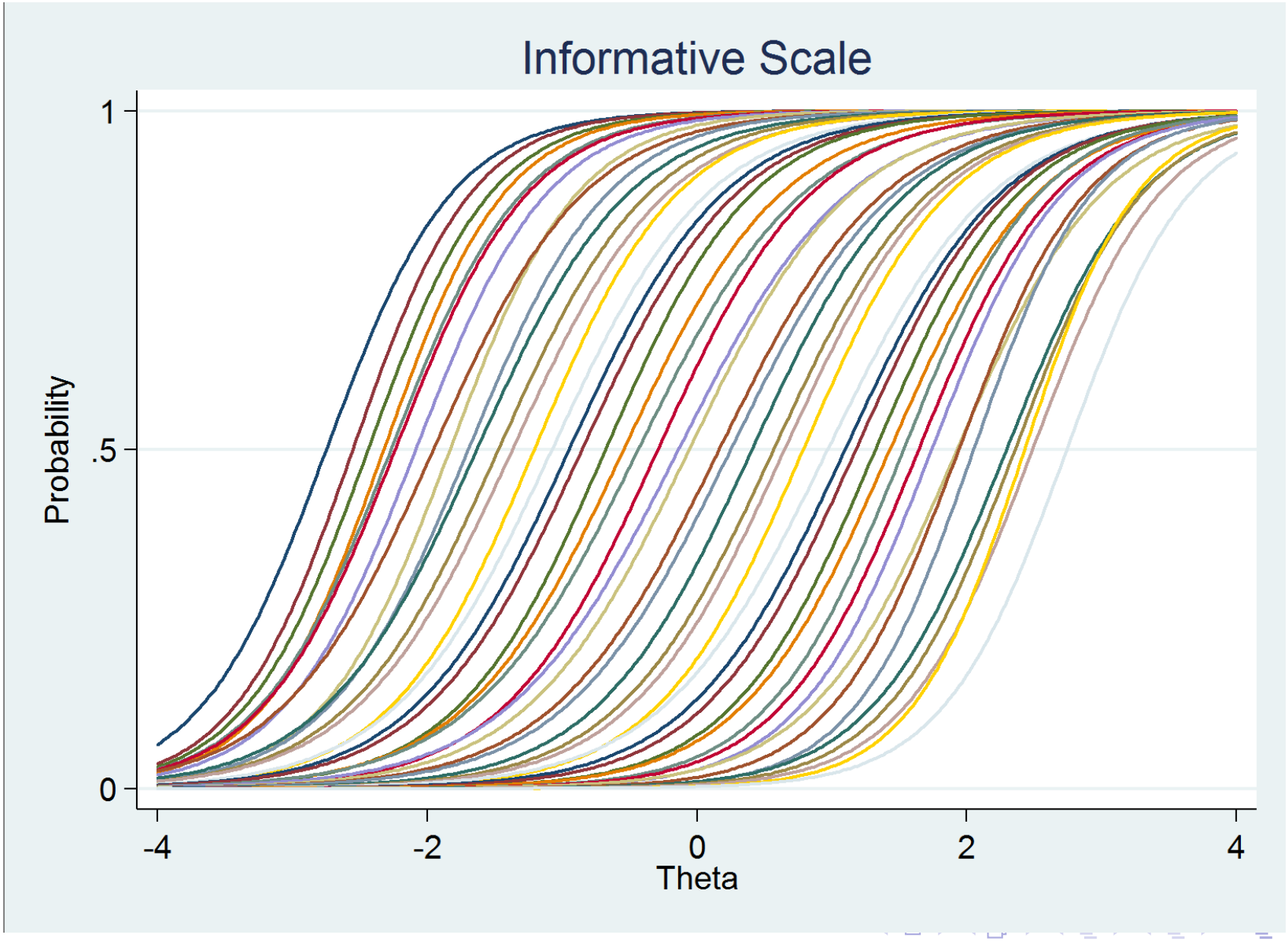
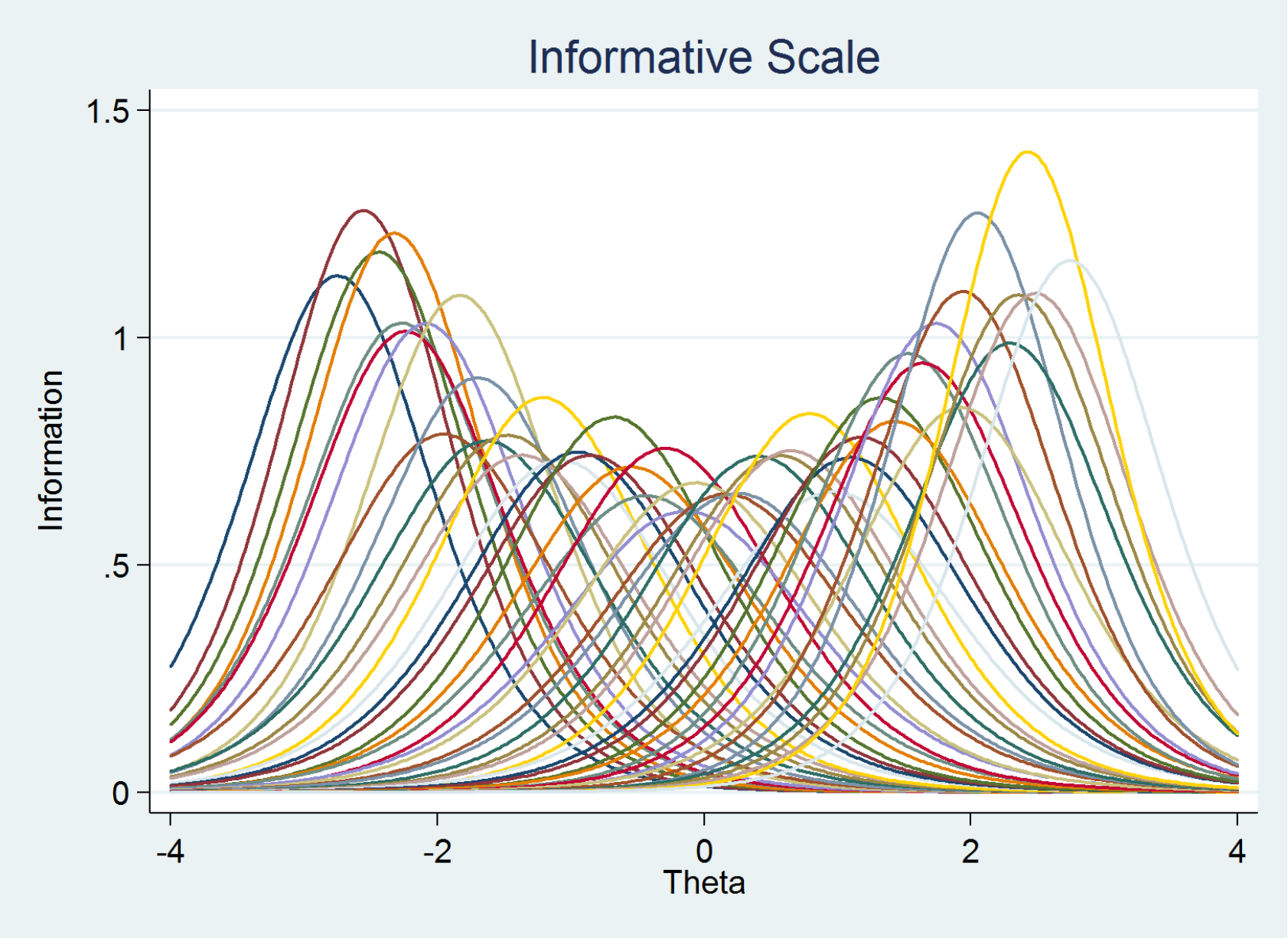


Figure 13: The item response functions of three 2PL items

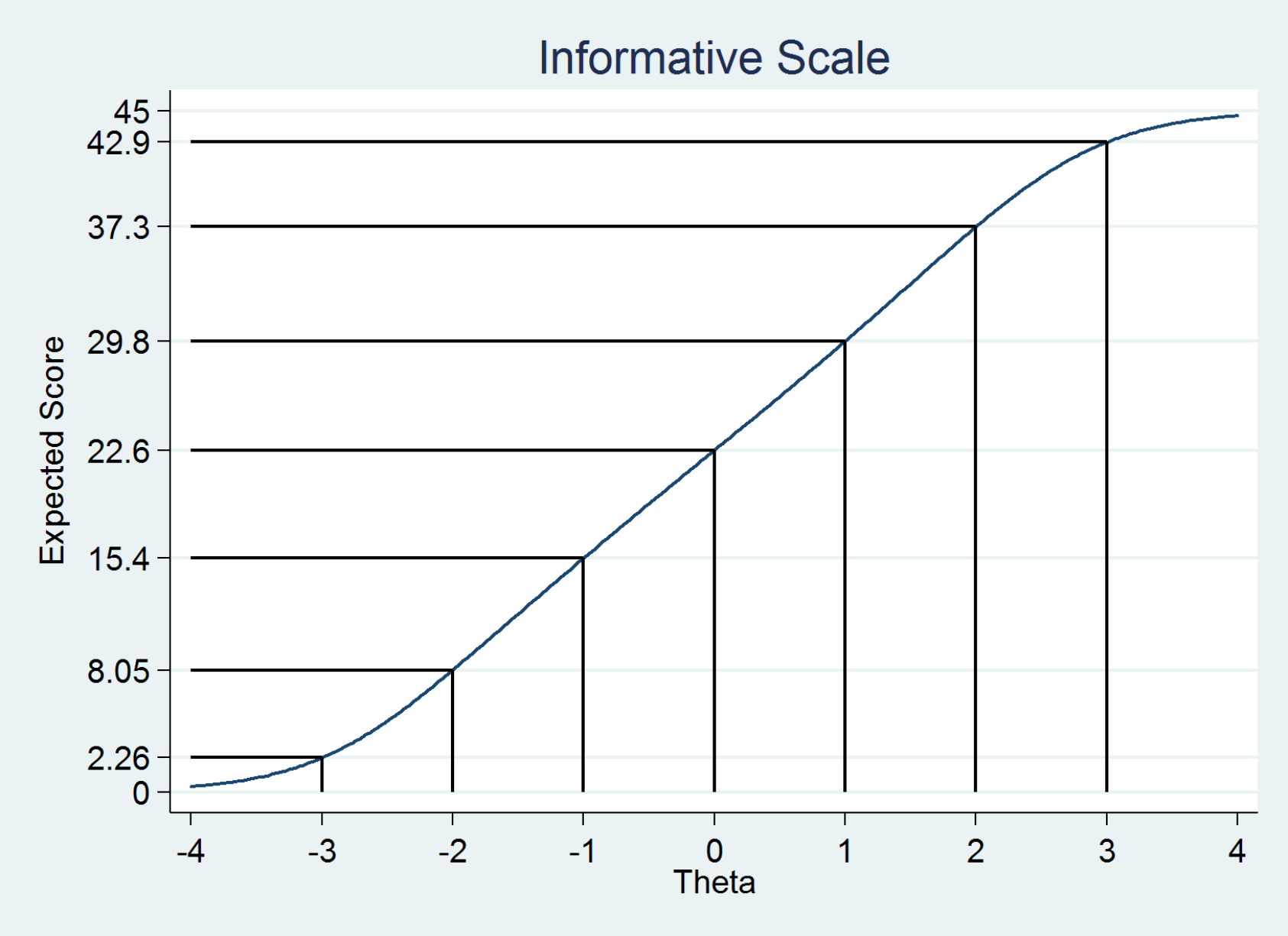
Properties of an Informative IRT Scale



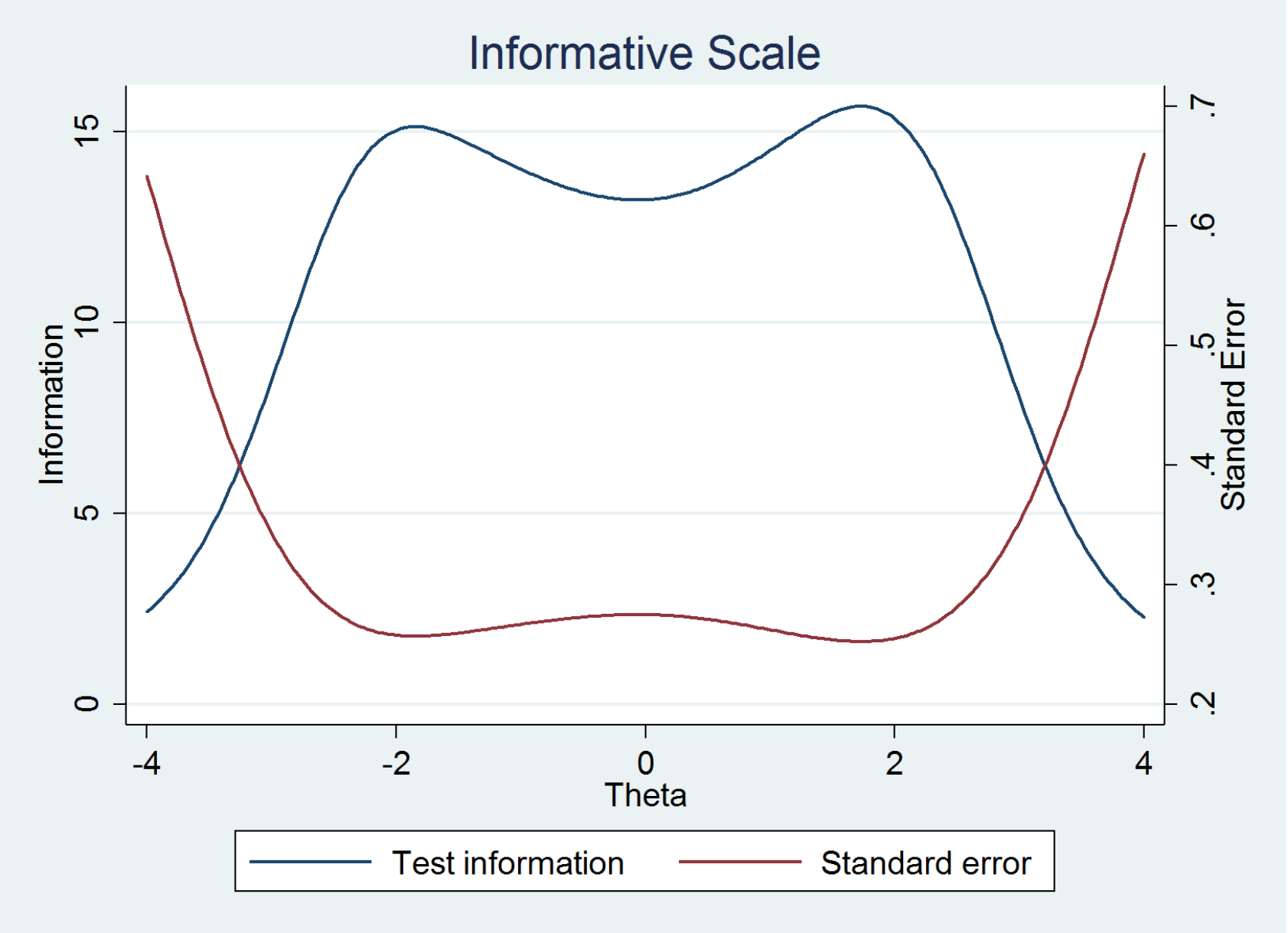
Properties of an Informative IRT Scale: IIF



Properties of an Informative IRT Scale: TCC



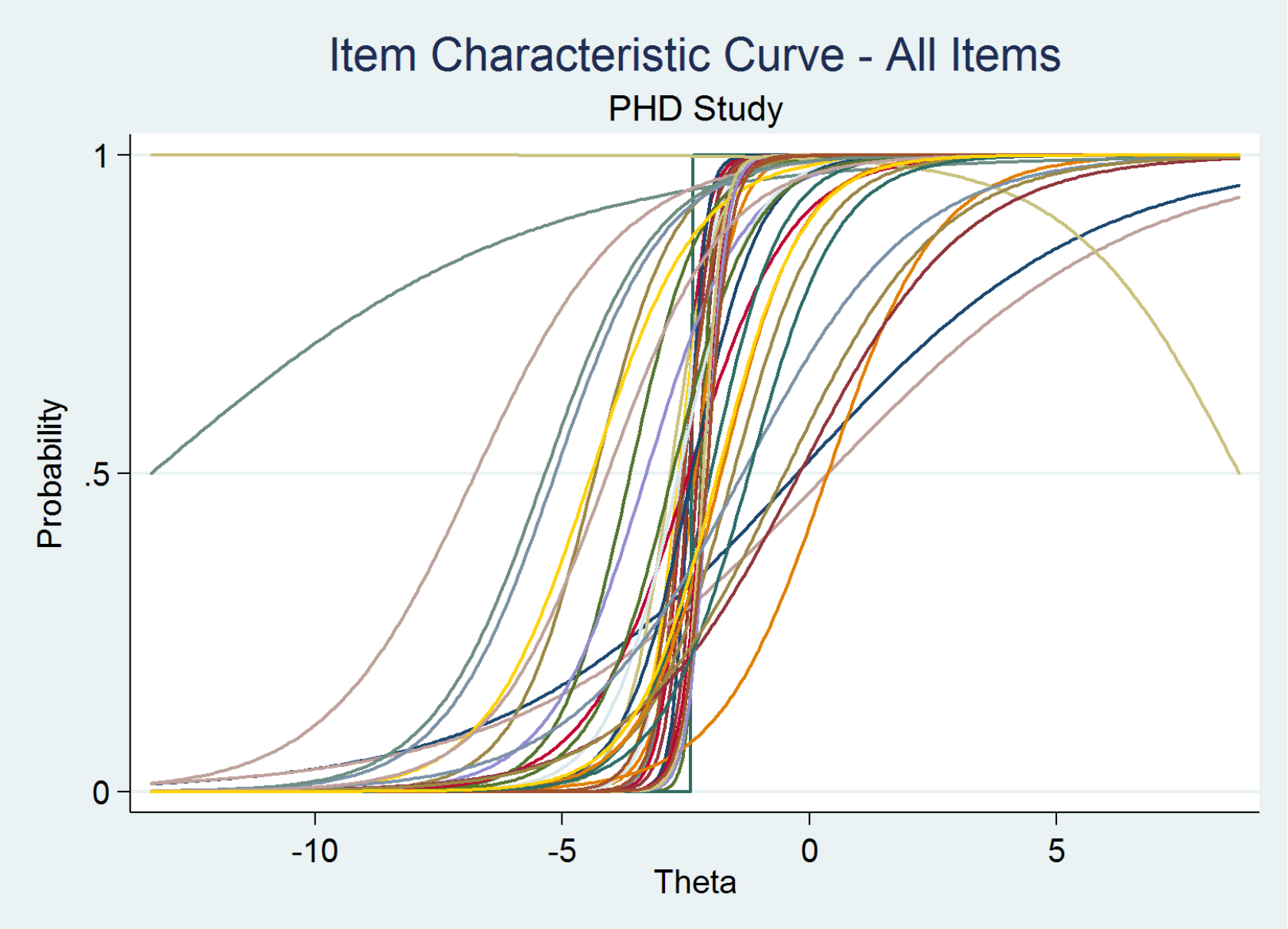
Properties of an Informative IRT Scale: IIF



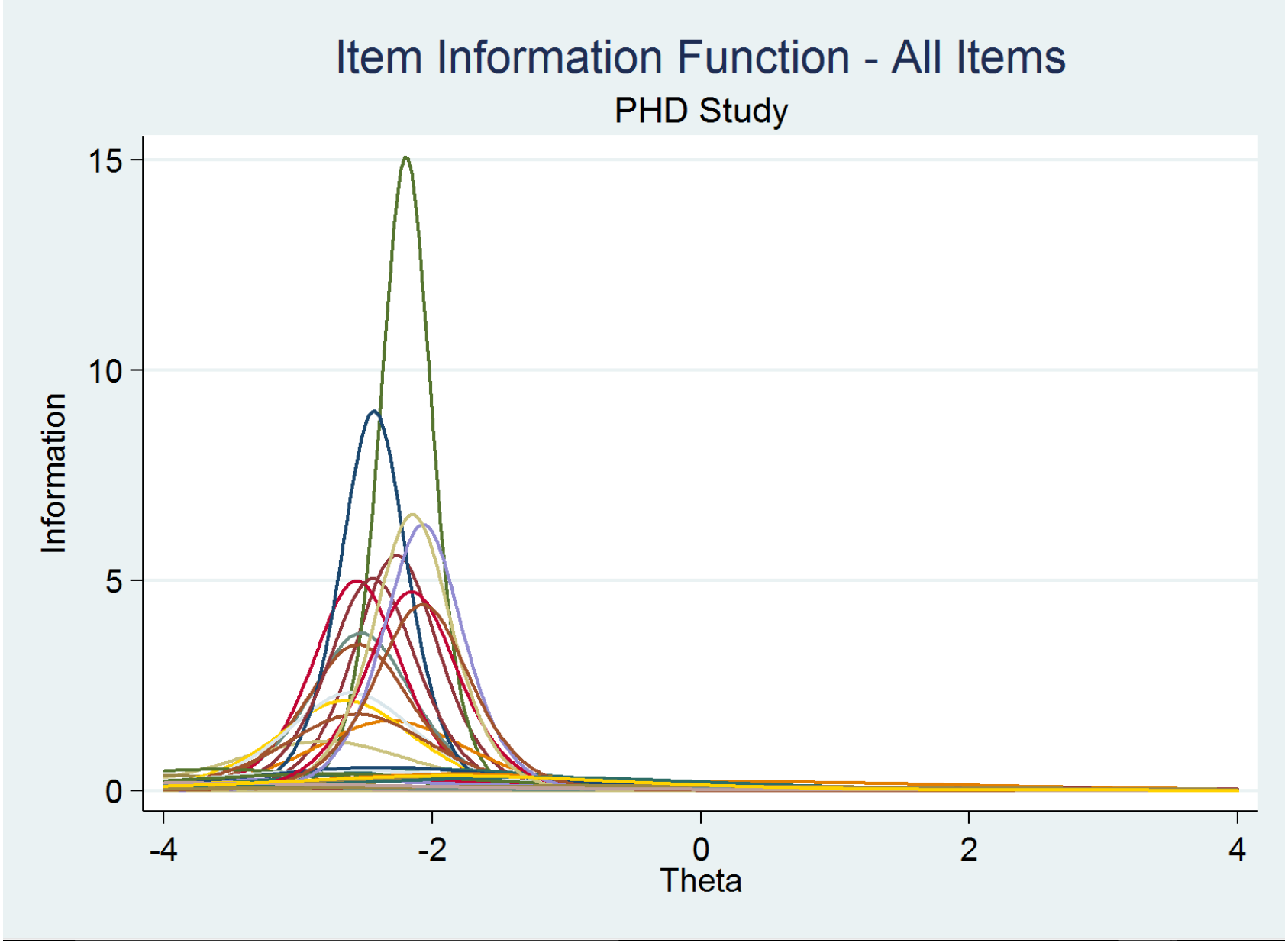
HOME: IRT Analysis

- In few words, an informative scale (as presented in the last four graphs) would have items that have good discriminatory power as well as variability in difficulty.
- This combination allows us to identify, with a lot of precision, households that have low, medium, and high quality environments.
- Unfortunately, the HOME Scale does not have this property.
- As I will show below, that are "too many" easy items and "too few" medium and difficult items.
- For this reason, the HOME Scale will be able to separate very low quality home environments from okay ones, but it will not have power to separate okay from great home environments.

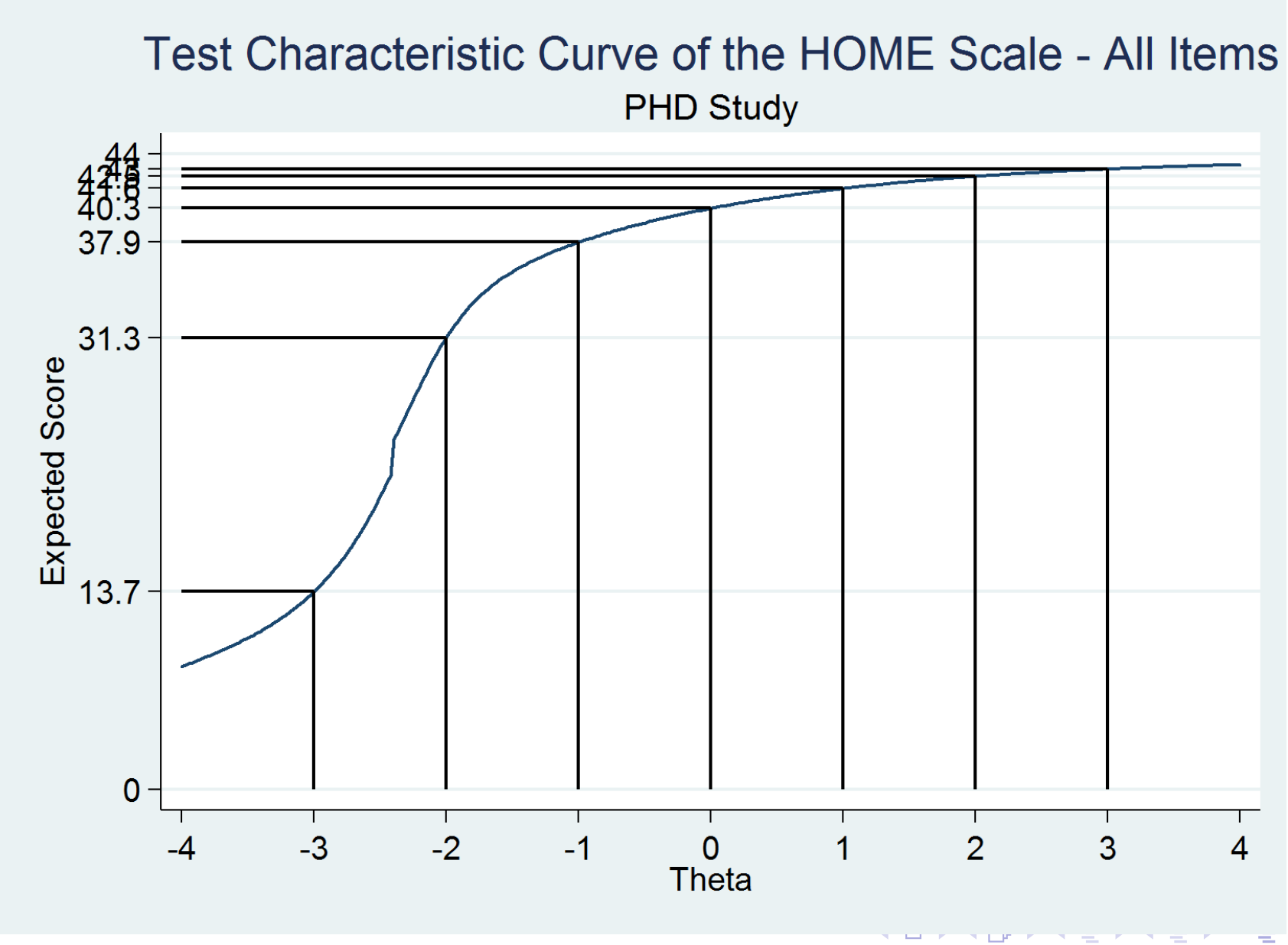
IRT Properties of Full Scale HOME



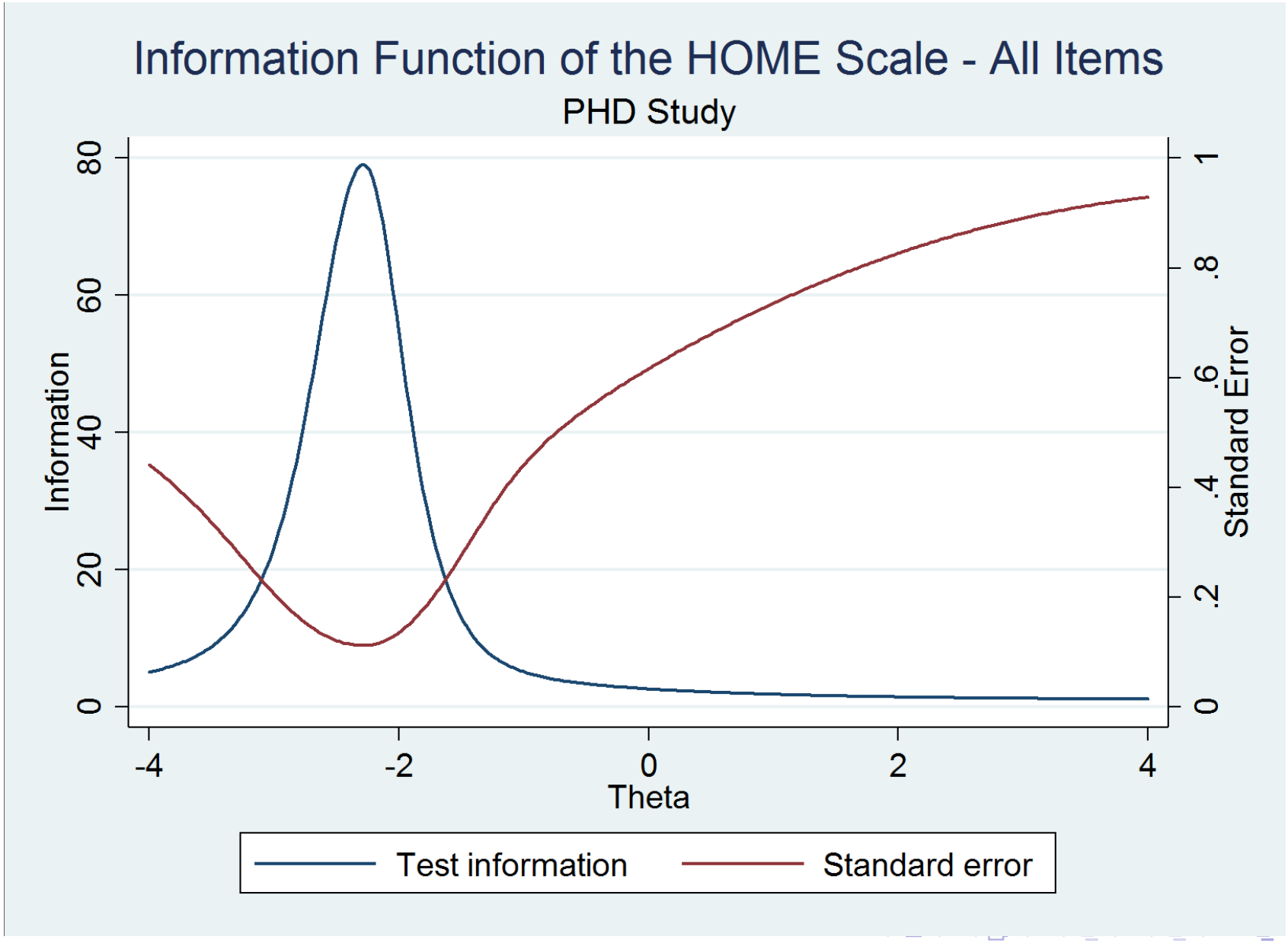
IRT Properties of Full Scale HOME



IRT Properties of Full Scale HOME



IRT Properties of Full Scale HOME



Why does the IRT Properties of the HOME Matter?

- It probably affects the estimation of the technology of skill formation.
- Why? Medium and high quality environments are difficult to separate.
- It is possible that differences between medium and high quality environments are more (or less) important for child development than differences between medium and low quality environments.
- Either case may lead to biases in the estimation of the technology of skill formation.

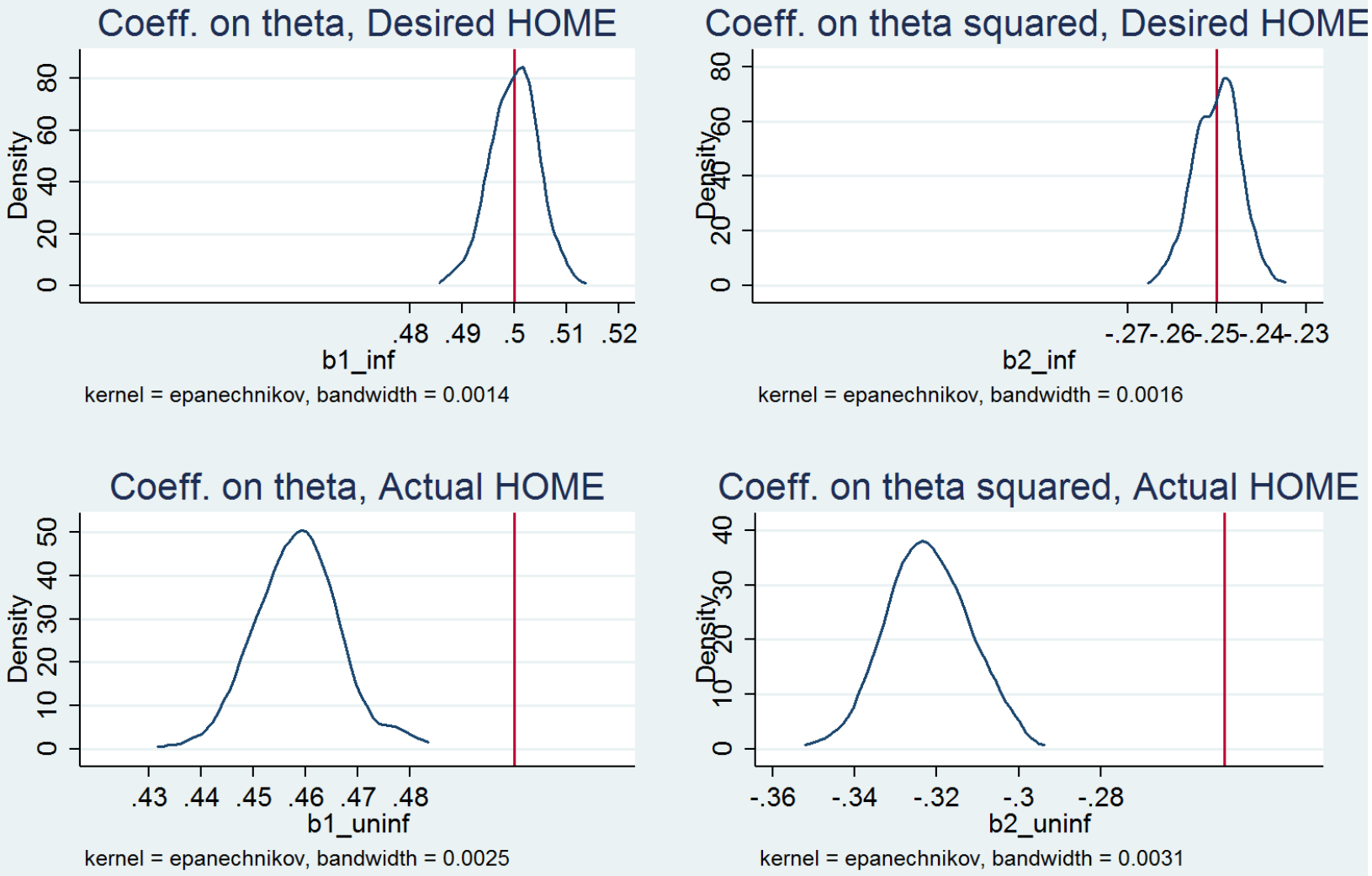
Monte Carlo Exercise

- Let h_1 denote human capital, θ denote investments, and ζ denote uncorrelated shocks. Consider the simple technology of skill formation:

$$h_1 = 1.0 + 0.5\theta - 0.25\theta^2 + \zeta \tag{1}$$

- To obtain an idea about potential problems of using the HOME as a measure of investment to be used in the estimation of (1):
  - Generate a HOME Scale with desirable IRT properties as the "desired" HOME Scale;
  - Generate a HOME Scale that has "flawed" IRT properties as the "actual" HOME Scale;
  - Estimate $\theta_{desired}$ from "desired" HOME Scale and θ_{flawed} from "actual" HOME Scale;
  - Regress h_1 on quadratic function of $\theta_{desired}$ and compare estimated with true coefficients;
  - Regress h_1 on quadratic function of θ_{flawed} and compare estimated with true coefficients.

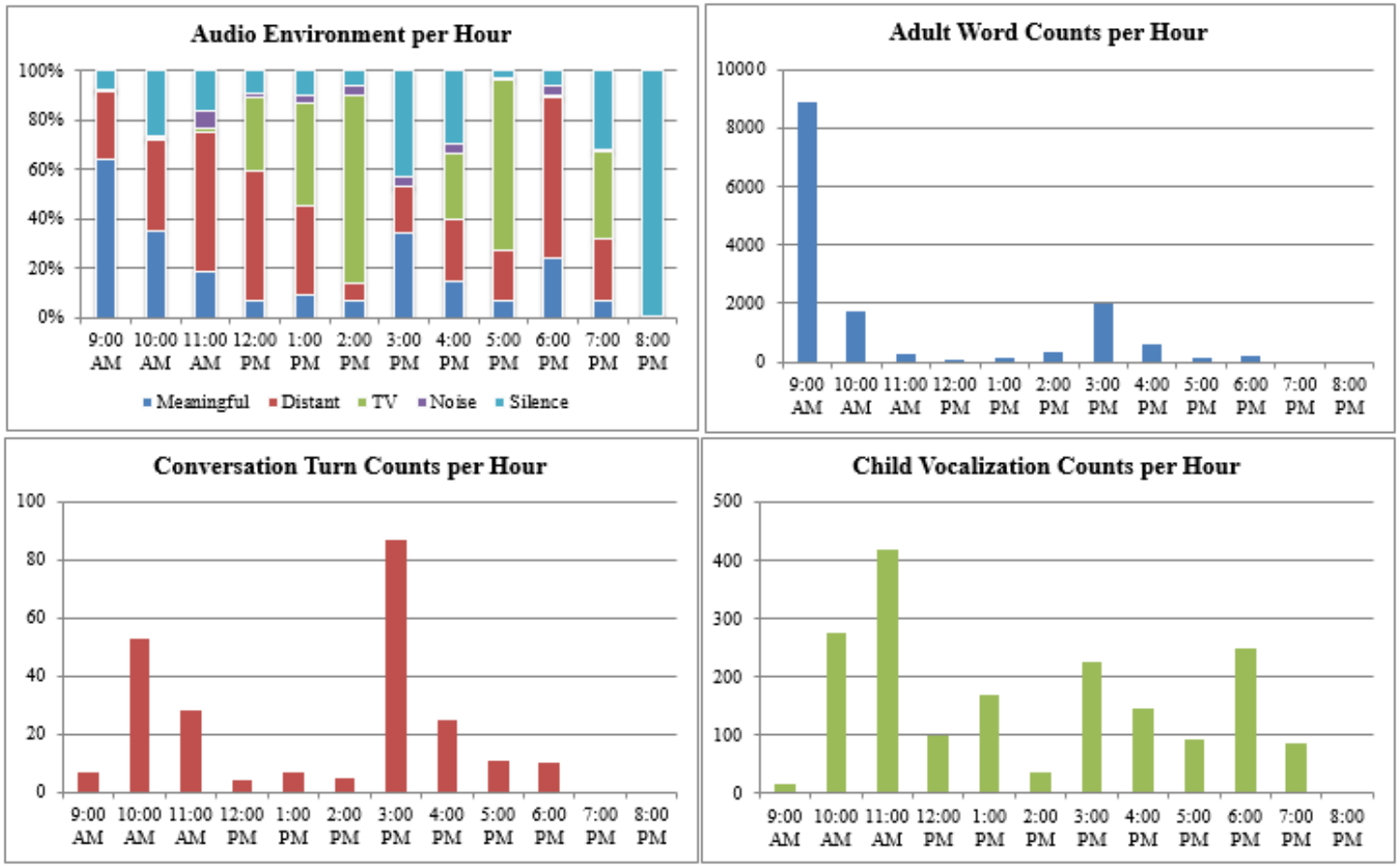
Monte Carlo Exercise



Measuring Quality and Quantity of Time: LENA Pro

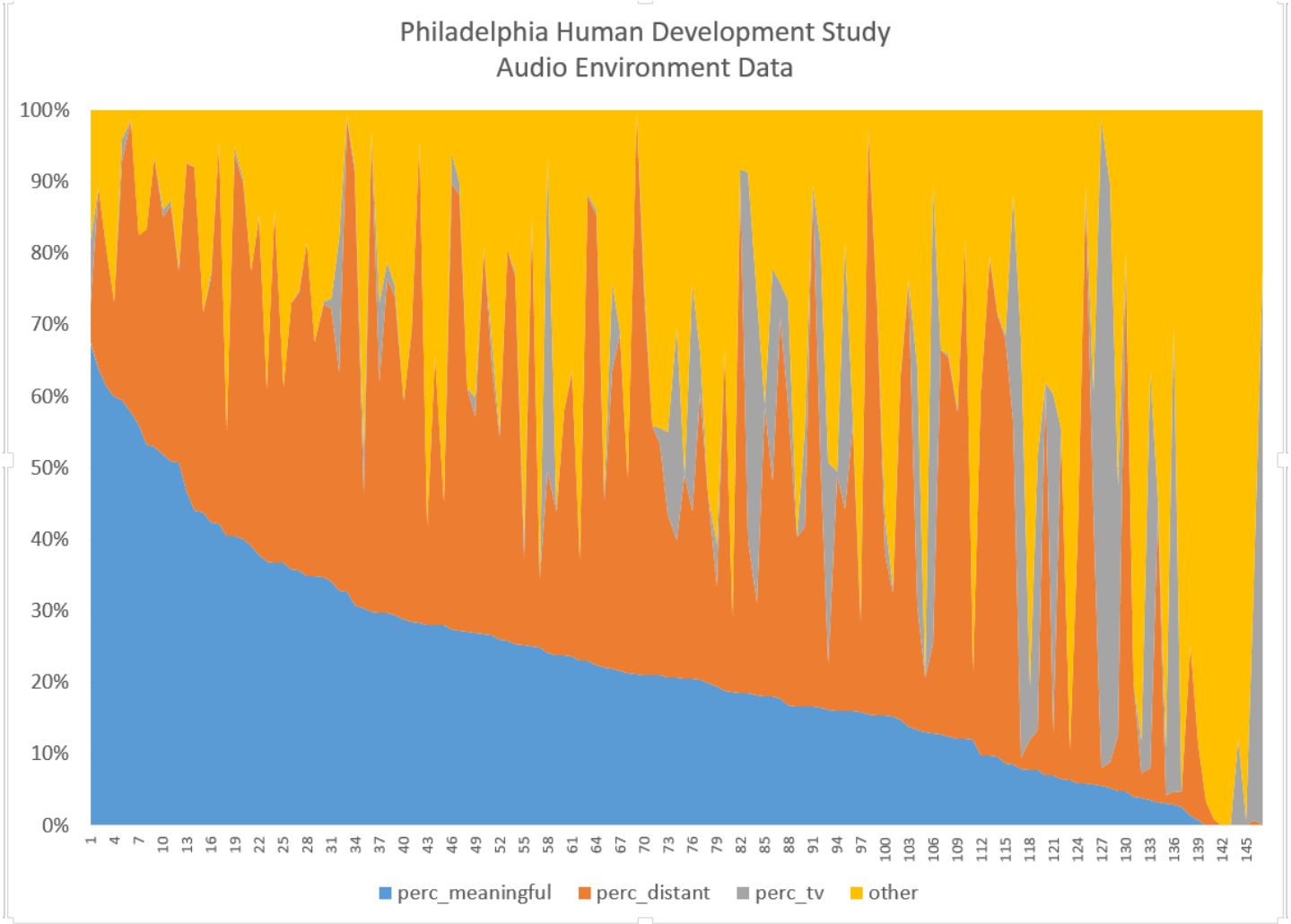
Measuring Quality and Quantity of Time: LENA Pro

Figure 2
Pilot LENA Pro System Study in the HIPPY Program - HISD



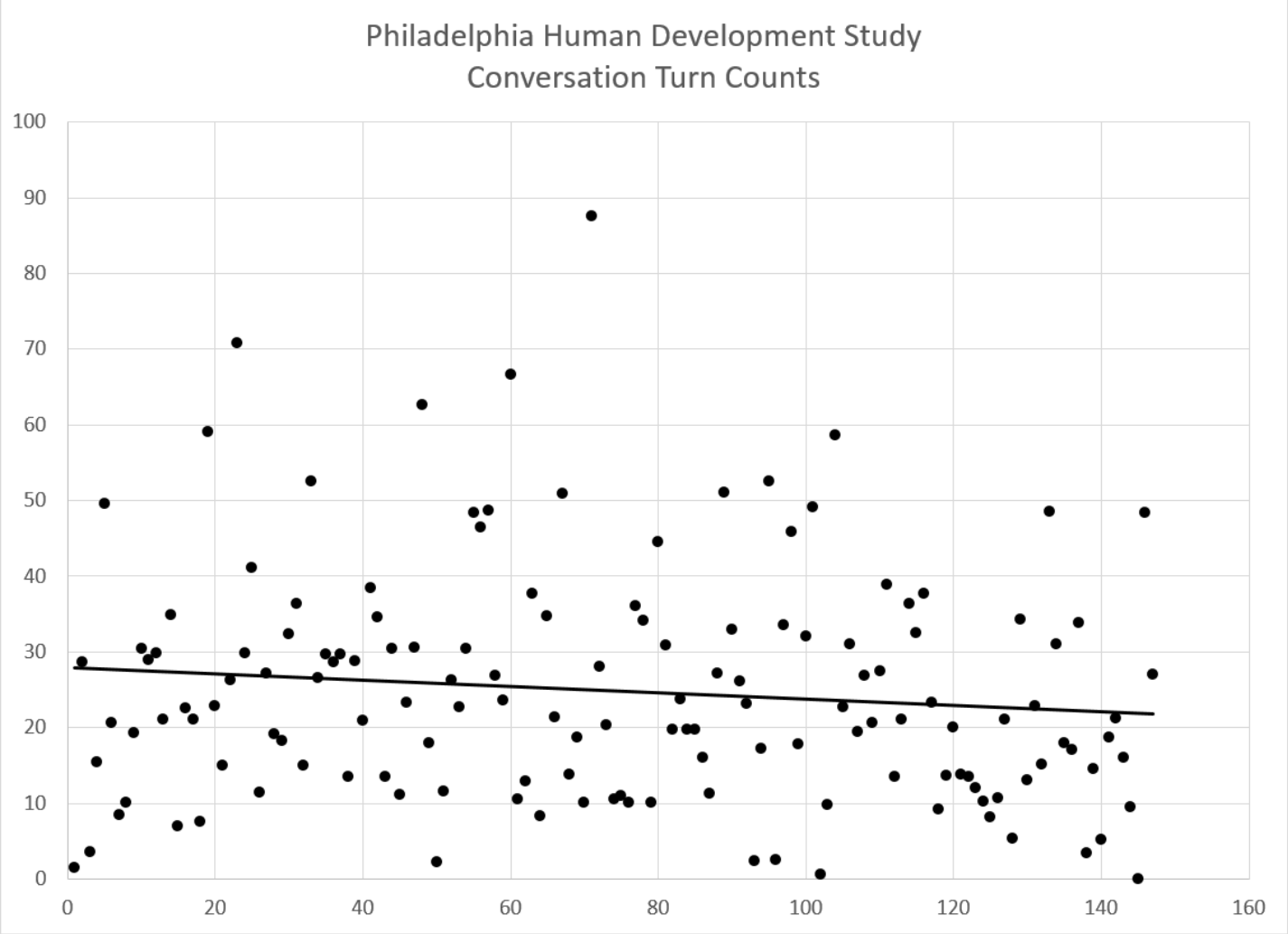
Measuring Quantity of Time: Meaningful Time

Philadelphia Human Development Study



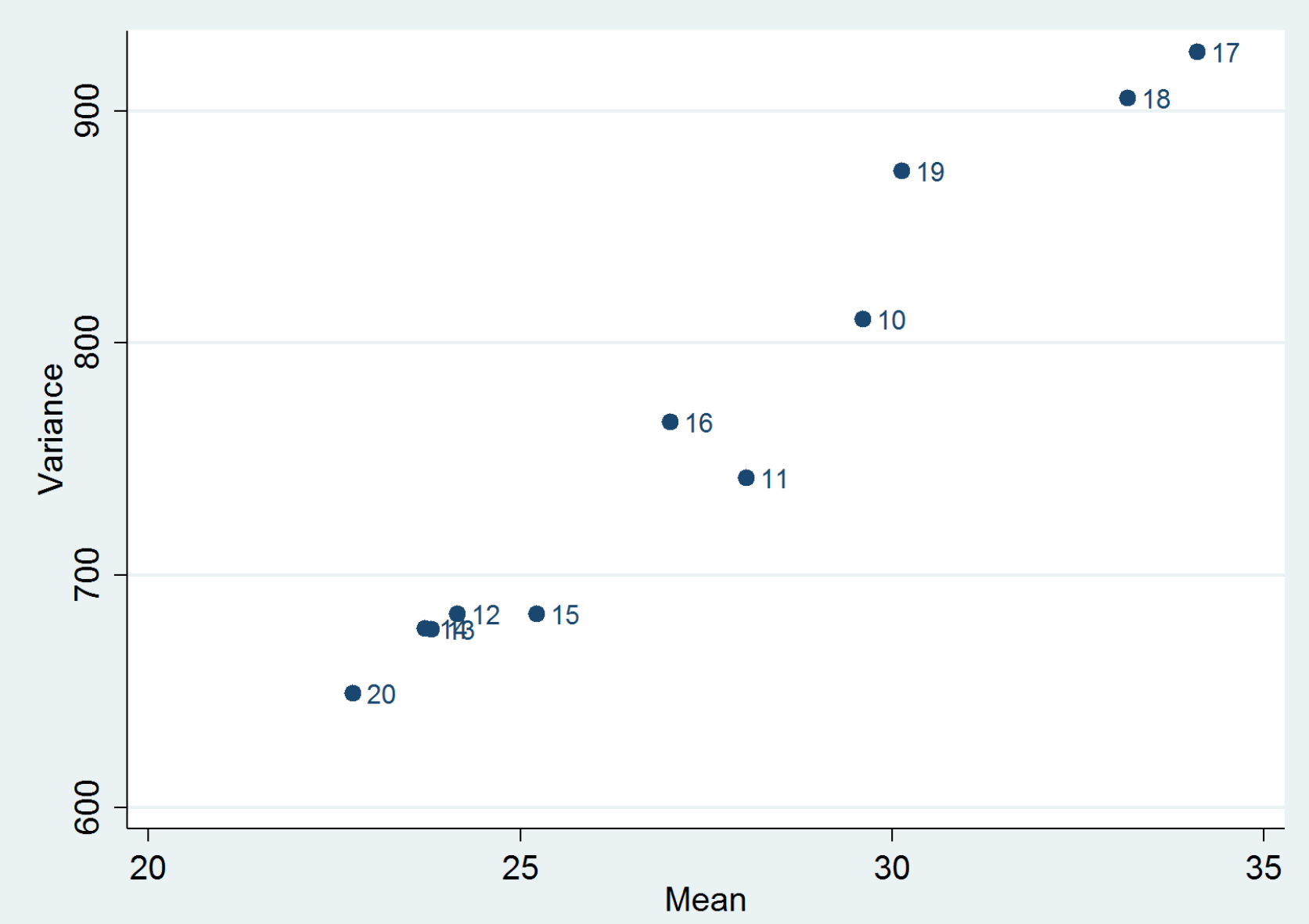
Measuring Quality of Time: Conversation Turn Counts

Philadelphia Human Development Study



Measuring Quality and Quantity of Time: LENA Pro

Philadelphia Human Development Study



Analysis of LENA Conversation Turn Counts Data

- Let $Y_{i,j}$ denote the jth observation on conversation turn counts between an adult and child i.
- Because these are counts, we model each observation as a Poisson random variable with parameter $\epsilon_i \lambda_{i,j}$ where ϵ_i is a random effect term and $\lambda_{i,j}$ is such that:

$$\ln \lambda_{i,j} = X_{i,j}\delta_j + \ln s_{i,j} \tag{2}$$

- Vector $X_{i,j}$ contains variables that describe the context of measurement and $s_{i,j}$ is "exposure" (i.e., number of seconds that the LENA device was on during the jth measurement).

Analysis of LENA Conversation Turn Counts Data

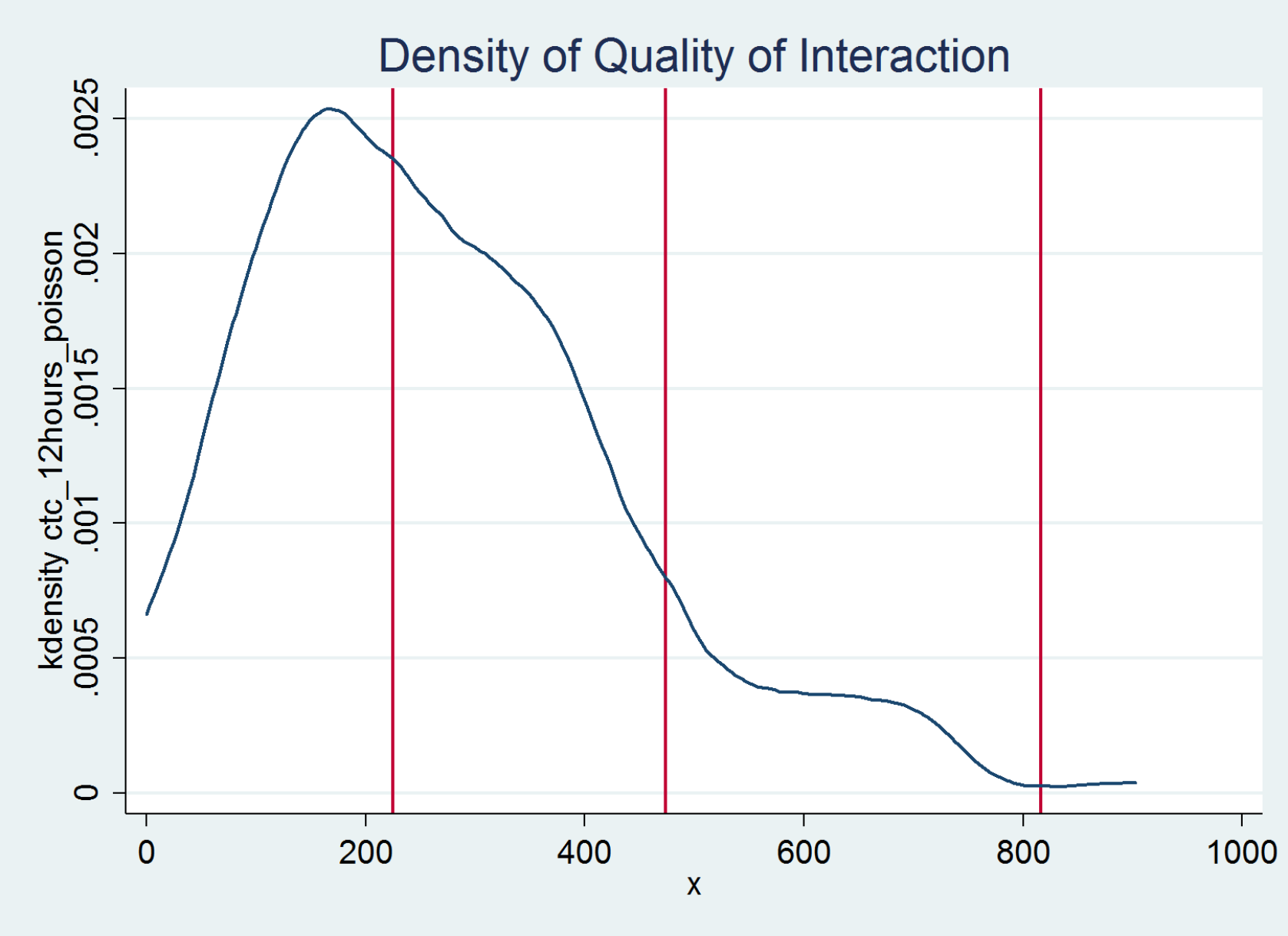
- Conditional on ϵ_i, the probability of observing a count equal to:

$$\Pr\left(y_{i,j} \middle| \epsilon_i\right) = \frac{\left(\epsilon_i \lambda_{i,j}\right)^{y_{i,j}}}{y_{i,j}!} e^{-\epsilon_i \lambda_{i,j}}$$

  where $\Pr\left(y_{i,j} \middle| \epsilon_i\right) = \Pr\left(Y_{i,j} = y_{i,j} \middle| \epsilon_i\right)$ is the probability that the count of variable $Y_{i,j}$ is equal to $y_{i,j}$ conditional on ϵ_i.
- Assume that, conditional on ϵ_i, the events are independent. Thus:

$$\Pr\left(y_{i,1}, \ldots, y_{i,J} \middle| \epsilon_i\right) = \left\{ \left[\prod_{j=1}^{J} \frac{\left(\lambda_{i,j}\right)^{y_{i,j}}}{y_{i,j}!} \right] \epsilon_i^{\sum_{j=1}^{J} y_{i,j}} e^{-\epsilon_i \sum_{j=1}^{J} \lambda_{i,j}} \right\} \tag{3}$$

- Because we don't observe the random effect ϵ_i, we need to integrate it out.
- We assume that ϵ_i has gamma distribution with mean one and variance $\frac{1}{\alpha}$

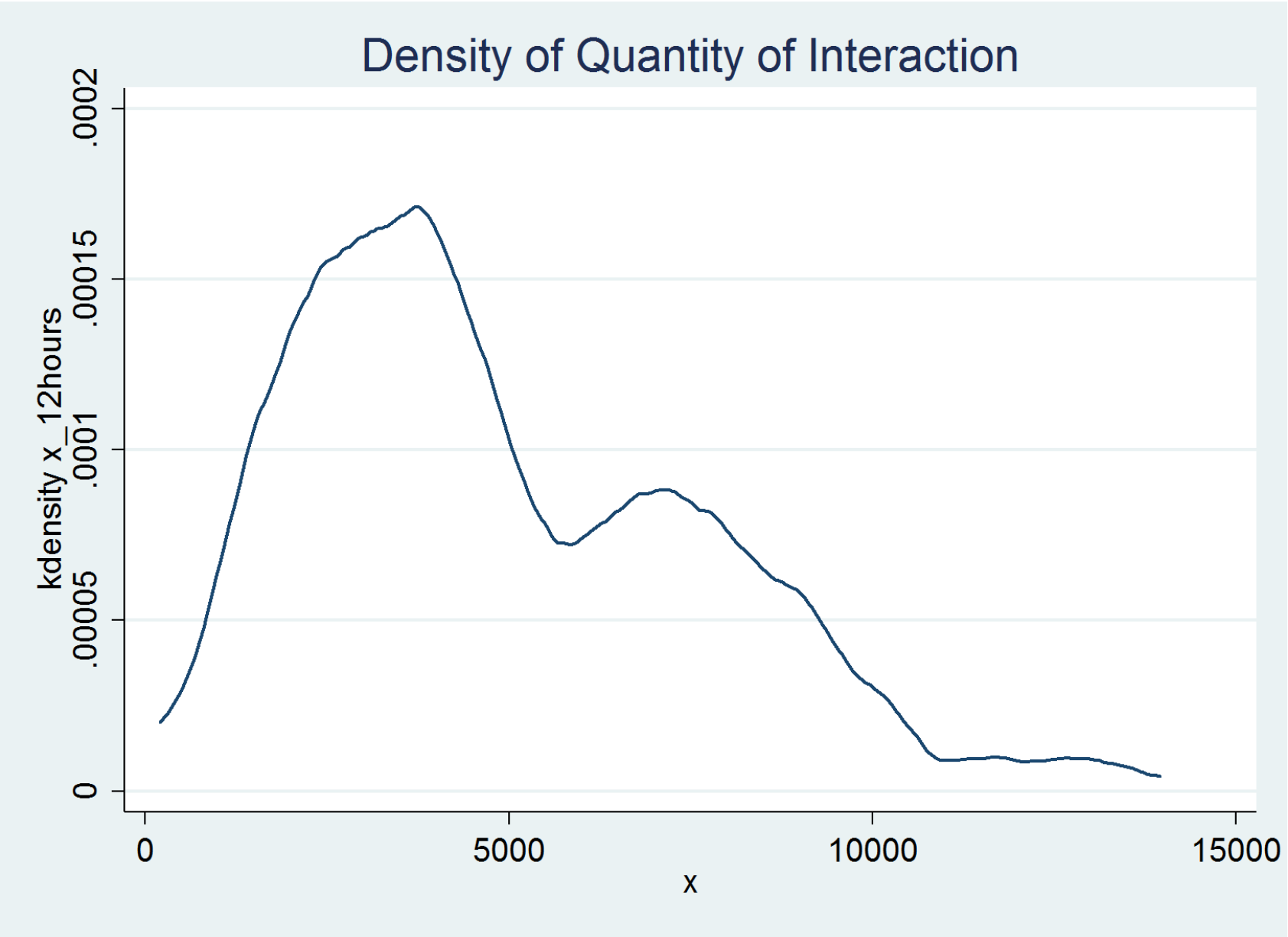

Density of Quality of Interaction
kdensity ctc_12hours_poisson
0
.0005
.001
.0015
.002
.0025
0
200
400
600
800
1000
x

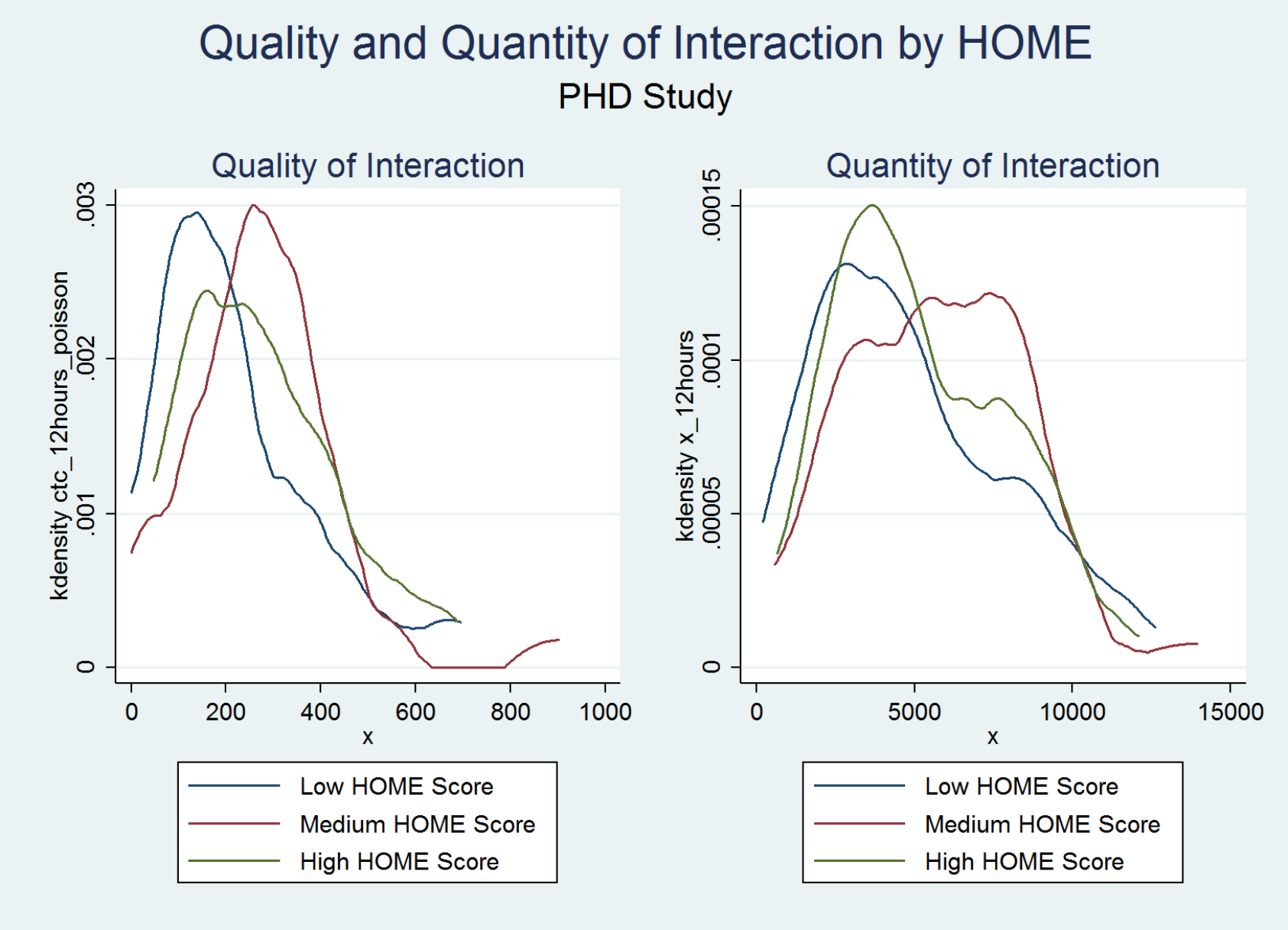
Analysis of LENA Conversation Turn Counts Data

- Let $M_{i,j}$ denote the share of meaningful time of adult-child interaction in jth observation.
- Because these are proportion data, we model each observation as the following logistic regression:

$$\ln\left\{\frac{M_{i,j}}{1-M_{i,j}}\right\} = X_{i,j}\rho_j + \mu_i + \nu_{i,j}$$

  where μ_i is a random effect with mean zero and variance σ_μ^2.
- We are interested in estimating the unobserved heterogeneity captured by μ_i across families.


Density of Quantity of Interaction
kdensity x_12hours
0
.00005
.0001
.00015
.0002
0
5000
10000
15000
x


Quality and Quantity of Interaction by HOME
PHD Study
Quality of Interaction
kdensity ctc_12hours_poisson
0
.001
.002
.003
0
200
400
600
800
1000
x
Low HOME Score
Medium HOME Score
High HOME Score
Quantity of Interaction
kdensity x_12hours
0
.00005
.0001
.00015
0
5000
10000
15000
x
Low HOME Score
Medium HOME Score
High HOME Score

Conclusion

- Previous literature shows that early investments matter.
- Difficult to simultaneously and precisely measure quantity and quality of parent-child interaction.
- New forms of measurement will help us understand:
  - What types of investment are more important for each dimension of human capital.
  - How each type of investment is influenced by existing stock of human capital of the child.
  - Whether each type is sensitive to opportunity costs, parental beliefs, and household characteristics.
- Design new interventions that can directly or indirectly influence each type of investment.

The Economics and the Econometrics of Human Development: Testing and Operationalizing the Theory Part IIIB

James J. Heckman
University of Chicago

CeMMAP Masterclass
University College of London
June 23, 2016

THE UNIVERSITY OF CHICAGO

Testing and Operationalizing the Theory

- Decompose the $\boldsymbol{\theta_t}$ vector into three subvectors:

$$\boldsymbol{\theta_t} = (\boldsymbol{\theta_{C,t}}, \boldsymbol{\theta_{N,t}}, \boldsymbol{\theta_{H,t}}) \tag{1}$$

where

- $\boldsymbol{\theta_{C,t}}$ is a vector of cognitive abilities (e.g., IQ) at age t,
- $\boldsymbol{\theta_{N,t}}$ is a vector of noncognitive abilities (e.g., patience, self-control, temperament, risk aversion, discipline, and neuroticism) at age t, and
- $\boldsymbol{\theta_{H,t}}$ is a vector of health stocks for mental and physical health at age t.

- Functionings (task j) at age t:

$$Y_{j,t} = \psi_{j,t}(\boldsymbol{\theta_t}, e_{j,t}, \boldsymbol{X_{j,t}}), \quad j \in \{1, \ldots, J_t\} \quad \text{and} \quad t \in \{1, \ldots, 2T\} \tag{2}$$

- $Y_{j,t}$: outcome from activity j at time t
- $\boldsymbol{\theta_t}$ is the vector of skills at age t
- $\boldsymbol{X_{j,t}}$ is a vector of purchased inputs that affect the functionings
- $e_{j,t}$ is effort in task
- T is the length of childhood
- T is the length of adulthood
- $2T$ is total **lifetime**

Effort: $e_{j,t}$

$$e_{j,t} = \delta_j(\boldsymbol{\theta_t}, \boldsymbol{A_t}, \boldsymbol{X_{j,t}}, \boldsymbol{R^a_{j,t}}(\mathcal{I}_{t-1}) \mid u). \tag{3}$$

- $\boldsymbol{A_t}$: environment
- $\boldsymbol{R^a_{j,t}}$: incentives

Estimating and Interpreting the Distribution of Skills, the Maps Between Skills and Functionings and the Technology of Skill Formation

- **Low dimensional** skills ("factors") generate a high dimensional set of functionings.
- Dimension and factor structures selected through a variety of methods.
- Exploratory factor analysis.
- Novel Bayesian procedures—avoid arbitrary methods in Exploratory Factor Analysis.

- Cunha et al. (2010) present conditions under which the outcome equation and technology equation are non-parametrically identified.
- They develop methods for accounting for the measurement error of inputs, anchoring estimated skills on adult outcomes (so that scales are defined in meaningful units), and accounting for the endogeneity of investments.

- Heckman et al. (2013) develop and apply simple and easily implemented least-squares estimators of linear factor models to estimate equations for outcomes.

Linear Factor Example

Link to "Formulating, Identifying"

(Cunha & Heckman 2008, *JHR*)

Nonparametric Factor Models Are Natural Frameworks for Estimating Skills and Determining Frontier Skill Sets

Estimating the Technology of Production of Cognitive and Noncognitive Skills

(builds on Cunha, Heckman, & Schennach, 2010)

Link to "Bayesian Exploratory Factor Analysis"

by Frühwirth-Schnatter et al. (*JOE*, 2014)

Skills as Determinants of Outcomes

Multiple Skills Shape Human Achievement Across a Variety of Dimensions

Estimating and Interpreting the Distribution of Skills, Maps Between Skills and Outcomes and the Technology of Skill Formation

Augmented Factor Models Are Natural Frameworks for Estimating Skills

- Figure 1 from (Eisenhauer et al., 2014) plots the probability and the return of enrolling in college immediately after having graduated high school as a function of the deciles of scalar summaries of cognitive and noncognitive skills.

Figure 1: The Probability and Returns of College Enrollment by Endowments Levels

Figure: Choice Probability, College Enrollment

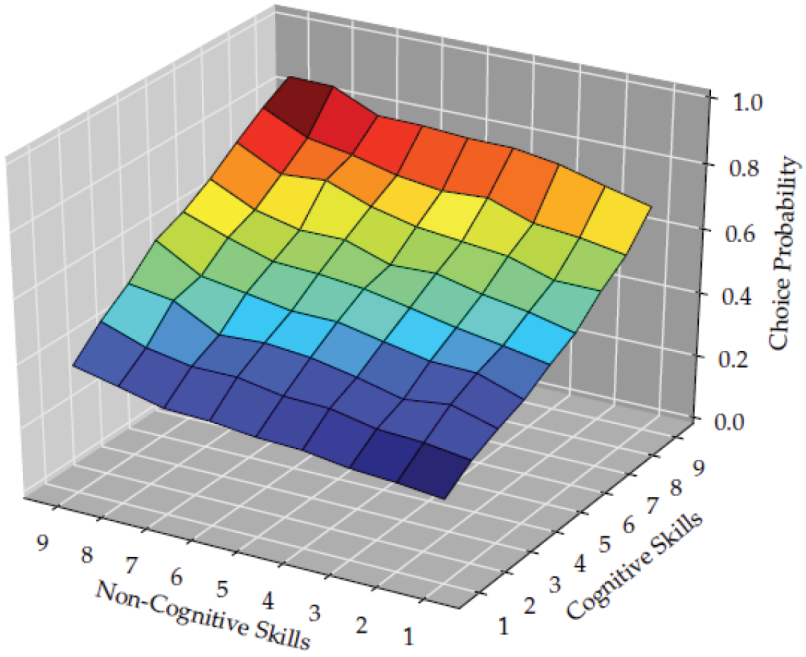
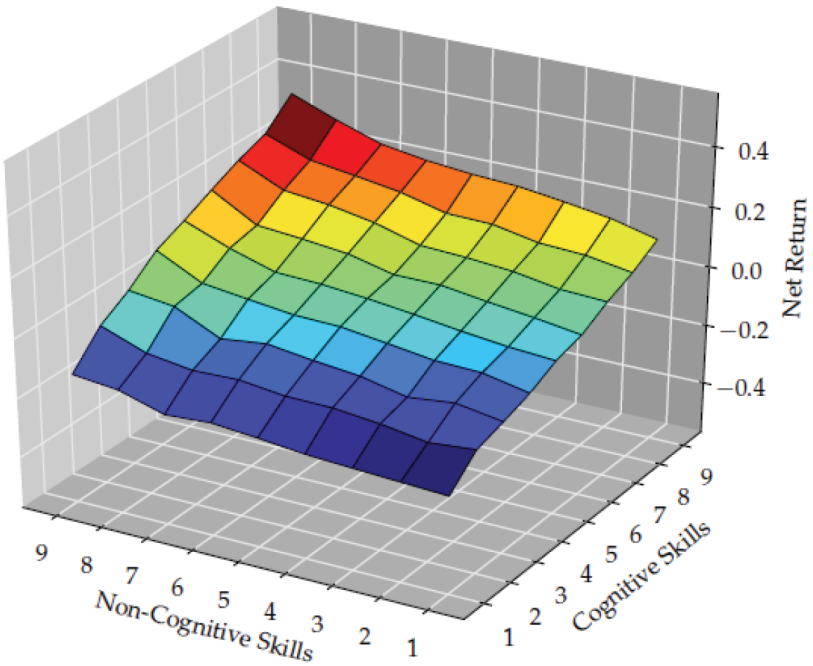


Figure: Net Return, College Enrollment



Source: Eisenhauer et al. (2014)
Note: College enrollment refers to the individuals who enroll in college immediately after having finished high school. Returns are expressed in units of millions of dollars. Higher deciles correspond to higher levels. See Eisenhauer et al. (2014) for greater details.

- The return is calculated over a 65-year-long working life. Lifecycle earning profiles are simulated using the estimated parameters.
- See Eisenhauer et al. (2014) for a precise description of the model, data, and computations.

See Appendix on **Evidence on The Predictive Power of Cognitive and Socioemotional Traits**

Link to Appendix

Similar Patterns for a Variety of Diverse Outcomes

A low dimensional vector of capabilities predicts a wide variety of outcomes

- Crime
- Wages
- Health
- Healthy behaviors (smoking, drug use)
- Trust
- Voting behavior
- Employment
- Participation in welfare

Technology of Skill Formation

$$\theta_{k,t+1} = f_{s,k}\left(\theta_t, I_{k,t}, \theta_{P,t}\right) \tag{4}$$

for $k \in \{C, N, H\}, t \in \{1, 2, \ldots, T\}$.

Identification of the Technology of Skill Formation

- To estimate the technology of skill formation: Have to solve three problems.

1. Don't observe $(\theta_t, I_{k,t}, \theta_{P,t})$ directly, but have many measurements on it. **Measurement error in nonlinear systems.**
2. Don't know which scale to use to measure components of $\boldsymbol{\theta_t}$. **Anchor test scores on adult outcomes.**
3. Investment $I_{k,t}$ may be chosen by parents based on information that may be unobserved by the econometrician $(\eta_{k,t})$. **Endogeneity of investment.**

Estimates of Technologies of Skill Formation and Some Implications

- Technology equations with shocks:

$$\theta_{k,t+1} = \Big[\gamma_{t,k,1}\theta_{C,t}^{\phi_{t,k}} + \gamma_{t,k,2}\theta_{N,t}^{\phi_{t,k}} + \gamma_{t,k,3}I_t^{\phi_{t,k}} + \gamma_{t,k,4}\theta_{C,P}^{\phi_{t,k}} + \gamma_{t,k,5}\theta_{N,P}^{\phi_{t,k}}\Big]^{\frac{\rho_{t,k}}{\phi_{t,k}}} e^{\eta_{k,t}}, \tag{5}$$

where $\gamma_{t,k,l} \geq 0$ and $\sum_{l=1}^{5} \gamma_{t,k,l} = 1$, $0 \leq \rho_{t,k} \leq 1$, $\phi_{t,k} < 1$, $k \in \{C, N\}$, $t \in \{1, 2\}$.

Cunha, Heckman, and Schennach (2010)

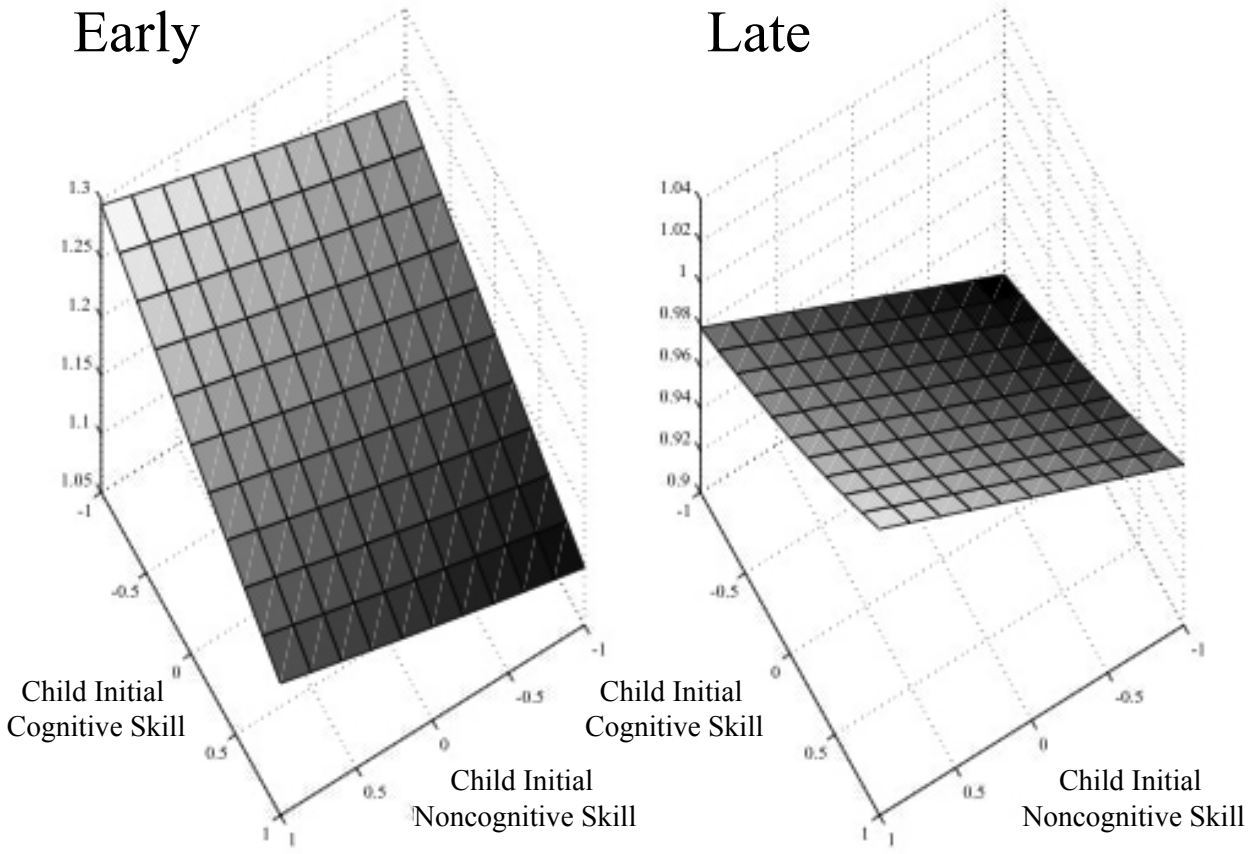
- Skills evolve over the life cycle
- Parental investments explain 34% of variance of educational attainment
- Self-productivity becomes stronger as children become older, for both cognitive and noncognitive skill formation (i.e., $\frac{\partial \theta_{t+1}}{\partial \theta_t} \uparrow t$).
- Strong cross effects (noncognitive skills foster cognitive investment)
- Complementarity between cognitive skills and investment becomes **stronger** as children become older. The elasticity of substitution for cognitive production is *smaller* in second stage production

- **Emerging dynamic complementarity.**
- It is more difficult to compensate for the effects of adverse environments on **cognitive** endowments at later ages than it is at earlier ages. This pattern of the estimates helps to explain the evidence on ineffective cognitive remediation strategies for disadvantaged adolescents reported in Cunha et al. (2006), Cunha (2007), and later papers.
- Complementarity between **noncognitive** skills and investments stays roughly constant over the life cycle.
- Suggests that later life investments should be more focused on promoting noncognitive—personality—skills.
- The evidence on which adolescent interventions are successful is consistent with this evidence.

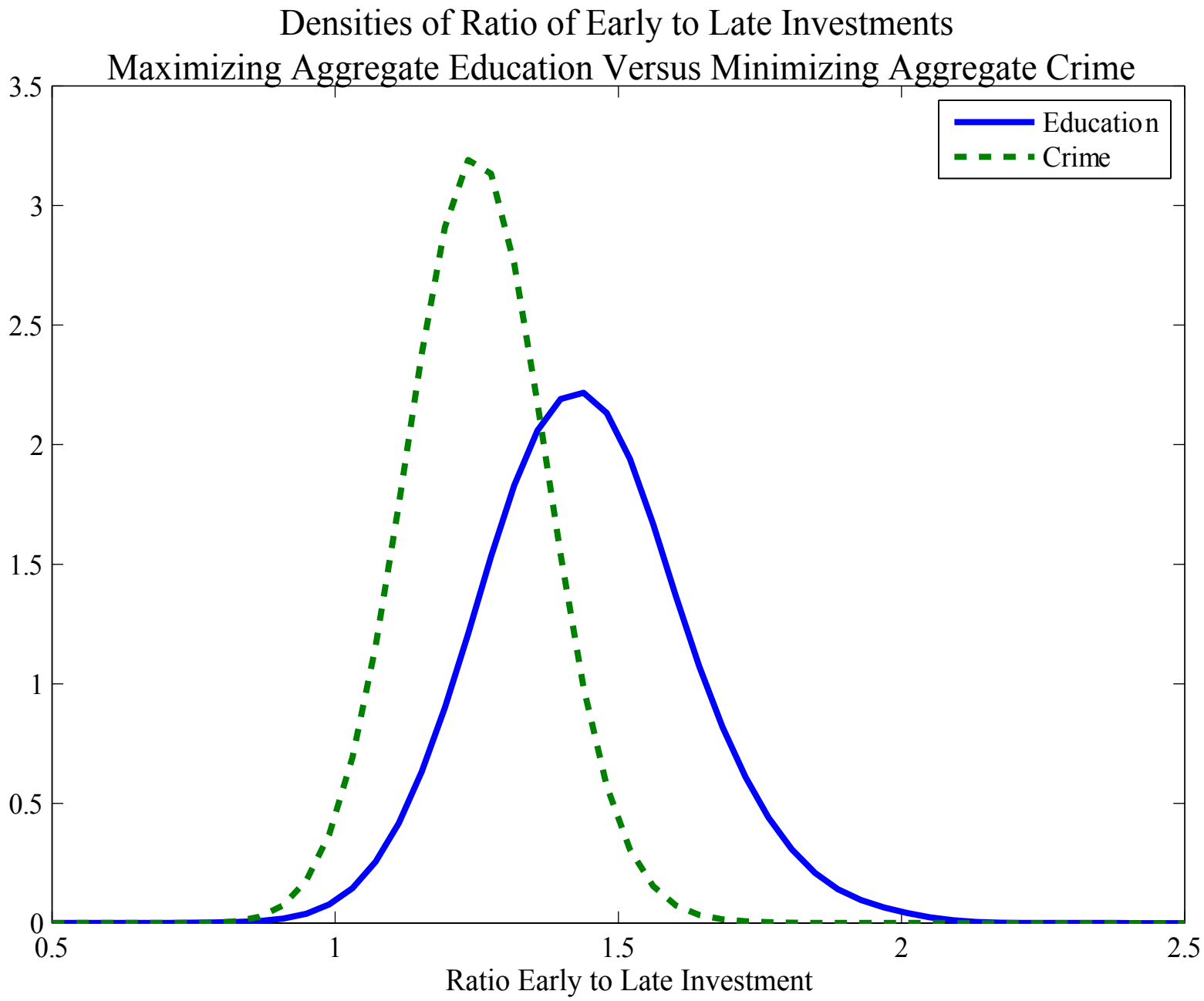
The Implications of the Estimates for Design of Policy

- Targeted strategies
- Consider a policy for a social planner to optimize the stock of education in society.
- Assume (for simplicity) full control of investment (ignores parental responses)
- The bulk of the evidence in the child development literature shows **reinforcement** of investment by parents.
- No consideration of social fairness, equality of opportunity or equality of final outcomes—just efficiency.
- Yet with these estimates the optimal policy invests the most in the disadvantaged.
- As an empirical matter, social justice is enhanced by what is productively efficient.

Figure 2: Socially Optimal Early and Late **Levels** of Investment by Initial Skills



Source: Cunha et al. (2010). Optimal investments to maximize aggregate education in society.


Densities of Ratio of Early to Late Investments
Maximizing Aggregate Education Versus Minimizing Aggregate Crime
Education
Crime
3.5
3
2.5
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
Ratio Early to Late Investment

Review of the Literature

Table 1: Skill Production Functions

	Skill Output			Functional	
	Cognitive[a]	**Non-cognitive**[a]	**Health**	**Form**	**Anchoring**
Todd and Wolpin (2003)	✓	X	X	Linear	X
Bernal and Keane (2010)	✓	X	X	Linear	X
Cunha and Heckman (2008)	✓	✓	X	Linear	✓[b]
Cunha et al. (2010)	✓	✓	X	CES	✓
Todd and Wolpin (2007)	✓	X	X	Linear	X
Cunha (2007)	✓	X	X	CES	✓
Del Boca et al. (2014)	✓	X	X	Log-Linear	X
Caucutt and Lochner (2012)	✓	X	X	CES	✓[j]
Bernal (2008)	✓	X	X	Linear	X
Gayle et al. (2015)	X[g]	X[g]	X	N/S	X
Bernal and Keane (2011)	✓	X	X	Linear	X
Attanasio et al. (2015)	✓	✓	X	CES	X
Meghir et al. (2015)	✓	X	✓	CES	X

Table 1 (continued): Skill Production Functions

	Self Productivity[a]		
	Cognitive	**Non-cognitive**	**Health**
Todd and Wolpin (2003)	✓- N/A	X	X
Bernal and Keane (2010)	✓- N/A	X	X
Cunha and Heckman (2008)	0.977	0.884	X
Cunha et al. (2010)	0.487/0.902[c]	0.649/0.868[c]	X
Todd and Wolpin (2007)	0.21 - 0.34[d]	X	X
Cunha (2007)	0.735/0.799 /0.872[f]	X	X
Del Boca et al. (2014)	(0.14, 0.503)/(0.172, 0.922)[f]	X	X
Caucutt and Lochner (2012)	✓- N/A	X	X
Bernal (2008)	✓- N/A	N/A	X
Gayle et al. (2015)	N/S	N/S	X
Bernal and Keane (2011)	✓- N/A	X	X
Attanasio et al. (2015)	0.591/0.566	0.435/0.413	X
Meghir et al. (2015)	0.283/0.697/0.605 age8/age12/age15	X	0.671/0.763/0.861/0.785 age5/age8/age12/age15

Table 1 (continued): Skill Production Functions

	Cross Productivity[a]			Increasing Investments and Skill
	Cognitive	**Non-cognitive**	**Health**	**Complementarity over Time[h]**
Todd and Wolpin (2003)	X	X	X	U
Bernal and Keane (2010)	X	X	X	U
Cunha and Heckman (2008)	0.003	0.028	X	U
Cunha et al. (2010)	0.000/0.008[c]	0.083/0.011[c]	X	✓
Todd and Wolpin (2007)	X	X	X	U
Cunha (2007)	X	X	X	✓
Del Boca et al. (2014)	X	X	X	N/A
Caucutt and Lochner (2012)	X	X	X	N/A
Bernal (2008)	X	X	X	U
Gayle et al. (2015)	X	X	X	N/S
Bernal and Keane (2011)	X	X	X	U
Attanasio et al. (2015)	0.03/0.038 (not signif.)	0.11/0.122	X	✓
Meghir et al. (2015)	0.158/0.223/-0.054/0.018 age5/age8/age12/age15 (ages 12 and 15 not signif.)	X	-0.045/-0.054/0.061 age 8/age12/age15 (ages 8 and 12 not signif.)	✓

Appendices

Bayesian Exploratory Factor Analysis
by Frühwirth-Schnatter et al. (*JOE*, 2014)

Recent revival: New challenges raised by the empirical research

- Profusion of high-dimensional data: How to aggregate them and extract meaningful information?
- How many latent factors should be retained?
- How to specify the structural relationships between observed measurements and latent variables?

This paper

Pick dimensions of space and factor loadings simultaneously

Dedicated Factor Model

Factor model where each measurement loads on a single latent factor.

- Facilitates **interpretation**: Each latent factor is related to a well-defined subset of measurements.
  $\Longrightarrow$ cf. Thurstone's (1947) simple structure.

- Achieves **identification**: Rules out most rotation problems that are usually solved by imposing restrictions on the factor loading matrix.

Technical contributions

1. **Factor search procedure: Sampling to produce identified models.**
2. Sampling the latent factors and their correlation matrix in a dimension-varying model.
3. Block sampling of the correlation matrix jointly with the factors.

Model Specification

- Set of Q measurements $M = (M_1, \ldots, M_Q)'$ related to covariates X and to K latent factors $\theta = (\theta_1, \ldots, \theta_K)'$:

$$M = \beta X + \alpha\theta + \varepsilon$$

with $K << Q$.

- Independence assumptions:

$$\theta \perp\!\!\!\perp X \perp\!\!\!\perp \varepsilon \qquad\qquad \varepsilon_i \perp\!\!\!\perp \varepsilon_j \quad \forall i \neq j$$

- Distributional assumptions:

$$\theta \sim \mathcal{N}(0, \Omega)$$
$$\varepsilon \sim \mathcal{N}(0, \Sigma) \qquad\qquad \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_Q^2)$$

Identification

- Covariance structure:

$$\text{cov}(M \mid X) = \alpha\Omega\alpha' + \Sigma$$

- **Rotation problems!** Restrictions are required to achieve identification.

Identification: Dedicated Structure

- Each measurement is allowed to load on **at most** one latent factor.
- The **dedicated structure** achieves identification.
- **At least three dedicated measurements are required for each factor** (Anderson and Rubin, 1956)
  $\Longrightarrow$ Maximum number of factors is $Q/3$.
- Ω is restricted to be a **correlation matrix** R to set the scale of the factors: $\text{diag}(R) = (1, \ldots, 1)'$.

Identification: Dedicated Structure

Indicator matrix for dedicated structure: Δ, where each row indicates to which latent factor k measurement M_q is dedicated to:

$$\Delta_{M_q} = (0 \ldots 0 \quad \underbrace{1}_{\text{k}^{th}\text{ element}} \quad 0 \ldots 0)$$

An Example with $Q = 12$ and $K = 4$

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This model is identified:

- At least three measurements loading on each factor.
- Number of factors = Number of nonzero columns (3 here).
- Up to column-switching and sign-switching.
- Measurement M_8 is discarded.

Existing methods for dimension-varying models

- Reversible Jump MCMC (Green, 1995)
- Split and Merge moves (Richardson & Green, 1997)
- Birth and Death moves (Stephens, 2000)
- Evolutionary search algorithm (Carvalho et al., 2008)
- Nonparametric Factor Analysis (Griffiths & Ghahramani, 2005; Paisley and Carin, 2009)
- Sparse Bayesian Infinite Factor Models (Bhattacharya & Dunson, 2011)

$\Longrightarrow$ **How to design a sampling scheme that secures the identification of the dedicated factor model and works in practice?**

Prior Specification

- Indicators: $\Pr(\Delta_{M_q} = e_k) = \tau_k$, for $k = 0, \ldots, K$

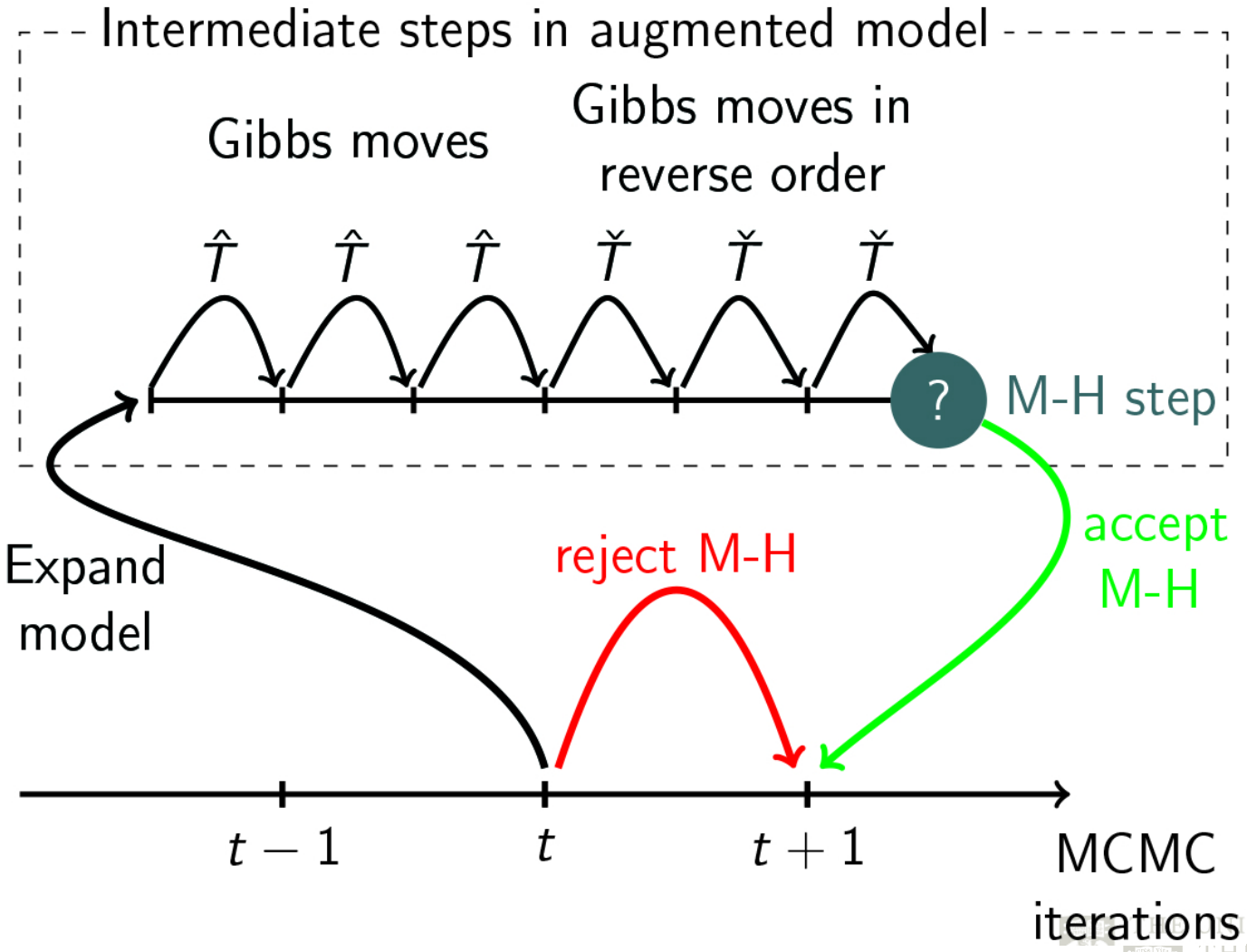
$$\begin{aligned}
\tau_{0k} &\sim \mathcal{B}eta(\kappa_0, \xi_0) \\
\tau_k &= (1 - \tau_{0k})\tau_k^\star \\
\tau^\star &= (\tau_1^\star, \ldots, \tau_K^\star) \sim \mathcal{D}ir(\kappa_1, \ldots, \kappa_K)
\end{aligned}$$

- Idiosyncratic variances: $\sigma_q^2 \sim \mathcal{IG}(c_{0q}, C_{0q})$
- Factor loadings: $\alpha_q \mid \sigma_q^2, \Delta_{M_q} \neq e_0 \sim \mathcal{N}(0, A_0\sigma_q^2)$
- Regression coefficients: $\beta_q \sim \mathcal{N}(0, B_0)$
- Factor correlation matrix: Inverse-Wishart in augmented model (MDA).

Intuition of the new sampler

- Based on the **Metropolis-Hastings algorithm** (Hastings, 1970).
- Use **intermediate Gibbs sweeps** to generate a proposal.
- Relax identifying restrictions temporarily to allow bigger moves, but accept **identified models** only.
- Approach inspired by the tempered transitions (Neal, 1996).

Factor Selection: Sampling Identified Models



Advantages of the approach

- Symmetry of the intermediate Gibbs sweeps guarantees the **detailed balance condition** of the Markov chain is verified and greatly simplifies computations.
- **Easy to implement:** Only unrestricted Gibbs sweeps are required to construct the algorithm.
- **Good performance** in practice in recovering the 'true' dedicated structure.

Empirical Application

- High-dimensional data on cognitive abilities, personality traits and physical health measurements.
- Application of BEFA: Exploratory analysis to unravel a small set of meaningful factors.
- Comparison to results obtained form classical approaches.

Data: The British Cohort Study

- Survey of all babies born (alive or dead) in one particular week of April 1970 in England, Scotland, Wales and Northern Ireland.
- Information drawn from different sweeps:

birth: background characteristics
age 10: cognitive, mental and physical health measurements

- Sample size: 2,080 males.

Data: The BCS — 131 items

Cognitive tests (cont.)

- Picture Language Comprehension Test
- Friendly Math Test
- Edinburgh Reading Test

British Ability Scales (BAS):

- BAS - Matrices
- BAS - Recall Digits
- BAS - Similarities
- BAS - Word Definition

Personality trait scales

- Rutter's Behavioral Disorder Scale (19 continuous items)
- Conners Hyperactivity Scale (19 continuous items)
- Child Developmental Scale (53 continuous items)
- Locus of Control Scale (16 binary items)
- Self-Esteem Scale (12 binary items)

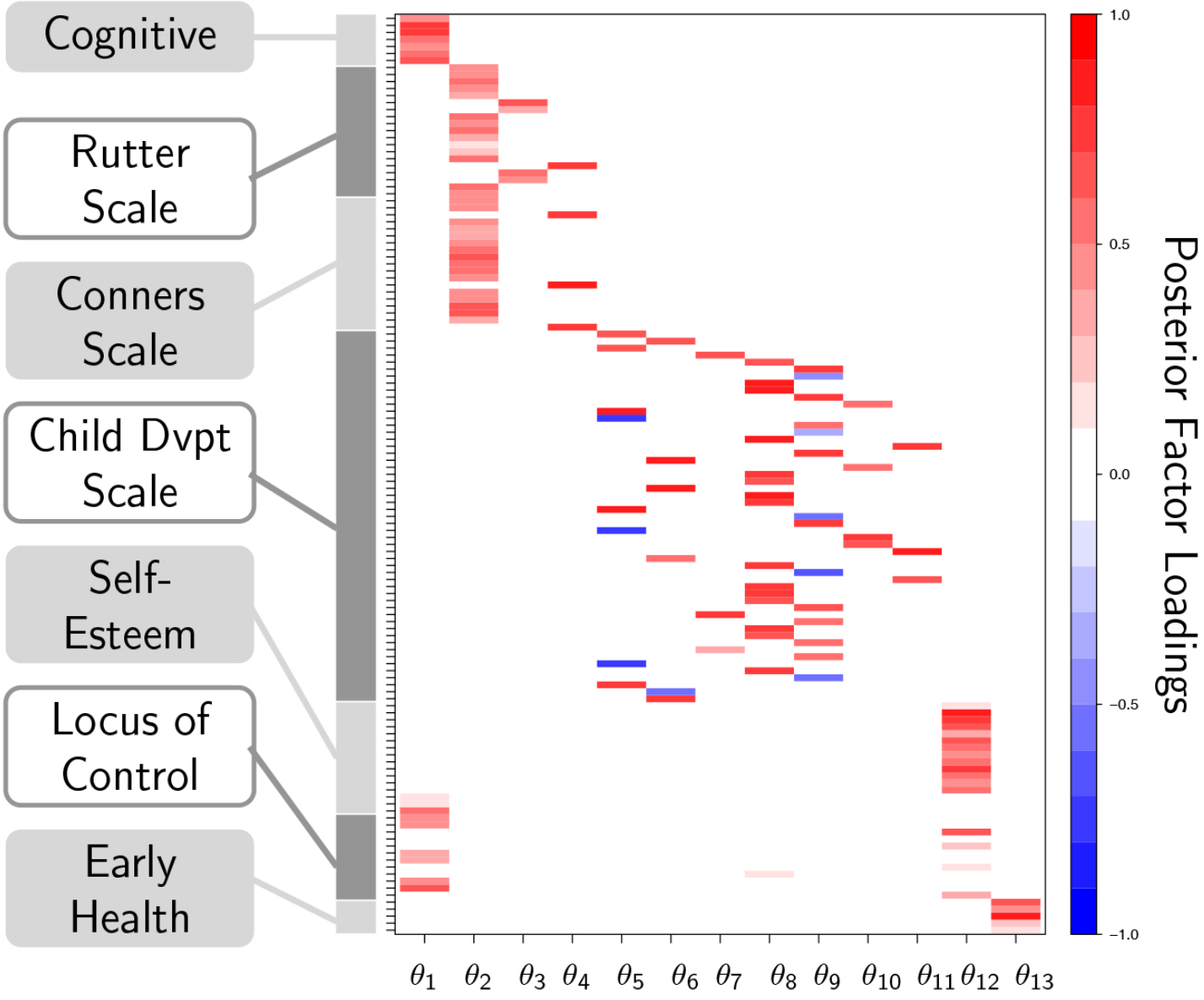
Physical health measures (continuous)

height, head circumference, weight, diastolic and systolic blood pressure

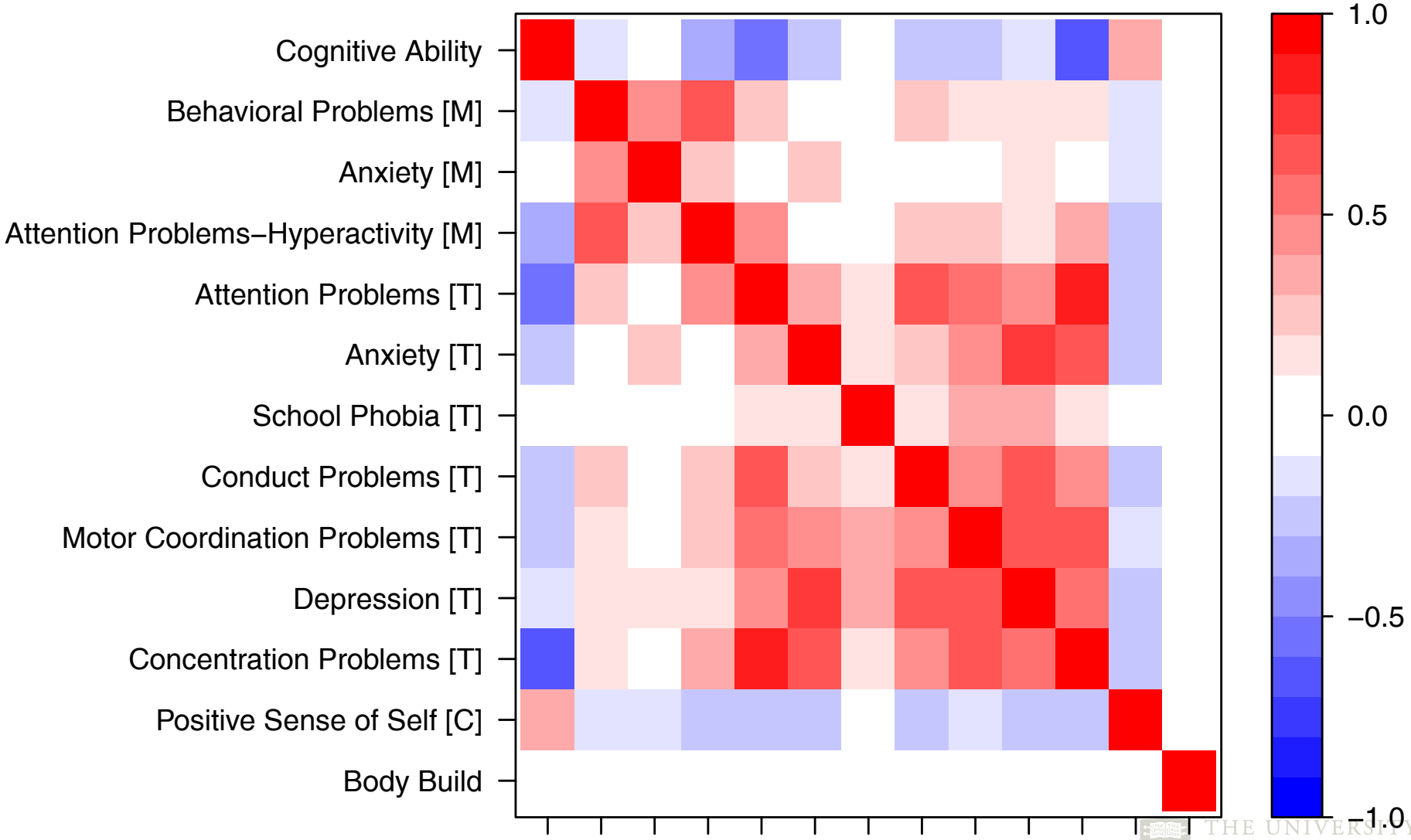
Table 2: Classical Methods to Select the Number of Factors

Method	*# of factors*
Cattell's Scree Plot	5–6
Onatski Test	6
Optimal Coordinates	11
Velicer's Rule	11
Kaiser's Rule	25
Horn's Parallel Analysis	(a) 15 or (b) 36
Bayesian Information Criterion	17
Akaike Information Criterion	47

(a) applies the criterion $\lambda > \bar{\lambda}$; (b) applies the criterion $\lambda > 0$, where λ denotes the eigenvalues from the reduced correlation matrix.


Cognitive
Rutter Scale
Conners Scale
Child Dvpt Scale
Self-Esteem
Locus of Control
Early Health
θ_1 θ_2 θ_3 θ_4 θ_5 θ_6 θ_7 θ_8 θ_9 θ_{10} θ_{11} θ_{12} θ_{13}
1.0
0.5
0.0
−0.5
−1.0
Posterior Factor Loadings

Posterior Correlation Matrix of the Factors



Summary

- New approach for dedicated factor models with unknown dimension and unknown structure *a priori*.
- Simple to implement and good results in practice, cf. Monte Carlo simulations in paper.
- BEFA uncovers the rich structure of the BCS data: 13 factors are identified, their correlation matrix is estimated.

Return to text from
"Bayesian Exploratory Factor Analysis"

"Notes on: Formulating, Identifying and Estimating the Technology of Cognitive and Noncognitive Skill Formation"

by Flávio Cunha and James J. Heckman (2008)

Journal of Human Resources

Estimating the Technology of Production of Cognitive and Noncognitive Skills

(1) Analyze the evolution of both cognitive and noncognitive outcomes using the equation system

$$\begin{bmatrix} \theta_{t+1}^N \\ \theta_{t+1}^C \end{bmatrix} = \begin{bmatrix} \gamma_1^N & \gamma_2^N \\ \gamma_1^C & \gamma_2^C \end{bmatrix} \begin{bmatrix} \theta_t^N \\ \theta_t^C \end{bmatrix} + \begin{bmatrix} \gamma_3^N \\ \gamma_3^C \end{bmatrix} \theta_t^I + \begin{bmatrix} \eta_t^N \\ \eta_t^C \end{bmatrix}, \quad (6)$$

where θ_t^I can be a vector.

- In more compact notation, $\theta_{t+1} = A\theta_t + B\theta_t^I + \eta_t$, where $A = \begin{bmatrix} \gamma_1^N & \gamma_2^N \\ \gamma_1^C & \gamma_2^C \end{bmatrix}$ and $B = \left(\gamma_3^N, \gamma_3^N\right)'$.
- Allowing for stage-specific coefficients, this equation becomes $\theta_{t+1} = A_t\theta_t + B_t\theta_t^I + \eta_t$

(2) Define $\eta_t' = \left(\eta_t^N, \eta_t^C\right)$

(3) Determine how stocks of cognitive and noncognitive skills at date t affect the stocks at date $t+1$, examining both self productivity (the effects of θ_t^N on θ_{t+1}^N, and θ_t^C on θ_{t+1}^C) and cross productivity (the effects of θ_t^C on θ_{t+1}^N and the effects of θ_t^N on θ_{t+1}^C) at each stage of the life cycle.

(4) Develop a dynamic factor model where we proxy $\theta_t' = \left(\theta_t^N, \theta_t^C, \theta_t^I\right)$ by vectors of measurements on skills and investments which can include test scores as well as outcome measures. In our analysis, test scores and parental inputs are indicators of the latent skills and latent investments. We account for measurement errors in output and input variables. We find substantial measurement errors in the proxies for parental investment and in the proxies for cognitive and noncognitive skills.

(5) Instead of imposing a particular index of parental input based on components of the home score, we estimate an index that best fits the data.

(5) Instead of relying solely on exclusion restrictions to generate instruments to correct for measurement error in the proxies for θ_t, and for endogeneity, we use covariance restrictions that exploit a feature of our data that there are many more measurements on θ_{t+1} and θ_t than the number of latent factors. This allows us to secure identification from cross equation restrictions using multiple indicator-multiple cause (MIMIC) (Jöreskog and Goldberger, 1975) and linear structural relationship (LISREL) (Jöreskog, Sörbom, and Magidson, 1979) models. When instruments are needed, they arise from the internal logic of the model developed in Cunha et al. (2006) and Cunha and Heckman (2007), using methods developed by Madansky (1964) and Pudney (1982).

(6) Instead of relying on test scores as measures of output and change in output due to parental investments, we anchor the scale of the test scores using adult outcome measures: log earnings and the probability of high school graduation. We thus estimate the effect of parental investments on the adult earnings of the child and on the probability of high school graduation.

A Model for the Measurements

- Assume access to measurement systems that can be represented by a dynamic factor structure:

$$Y_{j,t}^k = \mu_{j,t}^k + \alpha_{j,t}^k \theta_t^k + \varepsilon_{j,t}^k, \text{ for } j \in \left\{1, \ldots, m_t^k\right\}, k \in \{C, N, I\}$$

- m_t^k is the number of measurements on cognitive skills, noncognitive skills, and investments in period t; and where θ_t^k is a dynamic factor for component k, $k \in \{C, N, I\}$.
- $\text{Var}(\varepsilon_{j,t}^k) = \sigma_{k,j,t}^k$
- Initial conditions of the process, $\left(\theta_1^C, \theta_1^N\right)$, which correspond to endowment of abilities.

- Multiple measurements of abilities in the first period of our data, we can identify the distribution of the latent initial conditions.
- We also identify the distribution of each $\theta_t = \left(\theta_t^C, \theta_t^N, \theta_t^I\right)$, as well as the dependence across θ_t and $\theta_{t'}$, $t \neq t'$.
- The $\mu_{j,t}^k$ and the $\alpha_{j,t}^k$ can depend on regressors which we keep implicit.

- As above, let θ_t^C denote the stock of cognitive skill of the agent in period t.
- We do not observe θ_t^C directly.
- Instead, we observe a vector of measurements, such as test scores, $Y_{j,t}^C$, for $j \in \{1, 2, \ldots, m_t^C\}$.
- Assume that:

$$Y_{j,t}^C = \mu_{j,t}^C + \alpha_{j,t}^C \theta_t^C + \varepsilon_{j,t}^C, \text{ for } j \in \left\{1, 2, \ldots, m_t^C\right\} \tag{7}$$

  and set $\alpha_{1,t}^C = 1$ for all t.
- Some normalization is needed to set the scale of the factors.
- The $\mu_{j,t}^C$ may depend on regressors.

- Similar equation for noncognitive skills at age t, relating θ_t^N to proxies for it:

$$Y_{j,t}^N = \mu_{j,t}^N + \alpha_{j,t}^N \theta_t^N + \varepsilon_{j,t}^N, \text{ for } j \in \{1, \ldots, m_t^N\} \tag{8}$$

  and we normalize $\alpha_{1,t}^N = 1$ for all t.
- Measurement equations for investments, θ_t^I:

$$Y_{j,t}^I = \mu_{j,t}^I + \alpha_{j,t}^I \theta_t^I + \varepsilon_{j,t}^I, \text{ for } j \in \{1, \ldots, m_t^I\} \tag{9}$$

  and the factor loading $\alpha_{1,t}^I = 1$.
- The ε's are measurement errors that account for the fallibility of our measures of latent skills and investments.

- We analyze a linear law of motion for skills:

$$\theta_{t+1}^k = \gamma_0^k + \gamma_1^k \theta_t^N + \gamma_2^k \theta_t^C + \gamma_3^k \theta_t^I + \eta_t^k \tag{10}$$

  for $k \in \{C, N\}$ and $t \in \{1, \ldots, T\}$, where the error term η_t^k is independent across agents and over time for the same agents, but η_t^C and η_t^N are freely correlated.
- We assume that the η_t^k, $k \in \{C, N\}$, are independent of $\left(\theta_1^C, \theta_1^N\right)$.
- Can relax the independence assumption and allow for unobserved inputs (Cunha and Heckman, 2008).
- The γ_1^k, $I = 0, \ldots, 3$ may depend on regressors which we keep implicit.

- We allow the components of θ_t to be freely correlated for any t and with any vector $\theta_{t'}$, $t' \neq t$, and we can identify this dependence.
- Assume that any variables in the $\mu_{j,t}^k$ are independent of $\theta_t, \varepsilon_{j,t}^k$, and η_t^k for $k \in \{C, N, I\}$ and $t \in \{1, \ldots, T\}$.
- We now establish conditions under which the technology parameters are identified.

Semiparametric Identification

- The goal of the analysis is to recover the joint distribution of $\left\{\theta_t^C, \theta_t^N, \theta_t^I\right\}_{t=1}^T$, the distributions of $\left\{\eta_t^k\right\}_{t=1}^T$ and $\left\{\varepsilon_{j,t}^k\right\}_{j=1}^{m_t^k}$ nonparametrically, as well as the parameters $\left\{\alpha_{j,t}^k\right\}_{j=1}^{m_t^k}$, $\left\{\gamma_{j,t}^k\right\}_{j=1}^3$ for $k \in \{C, N\}$, and for $t \in \{1, \ldots, T\}$.
- Identification of the means of the measurements is straightforward under our assumptions.

1. Classical Measurement Error for the Case of Two Measurements Per Latent Factor:

$$m_t^C = m_t^N = m_t^I = 2$$

Assumption 1

$\varepsilon_{j,t}^k$ is mean zero and independent across agents and over time for $t \in \{1, \ldots, T\}$; $j \in \{1, 2\}$; and $k \in \{C, N, I\}$;

Assumption 2

$\varepsilon^k_{j,t}$ is mean zero and independent of $\left(\theta^C_\tau, \theta^N_\tau, \theta^I_\tau,\right)$ for all $t, \tau \in \{1, \ldots, T\}$; $j \in \{1, 2\}$; and $k \in \{C, N, I\}$;

Assumption 3

$\varepsilon^k_{j,t}$ is mean zero and independent from $\varepsilon^l_{j,t}$ for i, $j \in \{1, 2\}$ and $i \neq j$ for $k = l$; otherwise $\varepsilon^k_{j,t}$ is mean zero and independent from $\varepsilon^l_{j,t}$ for i, $j \in \{1, 2\}$;$k \neq l$, k, $l \in \{C, N, I\}$, and $t \in \{1, \ldots, T\}$.

Identification of the Factor Loadings

- Since we observe $\left\{ \left[Y_{j,t}^{k} \right]_{j=1}^{2} \right\}_{t=1}^{T}$ for every person, we can compute $\text{Cov}\left(Y_{1,t}^{k}, Y_{2,\tau}^{l} \right)$ from the data for all t, τ and k, l pairs, where $t, \tau \in \{1, \ldots, T\}$; $k, l \in \{C, N, I\}$.
- Consider, for example, measurements on cognitive skills.
- Recall that $\alpha_{1,t}^{C} = 1$.
- We know the left hand side of each of the following equations:

$$\text{Cov}\left(Y_{1,t}^{C}, Y_{1,t+1}^{C} \right) = \text{Cov}\left(\theta_{t}^{C}, \theta_{t+1}^{C} \right), \tag{11}$$

$$\text{Cov}\left(Y_{2,t}^{C}, Y_{1,t}^{C} \right) = \alpha_{2,t}^{C} \, \text{Cov}\left(\theta_{t}^{C}, \theta_{t+1}^{C} \right), \text{ and} \tag{12}$$

$$\text{Cov}\left(Y_{1,t}^{C}, Y_{2,t+1}^{C} \right) = \alpha_{2,t+1}^{C} \, \text{Cov}\left(\theta_{t}^{C}, \theta_{t+1}^{C} \right). \tag{13}$$

- We can identify $\alpha_{2,t}^{C}$ by taking the ratio of (12) to (11) and $\alpha_{2,t+1}^{C}$ from the ratio of (13) to (11).
- Proceeding in the same fashion, we can identify $\alpha_{j,t}^{k}$ for $t \in \{1, \ldots, T\}$ and $j \in \{1, 2\}$, up to the normalizations $\alpha_{1,t}^{k} = 1$, $k \in \{C, N, I\}$.
- We assume that $\alpha_{2,t}^{k} \neq 0$ for $k \in \{C, N, I\}$ and $t \in \{1, \ldots, T\}$.
- If $\alpha_{2,t}^{k} = 0$, we would violate the condition that states that there are exactly $m_t^k = 2$ valid measurements for the factor θ_t^k.

The Identification of the Joint Distribution of $\left\{\left(\theta_t^C, \theta_t^N, \theta_t^I\right)\right\}_{t=1}^T$

- Once the parameters $\alpha_{1,t}^k$ and $\alpha_{2,t}^k$ are identified (up to the normalization $\alpha_{1,t}^k = 1$), we can rewrite (7), (8), and (9) as

$$\frac{Y_{j,t}^k}{\alpha_{j,t}^k} = \frac{\mu_{j,t}^k}{\alpha_{j,t}^k} + \theta_t^k + \frac{\varepsilon_{j,t}^k}{\alpha_{j,t}^k},\ j \in \{1, 2\}$$

for $\alpha_{j,t}^k \neq 0$, $k \in \{C, N, I\}$; $t \in \{1, \ldots, T\}$.

- Now, define

$$Y_j = \left\{ \left(\frac{Y^C_{j,t}}{\alpha^C_{j,t}}, \frac{Y^N_{j,t}}{\alpha^N_{j,t}}, \frac{Y^I_{j,t}}{\alpha^I_{j,t}} \right) \right\}_{t=1}^{T} \quad \text{for } j = 1, 2.$$

- Similarly, define

$$\varepsilon_j = \left\{ \left(\frac{\varepsilon^C_{j,t}}{\alpha^C_{j,t}}, \frac{\varepsilon^N_{j,t}}{\alpha^N_{j,t}}, \frac{\varepsilon^I_{j,t}}{\alpha^I_{j,t}} \right) \right\}_{t=1}^{T} \quad \text{for } j = 1, 2,$$

  and

$$\mu_j = \left\{ \left(\frac{\mu^C_{j,t}}{\alpha^C_{j,t}}, \frac{\mu^N_{j,t}}{\alpha^N_{j,t}}, \frac{\mu^I_{j,t}}{\alpha^I_{j,t}} \right) \right\}_{t=1}^{T} \quad \text{for } j = 1, 2.$$

- Let θ denote the vector of all factors in all time periods:

$$\theta = \left\{ \left(\theta^C_t, \theta^N_t, \theta^I_t \right) \right\}_{t=1}^{T}.$$

- We rewrite the measurement equations as

$$Y_1 = \mu_1 + \theta + \varepsilon_1,$$
$$Y_2 = \mu_2 + \theta + \varepsilon_2.$$

- Under the assumption that measurement error is classical, we can apply Kotlarski's Theorem (Kotlarski, 1967) and identify the joint distribution of θ as well as the distributions of ε_1 and ε_2.
- Since $\alpha_{j,t}^k$ is identified, it is possible to recover the distribution of $\varepsilon_{j,t}^k$ for $j \in \{1, 2, \ldots, m_t^k\}$; $k \in \{C, N, I\}$ and $t \in \{1, 2, \ldots, T\}$.

The Identification of the Technology Parameters Assuming Independence of η.

- Assume that η_t^k is independent of $\left(\theta_t^C, \theta_t^N, \theta_t^I\right)$.
- Consider, for example, the law of motion for noncognitive skills,

$$\theta_{t+1}^N = \gamma_0^N + \gamma_1^N \theta_t^N + \gamma_2^N \theta_t^C + \gamma_3^N \theta_t^I + \eta_t^N \tag{14}$$

  for $t \in \{1, \ldots, T\}$.
- Assume that η_t^N is serially independent but possibly correlated with η_t^C.

Define

$$
\begin{aligned}
\widetilde{Y}^N_{1,t+1} &= Y^N_{1,t+1} - \mu^N_{1,t+1} \\
\widetilde{Y}^N_{1,t} &= Y^N_{1,t} - \mu^N_{1,t} \\
\widetilde{Y}^C_{1,t} &= Y^C_{1,t} - \mu^C_{1,t} \\
\widetilde{Y}^I_{1,t} &= Y^I_{1,t} - \mu^I_{1,t}
\end{aligned}
$$

- Substitute the measurement equations $\widetilde{Y}^N_{1,t+1}$, $\widetilde{Y}^N_{1,t}$, $\widetilde{Y}^C_{1,t}$, $\widetilde{Y}^I_t$ for θ^N_{t+1}, θ^N_t, θ^C_t, θ^I_t, respectively:

$$\begin{aligned}\widetilde{Y}^N_{1,t+1} = \gamma^N_0 + \gamma^N_1 \widetilde{Y}^N_{1,t} + \gamma^N_2 \widetilde{Y}^C_{1,t} + \gamma^N_3 \widetilde{Y}^I_{1,t} \\ + \left(\varepsilon^N_{1,t+1} - \gamma^N_{1,t}\varepsilon^N_{1,t} - \gamma^N_{2,t}\varepsilon^C_{1,t} - \gamma^N_{3,t}\varepsilon^I_{1,t} + \eta^N_{t+1}\right)\end{aligned} \tag{15}$$

  for $t \in \{1, \ldots, T\}$.

- If we estimate (15) by least squares, we do not obtain consistent estimators of γ^N_k for $k \in \{1, 2, 3\}$ because the regressors $\widetilde{Y}^N_{1,t}$, $\widetilde{Y}^C_{1,t}$, $\widetilde{Y}^I_{1,t}$ are correlated with the error term ω_{t+1}, where

$$\omega_{t+1} = \varepsilon^N_{1,t+1} - \gamma^N_{1,t}\varepsilon^N_{1,t} - \gamma^N_{2,t}\varepsilon^C_{1,t} - \gamma^N_{3,t}\varepsilon^I_{1,t} + \eta^N_{t+1}.$$

- Can instrument $\widetilde{Y}^N_{1,t}$, $\widetilde{Y}^C_{1,t}$, $\widetilde{Y}^I_{1,t}$ using $\widetilde{Y}^N_{2,t}$, $\widetilde{Y}^C_{2,t}$, $\widetilde{Y}^I_{2,t}$ as instruments by applying two-stage-least squares to recover the parameters γ^N_k for $k = 1, 2, 3$.
- See Madansky (1964) or Pudney (1982) for the precise conditions on the factor loadings.
- Our instruments are "internal instruments" justified by the model.
- The suggested instruments are also independent of η^N_t because of the assumed lack of serial correlation in η^N_t.

- We can repeat the argument for different time periods.
- In this way, we can identify stage-specific technologies for each stage of the child's life cycle.
- We can perform a parallel analysis for the cognitive skill equation.

2. Non-classical Measurement Error

We can replace Assumption 3 with the following assumption and still obtain full identification of the model.

Assumption 4

$\varepsilon_{1,t}^{k}$ is independent of $\varepsilon_{j,\tau}^{l}$ for $j \in \{2, \ldots, m_t^k\}$; $k, l \in \{C, N, I\}$ and $t, \tau \in \{1, 2, \ldots, T\}$, $m_t^k \geq 2$. $\varepsilon_{1,t}^{k}$ is independent of $\varepsilon_{1,\tau}^{k}$, for $t \neq \tau$. Otherwise the $\varepsilon_{j,\tau}^{l}$, for $j \in \{2, \ldots, m_t^k\}$; $k, l \in \{C, N, I\}$ and $t, \tau \in \{1, 2, \ldots, T\}$ can be arbitrarily dependent.

- Proof of identification is as follows.
- Let $Y_{j,t}^k = \alpha_{j,t}^k \theta_t^k + \varepsilon_{j,t}^k$, for $j \in \{1, \ldots, m_t^k\}$; $t \in \{1, \ldots, T\}$ and $k \in \{C, N, I\}$.
- Normalize $\alpha_{1,t}^k = 1$ for all $k \in \{C, N, I\}$ and $t \in \{1, \ldots, T\}$.
- Within a k system, for a fixed t, we can compute $\text{Cov}\left(Y_{j,t}^k, Y_{1,t}^k\right) = \alpha_{j,t}^k \text{Var}\left(\theta_t^k\right)$, $j \in \{1, \ldots, m_t^k\}$.
- For temporally adjacent systems, we can compute

$$\begin{aligned} \text{Cov}\left(Y_{1,t-1}^k, Y_{1,t}^k\right) &= \text{Cov}\left(\theta_{t-1}^k, \theta_t^k\right), \\ \text{Cov}\left(Y_{1,t-1}^k, Y_{j,t}^k\right) &= \alpha_{j,t}^k \text{Cov}\left(\theta_{t-1}^k, \theta_t^k\right), j \in \left\{2, \ldots, m_t^k\right\}. \end{aligned} \tag{16}$$

- Hence we can identify $\alpha_{j,t}^k$, $j \in \{1, \ldots, m_t^k\}$; $t \in \{1, \ldots, T\}$; and $k \in \{C, N, I\}$ and thus $\text{Var}\left(\theta_t^k\right)$, $t \in \{1, \ldots, T\}$; $k \in \{C, N, I\}$.
- With these ingredients in hand, we can identify $\text{Var}\left(\varepsilon_{j,t}^k\right)$, $t \in \{1, \ldots, T\}$, as well as

$$\text{Cov}\left(\varepsilon_{j,t}^k, \varepsilon_{j',t}^k\right) = \text{Cov}\left(Y_{j,t}^k, Y_{j',t}^k\right) - \alpha_{j,t}^k \alpha_{j',t}^k \,\text{Var}\left(\theta_t^k\right),$$

since we know every ingredient on the right hand side of the preceding equation.
- By a similar argument, we can identify

$$\text{Cov}\left(\varepsilon_{j,t}^k, \varepsilon_{j',\tau}^l\right) = \text{Cov}\left(Y_{j,t}^k, Y_{j',\tau}^k\right) - \alpha_{j,t}^k \alpha_{j',\tau}^l \,\text{Cov}\left(\theta_t^k, \theta_\tau^l\right). \quad (17)$$

- We can rewrite the measurement equations as a system:

$$\frac{Y_{j,t}^k}{\alpha_{j,t}^k} = \frac{\mu_{j,t}^k}{\alpha_{j,t}^k} + \theta_t^k + \frac{\varepsilon_{j,t}^k}{\alpha_{j,t}^k},$$

$j \in \left\{1, \ldots, m_t^k\right\}$; $t \in \{1, \ldots, T\}$; $k \in \{C, N, I\}$.

- Applying Schennach (2004), we can identify the joint distribution of $\left(\theta_1^C, \ldots, \theta_T^C, \theta_1^N, \ldots, \theta_T^N, \theta_1^I, \ldots, \theta_T^I\right)$ as well as the joint distribution of $\left\{\varepsilon_{j,t}^k\right\}$, $j \in \left\{1, \ldots, m_t^k\right\}$; $t \in \{1, \ldots, T\}$ and $k \in \{C, N, I\}$ using multivariate deconvolution.

The Identification of the Technology with Correlated Omitted Inputs

- It is unrealistic to assume that omitted inputs are serially independent.
- Fortunately, we can relax this assumption.
- Assume now that η_t^k is *not* independent of $\theta_t' = \left(\theta_t^C, \theta_t^N, \theta_t^I\right)$.
- Consider a model in which η_t^k can be decomposed into two parts:

$$\eta_t^N = \gamma_4^N \lambda + \nu_t^N \text{ and } \eta_t^C = \gamma_4^C \lambda + \nu_t^C$$

so that the equations of motion can be written as

$$\theta_{t+1}^N = \gamma_0^N + \gamma_1^N \theta_t^N + \gamma_2^N \theta_t^C + \gamma_3^N \theta_t^I + \gamma_4^N \lambda + \nu_t^N, \tag{18}$$

$$\theta_{t+1}^C = \gamma_0^C + \gamma_1^C \theta_t^N + \gamma_2^C \theta_t^C + \gamma_3^C \theta_t^I + \gamma_4^C \lambda + \nu_t^C. \tag{19}$$

- In this section, we normalize $\gamma_4^N = 1$.
- The term λ is a time-invariant input permitted to be freely correlated with θ_t.
- We allow λ to have a different impact on cognitive and non-cognitive skill accumulation.
- Let $\nu_t = \left(\nu_t^N, \nu_t^C\right)$.
- We make the following assumption.

Assumption 5

The error term ν_t is independent of θ_t, λ, ν_τ, conditional on regressors for any $\tau \neq t$.

- Under this assumption, we can identify both a stage-invariant technology and a stage-varying technology.
- We first analyze the stage-invariant case.
- Consider, for example, the law of motion for noncognitive skills.
- For any periods t, $t+1$ we can compute the difference

$$\begin{aligned}&\theta_{t+1}^N - \theta_t^N \\ &= \gamma_1^N \left(\theta_t^N - \theta_{t-1}^N\right) + \gamma_2^N \left(\theta_t^C - \theta_{t-1}^C\right) + \gamma_3^N \left(\theta_t^I - \theta_{t-1}^I\right) + \nu_t^N - \nu_{t-1}^N.\end{aligned} \tag{20}$$

- We use the measurement equations for to proxy θ_{t+1} and θ_t:

$$\begin{aligned}&\widetilde{Y}_{1,t+1}^N - \widetilde{Y}_{1,t}^N \\ &= \gamma_1^N \left(\widetilde{Y}_{1,t}^N - \widetilde{Y}_{1,t-1}^N\right) + \gamma_2^N \left(\widetilde{Y}_{1,t}^C - \widetilde{Y}_{1,t-1}^C\right) + \gamma_3^N \left(\widetilde{Y}_{1,t}^I - \widetilde{Y}_{1,t-1}^I\right) + \nu_t^N - \nu_{t-1}^N \\ &\quad + \left\{\left(\varepsilon_{1,t+1}^N - \varepsilon_{1,t}^N\right) - \gamma_1^N \left(\varepsilon_{1,t}^N - \varepsilon_{1,t-1}^N\right) - \gamma_2^N \left(\varepsilon_{1,t}^C - \varepsilon_{1,t-1}^C\right) - \gamma_3^N \left(\varepsilon_{1,t}^I - \varepsilon_{1,t-1}^I\right)\right\}.\end{aligned} \tag{21}$$

- OLS applied to (21) does not produce consistent estimates of γ_1^N, γ_2^N, γ_3^N and because the regressors $\left(\widetilde{Y}_{1,t}^k - \widetilde{Y}_{1,t-1}^k\right)$ are correlated with the error term ω, where

$$\omega = \left(\varepsilon_{1,t+1}^N - \varepsilon_{1,t}^N\right) - \gamma_1^N\left(\varepsilon_{1,t}^N - \varepsilon_{1,t-1}^N\right) - \gamma_2^N\left(\varepsilon_{1,t}^C - \varepsilon_{1,t-1}^C\right) - \gamma_3^N\left(\varepsilon_{1,t}^I - \varepsilon_{1,t-1}^I\right).$$

- However, we can instrument $\left(\widetilde{Y}_{1,t}^k - \widetilde{Y}_{1,t-1}^k\right)$ using $\left\{\left(Y_{j,t-1}^k - Y_{j,t-2}^k\right)\right\}_{j=2}^{m_t^k}$ as the instruments.
- These instruments are valid because of the generalization of investment equation (9) in Cunha and Heckman (2007) to a T period model.
- Using a two-stage least squares regression with these instruments allows us to recover the parameters γ_1^N, γ_2^N and γ_3^N.
- We can identify γ_0^N if we assume that $E(\lambda) = 0$.

- Following a parallel argument, we can identify γ_0^N, γ_1^N, γ_2^N and γ_3^N using the data on the evolution of cognitive test scores.
- Next, define

$$\psi_{t+1}^k = \theta_{t+1}^k - \left(\gamma_0^k + \gamma_1^k \theta_t^N + \gamma_2^k \theta_t^C + \gamma_3^k \theta_t^I\right).$$

- From the measurement equations, we know the joint distribution of $\left(\theta_{t+1}^k, \theta_t^N, \theta_t^C, \theta_t^I\right)$ for $k \in \{C, N\}$.
- We have established how to obtain the parameter values γ_0^N, γ_1^N, γ_2^N and γ_3^N.
- Consequently, we know the distribution of ψ_t^k for $k \in \{C, N\}$ and $t \in \{1, \ldots, T\}$.

- We have $2T$ equations:

$$\begin{aligned} \psi_T^N &= \lambda + \nu_T^N & \psi_T^C &= \gamma_4^C \lambda + \nu_T^C. \\ \psi_{T-1}^N &= \lambda + \nu_{T-1}^N & \psi_{T-1}^C &= \gamma_4^C \lambda + \nu_{T-1}^C. \\ &\vdots & &\vdots \\ \psi_1^N &= \lambda + \nu_1^N & \psi_1^C &= \gamma_4^C \lambda + \nu_1^C. \end{aligned}$$

- Under Assumption 5 we can apply Kotlarski's Theorem to this system and obtain the distribution of λ and ν_t for any t.
- Note that we can identify the parameter γ_4^C from the covariance:

$$\mathrm{Cov}\left(\psi_t^N, \psi_\tau^C\right) = \gamma_4^C \,\mathrm{Var}\left(\lambda\right)$$

  for any $t, \tau \in \{1, \ldots, T\}$ since the variance of λ is known.
- This approach solves the problem raised by correlated omitted inputs for stage-invariant technologies.

- For the stage-varying case, a similar argument applies. Recall that the first period of life is $t = 1$.
- In place of equation (20), we can write

$$\begin{aligned}
&\theta_{t+1}^N - \theta_t^N \\
&= \gamma_{0,t}^N - \gamma_{0,t-1}^N + \gamma_{1,t}^N \theta_t^N - \gamma_{1,t-1}^N \theta_{t-1}^N + \gamma_{1,t}^N \theta_t^C \\
&\quad - \gamma_{2,t-1}^N \theta_{t-1}^C + \gamma_{3,t}^N \theta_t^I - \gamma_{3,t-1}^N \theta_{t-1}^I + \nu_t^N - \nu_{t-1}^N.
\end{aligned} \tag{22}$$

- Using the measurement equations to proxy θ_{t+1} and θ_t, we obtain

$$\begin{aligned}
&\widetilde{Y}_{1,t+1}^N - \widetilde{Y}_{1,t}^N \\
&= \gamma_{0,t}^N - \gamma_{0,t-1}^N + \gamma_{1,t}^N \widetilde{Y}_{1,t}^N - \gamma_{1,t-1}^N \widetilde{Y}_{1,t-1}^N + \gamma_{2,t}^N \widetilde{Y}_{1,t}^C - \gamma_{2,t-1}^N \widetilde{Y}_{1,t-1}^C \\
&\quad + \gamma_{3,t}^N \widetilde{Y}_{1,t}^I - \gamma_{3,t-1}^N \widetilde{Y}_{1,t-1}^I + \nu_t^N - \nu_{t-1}^N \\
&\quad + \Big\{ \left(\varepsilon_{1,t}^N - \varepsilon_{1,t-1}^N\right) - \left(\gamma_{1,t}^N \varepsilon_{1,t}^N - \gamma_{1,t-1}^N \varepsilon_{1,t-1}^N\right) \\
&\quad \left(\gamma_{2,t}^N \varepsilon_{1,t}^C - \gamma_{2,t-1}^N \varepsilon_{1,t-1}^C\right) - \left(\gamma_{3,t}^N \varepsilon_{1,t}^I - \gamma_{3,t-1}^N \varepsilon_{1,t-1}^I, \right) \Big\}, t \geq 2.
\end{aligned}$$

- We can instrument $\widetilde{Y}^k_{1,t}$, $\widetilde{Y}^k_{1,t-1}$, $k \in \{C, N, I\}$, using $\left\{ Y^k_{j,t-1} \right\}_{j=2}^{m^k_t}$, $k \in \{C, N, I\}$ and $l \geq 2$, as instruments.
- The validity of the instruments is based on the generalization of investment equation (9) in Cunha and Heckman (2007), discussed in our analysis of stage-invariant technologies.
- Thus we can identify the coefficients of (22) except for the intercepts.
- We can identify relative intercepts $\left(\gamma^N_{0,t} - \gamma^N_{0,t-1}\right)$, $t \in \{2, \ldots, T\}$.
- With these intercepts in hand, we can identify the remaining parameters by the preceding proof provided we have enough proxies for each factor in each period.

Anchoring the Factors in the Metric of Earnings

- We can set the scale of the factors by estimating their effects on log earnings for children when they become adults.
- Let Y be adult earnings.
- We write

$$\ln Y = \mu_T + \delta_N \theta_T^N + \delta_C \theta_T^C + \varepsilon, \tag{23}$$

where ε is not correlated with θ_T or $\varepsilon_{j,T}^k$.

- Define

$$D = \begin{pmatrix} \delta_N & 0 \\ 0 & \delta_C \end{pmatrix}.$$

- Assume $\delta_N \neq 0$ and $\delta_C \neq 0$.
- For any given normalization of the test scores we can transform the θ_t to an earnings metric by multiplying equation system (6) by D:

$$D\ \theta_{t+1} = \left(DAD^{-1}\right)\left(D\theta_t\right) + \left(DB\right)\theta_t^I + \left(D\ \eta_t\right), \tag{24}$$

  and work with $D\theta_{t+1}$ and $D\theta_t$ in place of θ_{t+1} and θ_t.
- The cross terms in (DAD^{-1}) are affected by this change of units but not the self-productivity terms.
- The relative magnitude of θ_t^I on the outcomes can be affected by this change in scale.

- We can use other anchors besides earnings.
- Cunha and Heckman (2008) report results from two anchors in this paper:
  (a) log earnings
  (b) the probability of graduating from high school.
- For the latter, we use a linear probability model.

Table 3: Unanchored Technology Equations[a]: Measurement Error is Classical, Absence of Omitted Inputs Correlated with θ_t, White Males, CNLSY/79

Dependent Variable	Noncognitive Skill $\left(\theta_{t+1}^{N}\right)$			Cognitive Skill $\left(\theta_{t+1}^{C}\right)$		
	(1)	(2)	(3)	(4)	(5)	(6)
Lagged Noncognitive Skill, $\left(\theta_t^N\right)$	0.884	0.884	0.884	0.028	0.028	0.028
	(0.021)	(0.021)	(0.021)	(0.013)	(0.013)	(0.013)
Lagged Cognitive Skill, $\left(\theta_t^C\right)$	0.003	0.003	0.003	0.977	0.977	0.977
	(0.013)	(0.012)	(0.013)	(0.038)	(0.038)	(0.038)
Parental Investment, $\left(\theta_t^I\right)$	0.072	0.078	0.080	0.064	0.069	0.071
	(0.020)	(0.021)	(0.024)	(0.013)	(0.014)	(0.015)
Maternal Education, S	0.004	0.004	0.004	0.003	0.003	0.003
	(0.008)	(0.008)	(0.008)	(0.010)	(0.010)	(0.010)
Maternal Cognitive Skill, A	-0.006	-0.006	-0.006	0.025	0.025	0.025
	(0.006)	(0.006)	(0.006)	(0.009)	(0.009)	(0.009)

- Findings:
  (1) Both cognitive and noncognitive skills show strong persistence over time;
  (2) Noncognitive skills in one period affect the accumulation of next period cognitive skills, but cognitive skills in one period do not affect the accumulation of next period noncognitive skills;
  (3) The estimated parental investment factor affects noncognitive skills slightly more strongly than cognitive skills, but the differences are not statistically significant;
  (4) The mother's ability affects the child's cognitive ability but not noncognitive ability;
  (5) The mother's education plays no role in affecting the evolution of ability after controlling for parental investments, and mother's ability.

- We contrast the OLS estimates of this model (presented in Table 9) with our measurement-error corrected versions in subsection 6 below.

- The dynamic factors are statistically dependent.
- Table 4 shows the evolution of the correlation patterns across the dynamic factors.
- The correlation between cognitive and noncognitive skills is 0.18 at ages six and seven, and grows to around 0.28 at ages 12 and 13.
- There is a strong contemporaneous correlation among noncognitive skill and the home investment.

Table 4: Contemporaneous Correlation Matrices: Measurement Error is Classical, Absence of Omitted Inputs Correlated with θ_t, White Males, CNLSY/79

	Noncognitive	Cognitive	Investments
Period 1 – Children ages 6 and 7			
Noncognitive	1.0000	0.1892	0.3426
Cognitive	0.1892	1.0000	0.2921
Investments	0.3426	0.2921	1.0000
Period 2 - Children ages 8 and 9			
Noncognitive	1.0000	0.2334	0.4065
Cognitive	0.2334	1.0000	0.3835
Investments	0.4065	0.3835	1.0000
Period 3 - Children ages 10 and 11			
Noncognitive	1.0000	0.2643	0.4785
Cognitive	0.2643	1.0000	0.4892
Investments	0.4785	0.4892	1.0000
Period 4 - Children ages 12 and 13			
Noncognitive	1.0000	0.2845	0.5511
Cognitive	0.2845	1.0000	0.6111
Investments	0.5511	0.6111	1.0000

- The correlation starts off at 0.40 at ages six and seven and grows to 0.55 by ages 12 and 13.
- The same pattern is true for the correlation between cognitive skills and home investments.
- The correlation between these two variables goes from 0.38 at ages six and seven to 0.61 at ages 12 and 13.

Allowing for Non-Classical Measurement Error

- See Cunha and Heckman (2008).

Table 5: Unanchored Technology Equations[a]: Measurement Error is Classical, Allows for Omitted Input λ Correlated with θ_t, White Males, CNLSY/79

Dependent Variable	Noncognitive Skill $\left(\theta_{t+1}^{N}\right)$	Cognitive Skill $\left(\theta_{t+1}^{C}\right)$
Lagged Noncognitive Skill, $\left(\theta_t^N\right)$	0.8848	0.0276
	(0.021)	(0.013)
Lagged Cognitive Skill, $\left(\theta_t^C\right)$	0.0022	0.9891
	(0.013)	(0.039)
Parental Investment, $\left(\theta_t^I\right)$	0.0797	0.0844
	(0.020)	(0.017)
Omitted Correlated Inputs, λ	0.2835	1.0000
	(0.134)	(normalized)

(a) Let $\theta_t' = (\theta_t^N, \theta_t^C, \theta_t^I)$ denote the noncognitive, cognitive and investment dynamic factors, respectively. Let λ denote omitted inputs that are potentially correlated with θ_t. The technology equations are:

$$\theta_{t+1}^k = \gamma_1^k \theta_t^N + \gamma_2^k \theta_t^C + \gamma_3^k \theta_t^I + \gamma_4^k \lambda + \nu_t^k$$

In this table we show the estimated parameter values and standard errors (in parentheses) of γ_1^k, γ_2^k, γ_3^k and γ_4^k. Note that for identification purposes we normalize $\gamma_4^C = 1$. Investment is normalized on family income.

Anchoring estimates of the factor scale using adult outcomes

- Table 6 reports estimates of the time-invariant technology that use the earnings data for persons age 23-28 to anchor the output of the production function in a log dollar metric.
- We initially assume that η_t is serially uncorrelated and that measurement error is classical.
- We relax these assumptions below, when we report estimates of more general specifications.
- Our fitted earnings function is linear in age, and depends on the final level of the factors θ^C_{T+1} and θ^N_{T+1}.

Table 6: Anchored Technology Equations[a]: Anchoring on Log Earnings and Graduation from High School Measurement Error is Non-Classical, No Omitted Inputs Correlated with θ_t, White Males, CNLSY/79

Dependent Variable	Noncognitive Skill		Cognitive Skill	
	(1)	(2)	(4)	(5)
Lagged Noncognitive Skill, $\left(\theta_t^N\right)$	0.8844	0.8843	0.0100	0.0687
	(0.0210)	(0.0210)	(0.0046)	(0.0319)
Lagged Cognitive Skill, $\left(\theta_t^C\right)$	0.0084	0.0012	0.9777	0.9771
	(0.0364)	(0.0053)	(0.0380)	(0.0380)
Parental Investment, $\left(\theta_t^I\right)$	0.0101	0.0079	0.0032	0.0173
	(0.0028)	(0.0022)	(0.0007)	(0.0035)
Maternal Education, S	0.0006	0.0004	0.0002	0.0008
	(0.0011)	(0.0009)	(0.0005)	(0.0027)
Maternal Cognitive Skill, A	-0.0008	-0.0007	0.0013	0.0068
	(0.0008)	(0.0007)	(0.0005)	(0.0024)

- One problem that might arise in using log earnings as an anchor for this sample is that log earnings are observed for the children who are born to very young mothers, making it a very selected sample.
- To check the robustness of these conclusions with regard to the log earnings anchor, we also use high school graduation for a person at least 19 years-old to anchor the parameters of the technology equations.
- We model the probability of high school graduation as a linear probability equation.
- It is interesting to note that in the metric of the probability of graduating from high school, the estimated parental investment factor affects *cognitive* skills more strongly than noncognitive skills.

- This is because cognitive skills receive higher weight in the high school graduation equation than in the log earnings equation.
- The relative strength of these effects is reversed across the two metrics.
- **The choice of a metric is not innocuous**.

Table 7: Unanchored Stage Specific Technology Equations[a]: Measurement Error is Classical, No Omitted Inputs Correlated with θ_t, White Males, CNLSY/79

Dependent Variable	Noncognitive Skill $\left(\theta_{t+1}^{N}\right)$			Cognitive Skill $\left(\theta_{t+1}^{C}\right)$		
	Stage 1	Stage 2	Stage 3	Stage 1	Stage 2	Stage 3
Lagged Noncognitive Skill, $\left(\theta_t^N\right)$	0.9849	0.9383	0.7570	0.0605	0.0212	0.0014
	(0.014)	(0.015)	(0.010)	(0.012)	(0.008)	(0.008)
Lagged Cognitive Skill, $\left(\theta_t^C\right)$	0.0508	-0.0415	0.0412	0.9197	0.8845	0.9099
	(0.043)	(0.041)	(0.041)	(0.023)	(0.021)	(0.019)
Parental Investment, $\left(\theta_t^I\right)$	0.0533	0.1067	0.0457	0.1125	0.0364	0.0379
	(0.013)	(0.022)	(0.019)	(0.032)	(0.014)	(0.014)
Maternal Education, S	0.0034	-0.0028	0.0138	0.0050	0.0131	0.0021
	(0.007)	(0.007)	(0.008)	(0.010)	(0.012)	(0.014)
Maternal Cognitive Skill, A	0.0007	-0.0077	-0.0134	0.0506	0.0044	0.0194
	(0.001)	(0.001)	(0.002)	(0.013)	(0.008)	(0.007)

Table 8: Anchored Stage Specific Technology Equations[a]: Anchor: Log Earnings of the Child Between Ages 23-28 Measurement Error is Classical, No Omitted Inputs Correlated with θ_t, White Males, CNLSY/79

Dependent Variable	Noncognitive Skill $\left(\theta_{t+1}^{N}\right)$			Cognitive Skill $\left(\theta_{t+1}^{C}\right)$		
	Stage 1	Stage 2	Stage 3	Stage 1	Stage 2	Stage 3
Lagged Noncognitive Skill, $\left(\theta_t^N\right)$	0.9849	0.9383	0.7570	0.0216	0.0076	0.0005
	(0.014)	(0.015)	(0.010)	(0.004)	(0.003)	(0.003)
Lagged Cognitive Skill, $\left(\theta_t^C\right)$	0.1442	-0.1259	0.1171	0.9197	0.8845	0.9099
	(0.120)	(0.115)	(0.115)	(0.023)	(0.021)	(0.019)
Parental Investment, $\left(\theta_t^I\right)$	0.0075	0.0149	0.0064	0.0056	0.0018	0.0019
	(0.002)	(0.003)	(0.003)	(0.002)	(0.001)	(0.001)
Maternal Education, S	0.0005	-0.0004	0.0019	-0.0003	0.0007	0.0001
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
Maternal Cognitive Skill, A	0.0001	-0.0011	-0.0019	0.0025	0.0002	0.0010
	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)

Estimating the Components of the Home Investment Dynamic Factor

- See Cunha and Heckman (2008).

Table 9: OLS Estimation of the Technology Equations: Measurement Error is Classical, Absence of Omitted Inputs Correlated with θ_t, White Males, CNLSY/79, Cont'd

Dependent Variable	Antisocial Score (t + 1)	PIAT Math (t+1)
Antisocial Score, t	0.6431	0.0333
	(0.0165)	(0.0096)
PIAT Math, t	0.0933	0.5909
	(0.0317)	(0.0184)
HOME Score, t	0.0147	-0.0137
	(0.0059)	(0.0034)
Maternal Education	0.0358	0.0208
	(0.0091)	(0.0053)
Maternal ASVAB Arithmetics	-0.0254	0.0658
	(0.0190)	(0.0110)

- Generally, OLS coefficients are downward-biased, showing much smaller self productivity, cross productivity and investment productivity effects.
- The estimated effect of the HOME score on the Math score is perverse.

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Estimating the Technology of Production of Cognitive and Noncognitive Skills

(builds on Cunha, Heckman, & Schennach, 2010)

$$
\begin{aligned}
\theta_{t+1}^{C} &= f_t^{C}\left(\theta_t^{C}, \theta_t^{N}, I_t^{C}, \eta_t^{C}\right) \\
\theta_{t+1}^{N} &= f_t^{N}\left(\theta_t^{C}, \theta_t^{N}, I_t^{N}, \eta_t^{N}\right)
\end{aligned}
$$

A Model for the Measurements

$$Y^C_{j,t}, \quad Y^N_{j,t}$$

$$Y_{j,t}^k = \mu_{j,t}^k + \alpha_{j,t}^k \theta_t^k + \varepsilon_{j,t}^k \text{ for } j = 1, \ldots, m_t^k, t = 1, \ldots, T, \text{ and } k = C, N \tag{25}$$

- $\alpha_{1,t}^{k} = 1$ and $\mu_{1,t}^{k} = 0$ for $k = C, N$

- Parental and schooling investments I_t^k not directly observed.
- Observe a vector of measurements $X_{j,t}^k$.

$$X_{j,t}^k = \mu_{j,t}^{X^k} + \beta_{j,t}^k I_t^k + \varepsilon_{j,t}^{I^k} \text{ for } j = 1, \ldots, m_t^{I^k}, t = 1, \ldots, T, \text{ and } k = C, N \tag{26}$$

Skill Measurements

$$M_j^k = \mu_j^{M^k} + \delta_j^k \theta_M^k + \varepsilon_j^{M^k} \tag{27}$$

Identification

- Identification of the factor loadings and of the joint distributions of the latent variables

- First establish identification of the factor loadings under the assumption that $m_t^k \geq 3$, $k = C, N, I^C, I^N, M^C, M^N$.
- Without loss of generality, we focus on $\alpha_{j,t}^C$ and note that similar expressions can be derived trivially for the loadings for the remaining latent factors.

- Since we observe $\left\{ Y_{j,t}^{C} \right\}_{j=1}^{m_t^C}$ for every person, we can compute from the data $Cov\left(Y_{i,t}^{C}, Y_{j,t}^{C}\right)$ for all i, j pairs.
- Let $Var\left(\theta_t^C\right) = \sigma_{\theta_t^C}^2$ denote the variance of θ_t^C, $t = 1, 2, \ldots, T$, and note that it may change over time.
- Recall that $\alpha_{1,t}^C = 1$. Consequently, we obtain:

$$Cov\left(Y_{1,t}^{C}, Y_{2,t}^{C}\right) = \alpha_{2,t}^{C} \sigma_{\theta_t}^2 \tag{28}$$

$$Cov\left(Y_{1,t}^{C}, Y_{3,t}^{C}\right) = \alpha_{3,t}^{C} \sigma_{\theta_t}^2 \tag{29}$$

$$Cov\left(Y_{2,t}^{C}, Y_{3,t}^{C}\right) = \alpha_{2,t}^{C} \alpha_{3,t}^{C} \sigma_{\theta_t}^2. \tag{30}$$

Taking ratios:

$$\frac{Cov\left(Y_{2,t}^{C},\, Y_{3,t}^{C}\right)}{Cov\left(Y_{1,t}^{C},\, Y_{3,t}^{C}\right)} = \alpha_{2,t}^{C}$$

$$\frac{Cov\left(Y_{2,t}^{C},\, Y_{3,t}^{C}\right)}{Cov\left(Y_{1,t}^{C},\, Y_{2,t}^{C}\right)} = \alpha_{3,t}^{C}.$$

- We can identify $\alpha^C_{2,t}$ and $\alpha^C_{3,t}$ from the ratios of covariances.
- Identify $\alpha^C_{j,t}$ for $j = 2, 3, \ldots, m^C_t$, $t = 1, \ldots, T$ up to the normalization $\alpha^C_{1,t} = 1$.
- Identify $\sigma^2_{\theta^C_t}$ for all $t = 1, 2, \ldots, T$.
- Once the parameters $\alpha^C_{1,t}, \alpha^C_{2,t}, \ldots, \alpha^C_{m^C_t,t}$ are identified (up to the normalization $\alpha^C_1 = 1$), we can rewrite (25) as:

$$\frac{Y^C_{j,t}}{\alpha^C_{j,t}} = \theta^C_t + \frac{\varepsilon^C_{j,t}}{\alpha^C_{j,t}}, j = 1, 2, \ldots, m^C_t. \tag{31}$$

- Can identify distribution of θ_t^C using results in Schennach (2004a,b)

$$\theta = \left(\left\{ \theta_t^C \right\}_{t=1}^T, \left\{ \theta_t^N \right\}_{t=1}^T, \left\{ I_t^C \right\}_{t=1}^T, \left\{ I_t^N \right\}_{t=1}^T, \theta_M^C, \theta_M^N \right).$$

$$
\begin{aligned}
\mathbf{z}_1 &= \left(\left\{ Y_{1,t}^{C} / \alpha_{1,t}^{C} \right\}_{t=1}^{T}, \left\{ Y_{1,t}^{N} / \alpha_{1,t}^{N} \right\}_{t=1}^{T}, \left\{ X_{1,t}^{I^C} / \beta_{1,t}^{C} \right\}_{t=1}^{T}, \left\{ X_{1,t}^{I^N} / \beta_{1,t}^{N} \right\}_{t=1}^{T}, M_1^C / \delta_1^C, M_1^N / \delta_1^N \right)' \\
\mathbf{z}_2 &= \left(\left\{ Y_{2,t}^{C} / \alpha_{2,t}^{C} \right\}_{t=1}^{T}, \left\{ Y_{2,t}^{N} / \alpha_{2,t}^{N} \right\}_{t=1}^{T}, \left\{ X_{2,t}^{I^C} / \beta_{2,t}^{C} \right\}_{t=1}^{T}, \left\{ X_{2,t}^{I^N} / \beta_{2,t}^{N} \right\}_{t=1}^{T}, M_2^C / \delta_2^C, M_2^N / \delta_2^N \right)' .
\end{aligned}
$$

These vectors consist, respectively, of the first and the second measurements for each factor. The corresponding measurement errors are

$$\varepsilon_1 = \left(\left\{ \varepsilon_{1,t}^{C} / \alpha_{1,t}^{C} \right\}_{t=1}^{T}, \left\{ \varepsilon_{1,t}^{N} / \alpha_{1,t}^{N} \right\}_{t=1}^{T}, \left\{ \varepsilon_{1,t}^{I^C} / \beta_{1,t}^{C} \right\}_{t=1}^{T}, \left\{ \varepsilon_{1,t}^{I^N} / \beta_{1,t}^{N} \right\}_{t=1}^{T}, \varepsilon_{1}^{M^C} / \delta_{1}^{C}, \right.$$

$$\varepsilon_2 = \left(\left\{ \varepsilon_{2,t}^{C} / \alpha_{2,t}^{C} \right\}_{t=1}^{T}, \left\{ \varepsilon_{2,t}^{N} / \alpha_{2,t}^{N} \right\}_{t=1}^{T}, \left\{ \varepsilon_{2,t}^{I^C} / \beta_{2,t}^{C} \right\}_{t=1}^{T}, \left\{ \varepsilon_{2,t}^{I^N} / \beta_{2,t}^{N} \right\}_{t=1}^{T}, \varepsilon_{2}^{M^C} / \delta_{2}^{C}, \right.$$

- Distribution θ identified by the following Theorem

Theorem 1

Let $\mathbf{Z}_1$, $\mathbf{Z}_2$, θ, ε_1, ε_2 *be random variables taking values in* $\mathbb{R}^L$ *and related through*

$$\begin{aligned} \mathbf{Z}_1 &= \theta + \varepsilon_1 \\ \mathbf{Z}_2 &= \theta + \varepsilon_2. \end{aligned}$$

If, for $k, k' = 1, \ldots, L$,

$$E\left[\varepsilon_{1,k} | \theta_k, \varepsilon_{2,k}\right] = 0 \tag{32}$$

$$\varepsilon_{2,k} \perp (\theta, \varepsilon_{2,k'}) \text{ for } k \neq k', \tag{33}$$

then the density of θ *is given by:*

$$p_\theta(\theta) = (2\pi)^{-L} \int e^{-i\chi\theta} \frac{E\left[e^{i\chi\cdot\mathbf{Z}_2}\right]}{\prod_{k=1}^{L} E\left[e^{i\chi_k \mathbf{Z}_{2,k}}\right]} \prod_{k=1}^{L} \exp\left(\int_0^{\chi_k} \frac{E\left[i\mathbf{Z}_{1,k} e^{i\zeta_k \mathbf{Z}_{2,k}}\right]}{E\left[e^{i\zeta_k \mathbf{Z}_{2,k}}\right]} d\zeta_k\right) d\chi, \tag{34}$$

provided all the above expectations exist.

- Note that θ can trivially include elements that are perfectly measured.
- Identification of the value added technology function

$$\theta_{t+1}^{k} = f_t^k \left(\theta_t^C, \theta_t^N, I_t^k, \theta_m^C, \theta_m^N\right) \tag{35}$$

for $k = C, N$.

- Note that, even if θ were perfectly observed, we could not separately identify the distribution of η_t^k and the function f_t^k because, without further normalization, a change in the density of η_t^k can be undone by a change in the function f_t^k.
- One approach is to assume that (35) is additively separable in η_t^k.
- Another way to avoid this ambiguity is to normalize η_t to have a uniform density on $[0, 1]$.
- As alternatives, any of the other normalizations suggested in Matzkin (2003) could be used.
- To show that f_t^k for $k = C$ or N is nonparametrically identified, we note that, from the knowledge of $p_\theta(\theta)$, we can calculate, for any $\bar{\theta} \in \mathbb{R}$,

$$P\left[\theta_{t+1}^k \leq \bar{\theta} | \theta_t^C, \theta_t^N, I_t^k\right] \equiv G\left(\bar{\theta} | \theta_t^C, \theta_t^N, I_t^k\right).$$

- Next, we set

$$f_t^k\left(\theta_t^C, \theta_t^N, I_t^k, \eta_t\right) = G^{-1}\left(\eta_t^k | \theta_t^C, \theta_t^N, I_t^k\right)$$

  where $G^{-1}\left(\eta_t^k | \theta_t^N, \theta_t^C, I_t^k\right)$ denotes the inverse of $G\left(\bar{\theta} | \theta_t^N, \theta_t^C, I_t^k\right)$ with respect to its first argument, i.e. the value $\bar{\bar{\theta}}$ such that $\eta_t^k = G\left(\bar{\theta} | \theta_t^N, \theta_t^C, I_t^k\right)$.
- By construction, this operation will produce a function f_t^k producing outcomes θ_{t+1}^k with the appropriate distribution, thanks to the well-known property that a random variable is mapped into a uniformly distributed variable under the mapping defined by its own cdf.

The more traditional separable technology function with zero mean disturbance can also be handled by our analysis,

$$\theta_{t+1}^k = f_t^k\left(\theta_t^C, \theta_t^N, I_t^k\right) + \eta_t^k,$$

simply by defining

$$f_t^k\left(\theta_t^C, \theta_t^N, I_t^k\right) \equiv E\left[\theta_{t+1}^k | \theta_t^C, \theta_t^N, I_t^k\right],$$

The density of η_t^k conditional on all variables can also be identified by

$$p_{\eta_t^k|\theta_t^C,\theta_t^N,I_t^k}\left(\eta_t^k|\theta_t^C,\theta_t^N,I_t^k\right) = p_{\theta_{t+1}^k|\theta_t^C,\theta_t^N,I_t^k}\left(\eta_t^k + E\left[\theta_{t+1}^k|\theta_t^C,\theta_t^N,I_t^k\right]|\theta_t^C,\theta_t^N,I_t^k\right)$$

since $p_{\theta_{t+1}^k|\theta_t^C,\theta_t^N,I_t^k}$ is known from $p_\theta(\theta)$.

Anchoring

- Anchoring value added in meaningful cardinal outcomes
- It is common in the literature on child investment to measure skills in units of the test score selected to have factor loading of one.

$$P\left[D=1|\theta_T^C,\theta_T^N\right]=g\left(\theta_T^C,\theta_T^N\right),$$

- Since the dummy D is perfectly observed, the corresponding element of the two repeated measurement vectors $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are identical and equal to D.
- Theorem 1 then implies that the joint density of D, θ_t^C and θ_t^N is identified, thus making it possible to identify $P\left[D = 1 | \theta_T^C, \theta_T^N\right]$.
- $g^C\left(\theta_T^C\right)$ and $g^N\left(\theta_T^N\right)$ from the function $g\left(\theta_T^C, \theta_T^N\right)$, by integrating over one of the variables, e.g.,

$$
\begin{aligned}
g^C\left(\theta_T^C\right) &\equiv \int g\left(\theta_T^C, \theta_T^N\right) p_{\theta_T^N}\left(\theta_T^N\right) d\theta_T^N \\
g^N\left(\theta_T^N\right) &\equiv \int g\left(\theta_T^C, \theta_T^N\right) p_{\theta_T^C}\left(\theta_T^C\right) d\theta_T^C
\end{aligned}
$$

The "anchored" skills, denoted by $\tilde{\theta}_t^k$, are then defined as

$$
\tilde{\theta}_t^k = g^k\left(\theta_t^k\right)
$$

for $k = C, N$.

$$\tilde{f}_t^k\left(\tilde{\theta}_t^C, \tilde{\theta}_t^N, I_t^k, \eta_t^k\right) \equiv g^k\left(f_t^k\left(\left[g^C\right]^{-1}\left(\tilde{\theta}_t^C\right), \left[g^N\right]^{-1}\left(\tilde{\theta}_t^N\right), I_t^k, \eta_t^k\right)\right),$$

$$
\begin{aligned}
\tilde{f}_t^k\left(\tilde{\theta}_t^C, \tilde{\theta}_t^N, I_t^k, \eta_t^k\right) &= \tilde{f}_t^k\left(g^C\left(\theta^C\right), g^N\left(\theta_t^N\right), I_t^k, \eta_t^k\right) \\
&= g^k\left(f_t^k\left(\left[g^C\right]^{-1}\left(g^C\left(\theta_t^C\right)\right), \left[g^N\right]^{-1}\left(g^N\left(\theta_t^N\right)\right)\right.\right. \\
&= g^k\left(f_t^k\left(\theta_t^C, \theta_t^N, I_t^k, \eta_t^k\right)\right) \\
&= g^k\left(\theta_{t+1}^k\right) = \tilde{\theta}_{t+1}^k,
\end{aligned}
$$

- Hence, $\tilde{f}_t^k$ provides the equation of evolution for the anchored skills $\tilde{\theta}_t^k$ that is consistent with the equation of evolution f_t^k of the original skills θ_t^k.

More General Factor Measurement Models

- The identification of a general nonlinear factor model
- A factor model of the general form

$$Y_j = a_j(\theta, \varepsilon_j) \text{ for } j = 1, \ldots, m, \tag{36}$$

Theorem 2

The distribution of θ in Model (36) is identified, if the following conditions are satisfied

1. *All ε_j for $j = 1, \ldots, m$ are mutually independent and independent from θ.*
2. *$p_{Y_1|Y_3}(Y_1|Y_3)$ and $p_{Y_1|\theta}(Y_1|\theta)$ each form a complete family of distributions (indexed by Y_3 and θ, respectively).*
3. *The density $p_{Y_2|\theta}(Y_2|\theta)$ is bounded uniformly in Y_2 and θ.*
4. *Whenever $\theta \neq \tilde{\theta}$, $p_{Y_2|\theta}(Y_2|\theta)$ and $p_{Y_2|\theta}\left(Y_2|\tilde{\theta}\right)$ differ at least over a set of nonzero Lebesgue measure.*
5. *There exists a known functional M, mapping a density to a real number, that has the property that $M\left[p_{Y_1|\theta}(\cdot|\theta)\right] = \theta$.*

Proof of Theorem 2.

Since $m \geq 3$, we can use Theorem 1 in Hu and Schennach (2006) to prove that the distribution of θ is identified, after setting $x = Y_1$, $y = Y_2$, $z = Y_3$, and $x^* = \theta$. Assumption 1 herein implies that the Assumption 1 in Hu and Schennach (2006) holds. All remaining assumptions are directly assumed. Theorem 1 in Hu and Schennach (2006) then implies that the joint density of Y_1 and θ is identified, which provides the density of θ, after integration over Y_1. $\square$

- Theorem 2 *does not* state that the distributions of the errors ε_j or that the functions $a_j(\cdot,\cdot)$ are fully identified.
- In fact, it is always possible to alter the distribution of ε_j and the dependence of the function $a_j(\cdot,\cdot)$ on its second argument in ways that cancel each other out, as has often been noted in the nonseparable literature (e.g. Matzkin, 2003).

Further results on skills "anchoring"

- We assume throughout that the factor loadings α_{1t}^{k} have been estimated before hand.
- We need to find the relationship between adult log income $\ln E$ and skills at the end of childhood θ_T^C, θ_T^N.
- The end result will be some normalization defined by some functions g^C and g^N or, for short:

$$g\left(\theta_t\right) \equiv \begin{pmatrix} g^C\left(\theta_t^C\right) \\ g^N\left(\theta_t^N\right) \\ I_t \end{pmatrix} \text{ where } \theta_t \equiv \begin{pmatrix} \theta_t^C \\ \theta_t^N \\ \theta_t^I \end{pmatrix} \equiv \begin{pmatrix} \theta_t^C \\ \theta_t^N \\ I_t \end{pmatrix}.$$

- Log linear separable case

$$\ln E = \mu_T + \delta^C \theta_T^C + \delta^N \theta_T^N + \xi$$

where ξ is an error term.

- A linear errors-in-variable models can be estimated with standard 2SLS using $\frac{Y_{j,T}^C}{\alpha_{jT}^C}$ for $j = 2, \ldots, m_T^C$ as instruments for the regressor $\frac{Y_{1,T}^N}{\alpha_{1T}^N}$ and using $\frac{Y_{j,T}^N}{\alpha_{jT}^N}$ for $j = 2, \ldots, m_T^N$ as instruments for the regressor $\frac{Y_{1,T}^N}{\alpha_{1T}^N}$.
- Of course, any observation other than $j = 1$ can also be singled out as the regressor.
- One can also use the average $\frac{1}{m_T^C/2} \sum_{j=1}^{m_T^C/2} \frac{Y_{j,T}^C}{\alpha_{jT}^C}$ as an instrument for $\frac{1}{m_T^C/2} \sum_{j=m_T^C/2+1}^{m_T^C} \frac{Y_{j,T}^C}{\alpha_{jT}^C}$, etc.

- Then, we set $g^C\left(\theta_t^C\right) \equiv \exp\left(\delta^C \theta_T^C\right)$ and $g^N\left(\theta_t^N\right) = \exp\left(\delta^N \theta_T^N\right)$.
- We are using exponentials here so that, in the limiting case of Cobb-Douglas technology, we would get a product of the form $\exp\left(\delta^C \theta_T^C\right) \exp\left(\delta^N \theta_T^N\right)$, which becomes the sum $\delta^C \theta_T^C + \delta^N \theta_T^N$ after taking logs, as in the earlier log linear paper.

Nonlinear case with separable error

Assume

$$E = g\left(\theta_T^C, \theta_T^N\right) + \xi \text{ with } \mathbb{E}\left[\xi | \theta_T^C, \theta_T^N\right] = 0.$$

In this case, the function of interest could be

$$g\left(\theta_T^C, \theta_T^N\right) \equiv \mathbb{E}\left[E | \theta_T^C, \theta_T^N\right].$$

If we specify $g\left(\theta_T^C, \theta_T^N\right)$ parametrically as $g\left(\theta_T^C, \theta_T^N; \delta\right)$, this can be estimated using moment conditions of the form

$$\mathbb{E}\left[\left(E - g\left(\theta_T^C, \theta_T^N; \delta\right)\right) \frac{\partial g\left(\theta_T^C, \theta_T^N; \delta\right)}{\partial \delta}\right] \equiv \mathbb{E}\left[u\left(E, \theta_T; \delta\right)\right] = 0$$

- How is it possible to extract two anchors $g^C\left(\theta_T^C\right)$ and $g^N\left(\theta_T^N\right)$ from one function $g\left(\theta_T^C, \theta_T^N; \delta\right)$?
- The most natural choice in this case is to average, e.g.:

$$g^C\left(\theta_T^C\right) = \int g\left(\theta_T^C, \theta_T^N, \delta\right) f\left(\theta_T^N\right) d\theta_T^N$$

- Another possible choice is to take $f\left(\theta_T^N|\theta_T^C\right)$ instead of $f\left(\theta_T^N\right)$, which would improve the predictive power of $g^C\left(\theta_T^C\right)$ when used alone to predict E.
- Alternatively, we can directly fit a separable function of the form $g^C\left(\theta_T^C;\delta^C\right)+g^N\left(\theta_T^N;\delta^N\right)$ by solving

$$\mathbb{E}\left[\left(E-\left(g^C\left(\theta_T^C;\delta^C\right)+g^N\left(\theta_T^N;\delta^N\right)\right)\right)\frac{\partial\left(g^C\left(\theta_T^C;\delta^C\right)+g^N\left(\theta_T^N;\delta^N\right)\right)}{\partial\left(\delta^C,\delta^N\right)}\right]=0.$$

Nonlinear nonseparable case

Nonlinear weakly separable case

Anchoring the Factors in the Metric of Earnings

$$\ln E_T = \mu_T + \delta_N \theta_T^N + \delta_C \theta_T^C + \varepsilon_T \tag{37}$$

Define

$$D = \begin{pmatrix} \delta_N & 0 \\ 0 & \delta_C \end{pmatrix}.$$

We assume $\delta_N \neq 0$ and $\delta_C \neq 0$.

For any given normalization of the test scores we can transform the θ_t to an earnings metric by multiplying the technology equation by D:

$$D\theta_{t+1} = \left(DA_t D^{-1}\right)\left(D\theta_t\right) + \left(DB\right) I_t + \left(D\eta_t\right), \tag{38}$$

Further results on skills "anchoring"

We assume throughout that the factor loadings α_{1t}^{k} have been estimated before hand.

$$g\left(\theta_t\right) \equiv \begin{pmatrix} g^C\left(\theta_t^C\right) \\ g^N\left(\theta_t^N\right) \\ I_t \end{pmatrix} \text{ where } \theta_t \equiv \begin{pmatrix} \theta_t^C \\ \theta_t^N \\ \theta_t^I \end{pmatrix} \equiv \begin{pmatrix} \theta_t^C \\ \theta_t^N \\ I_t \end{pmatrix}.$$

Log linear separable case

Nonlinear case with separable error

Nonlinear nonseparable case

The CES Specification

Assuming that the error term η_t^C enters the specification in the same way as the covariates (i.e., it's a "random input"), we have:

$$g^C\left(\theta_{t+1}^C\right) = f^C\left(g\left(\theta_t\right), \eta_t\right) = \left(\gamma_1^C\left(g^C\left(\theta_t^C\right)\right)^\alpha + \gamma_2^C\left(g^N\left(\theta_t^N\right)\right)^\alpha + \gamma_3^C\left(I_t\right)^\alpha + \gamma_4^C\left(\eta_t^C\right)^\alpha\right)^{1/\alpha}$$

Computational details:

$$\frac{\partial f^C\left(g\left(\theta_t\right), \eta_t^C\right)}{\partial \eta_t} = \left(\gamma_1^C\left(g^C\left(\theta_t^C\right)\right)^\alpha + \gamma_2^C\left(g^N\left(\theta_t^N\right)\right)^\alpha + \gamma_3^C\left(I_t\right)^\alpha + \gamma_4^C\left(\eta_t^C\right)^\alpha\right)^{(1/\alpha)-1}\gamma_4^C\left(\eta_t^C\right)^{\alpha-1}$$

$$\eta_t^C = \frac{1}{\gamma_4^C}\left(\left(g^C\left(\theta_{t+1}^C\right)\right)^\alpha - \gamma_1^C\left(g^C\left(\theta_t^C\right)\right)^\alpha - \gamma_2^C\left(g^N\left(\theta_t^N\right)\right)^\alpha - \gamma_3^C\left(I_t\right)^\alpha\right)^{1/\alpha}$$

Return to main text

Evidence on The Predictive Power of Cognitive and Socioemotional Traits

Figure 3: Sorting into Schooling

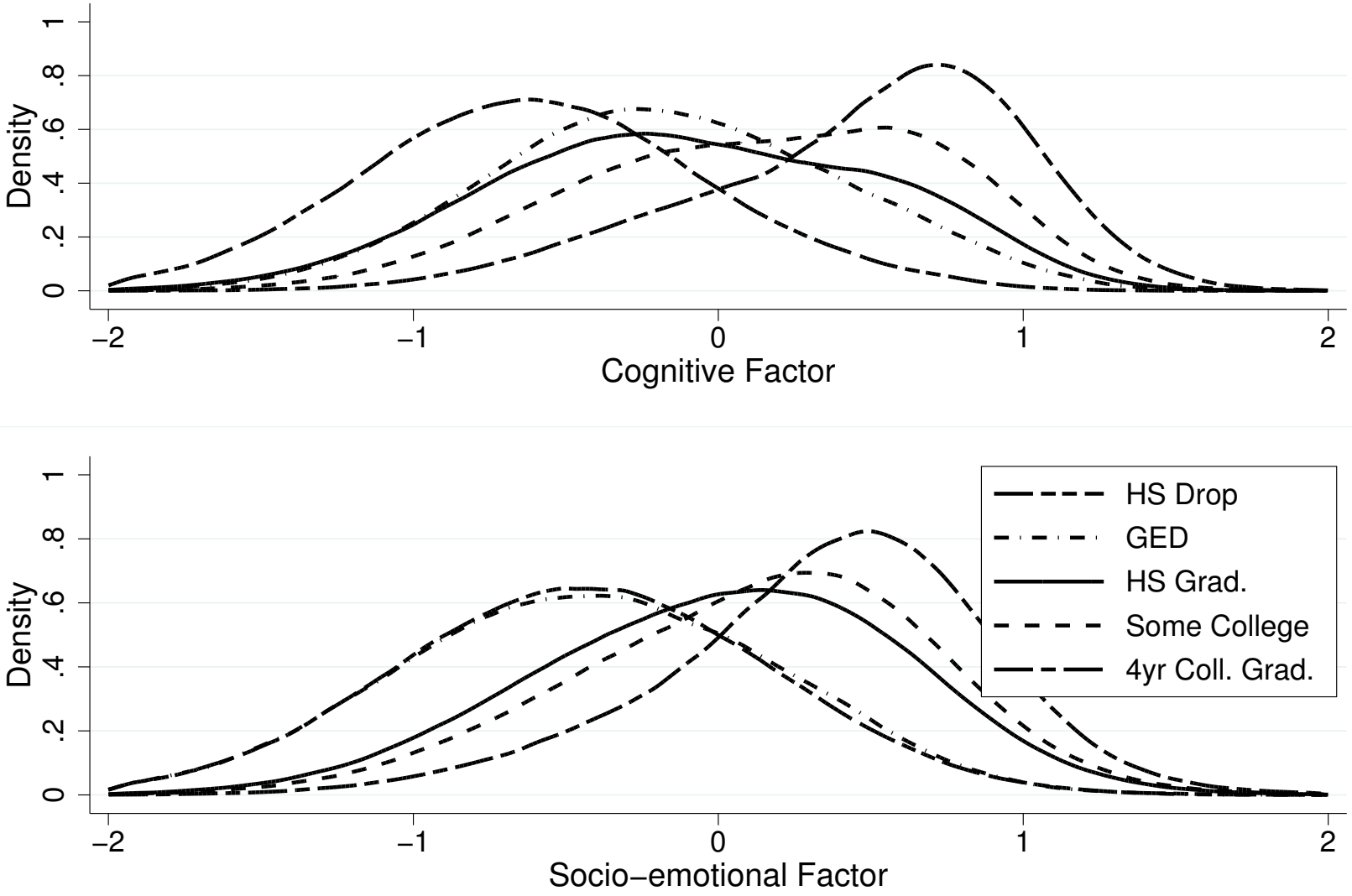
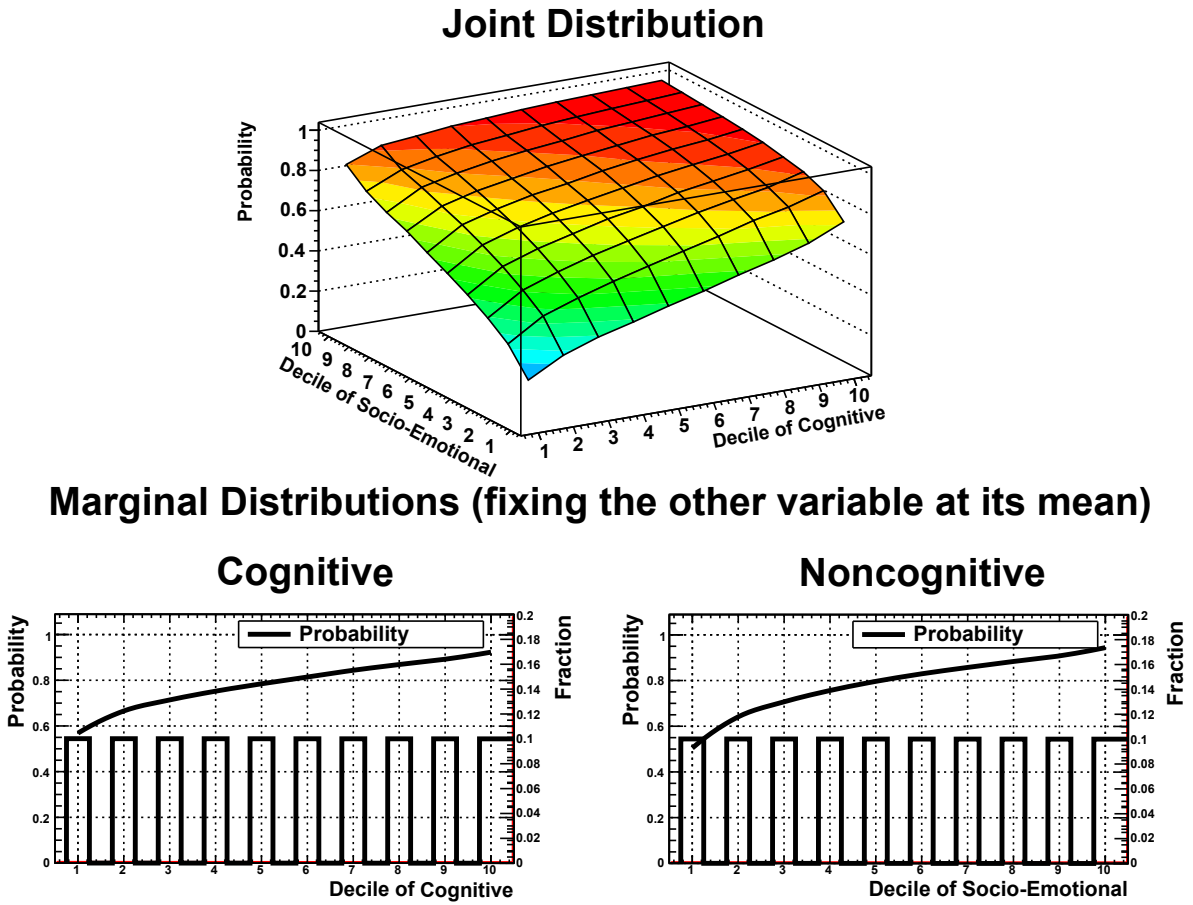
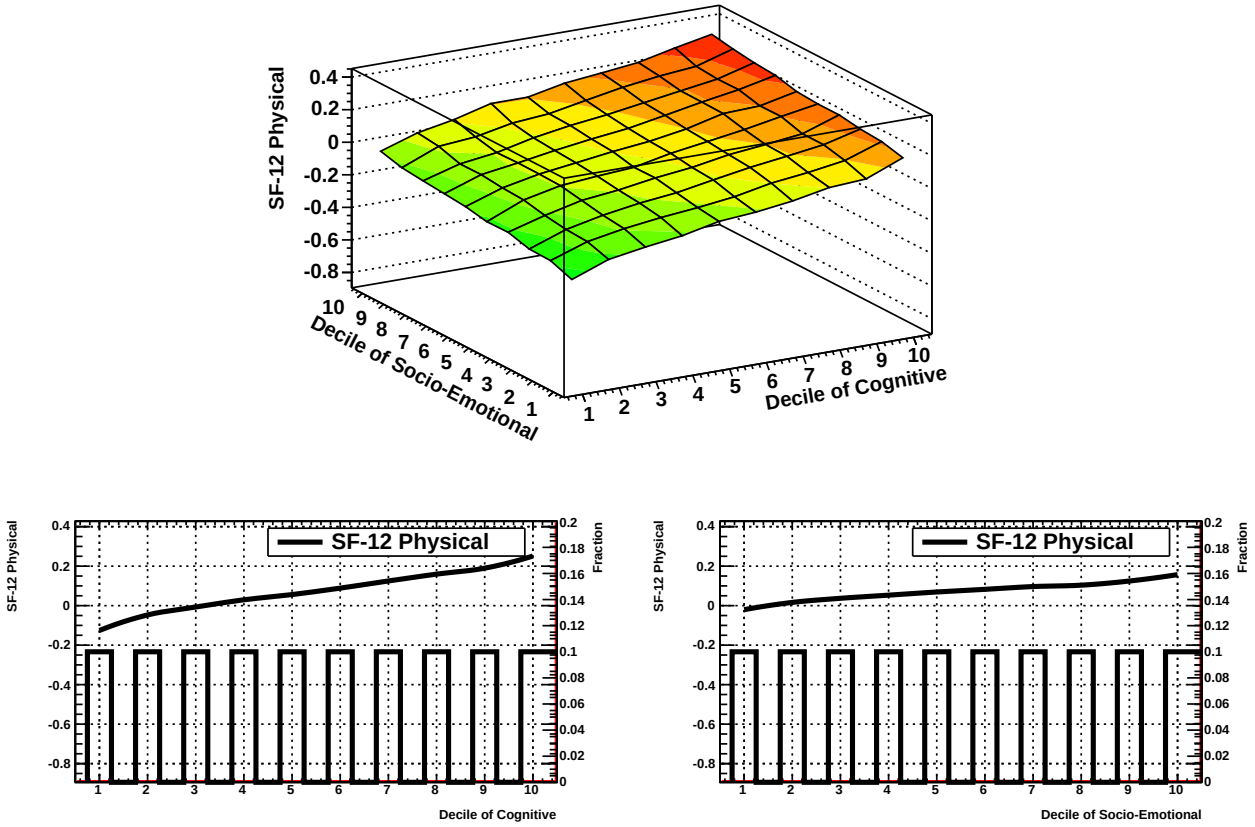


Figure 4: The Probability of Educational Decisions, by Endowment Levels, Dropping from Secondary School vs. Graduating



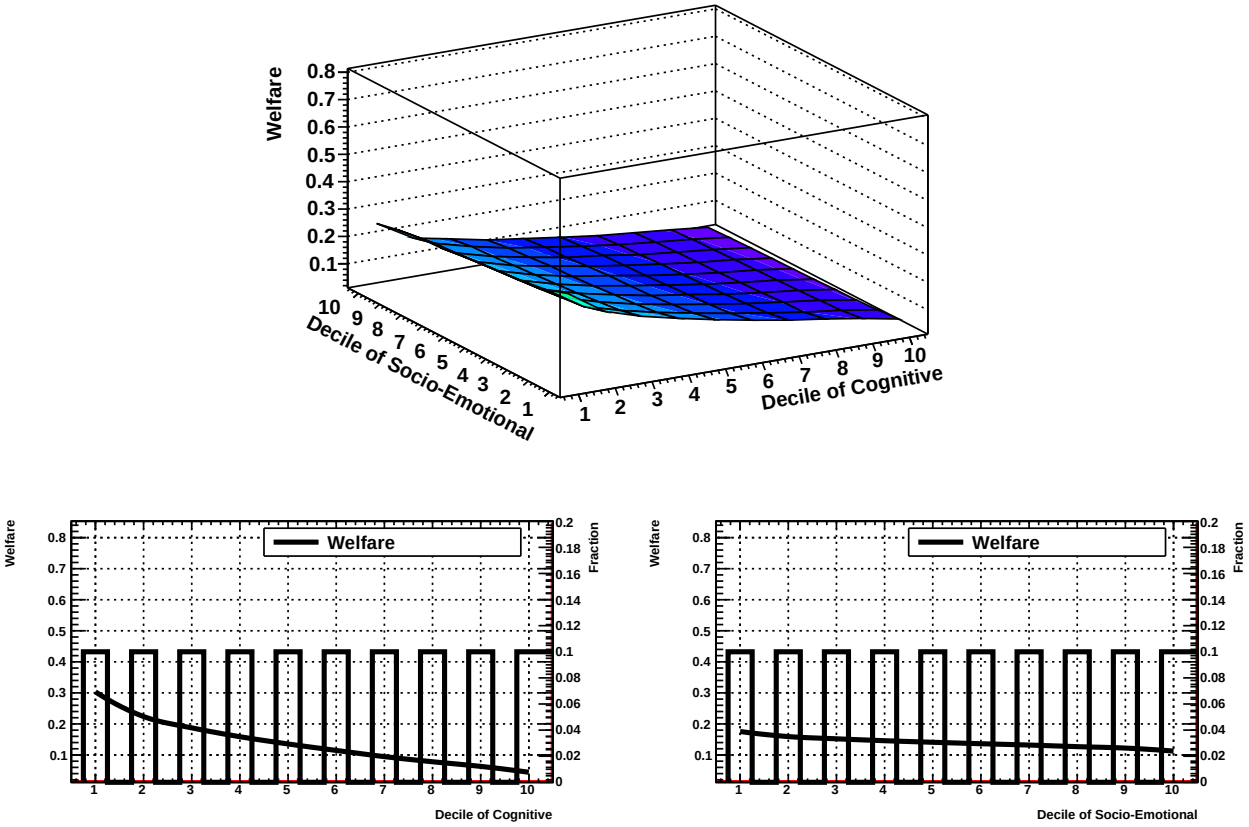
Source: Heckman, Humphries, and Veramendi (2016).

Figure 5: The Effect of Cognitive and Socio-emotional endowments on Physical Health at age 40 (PCS-12)



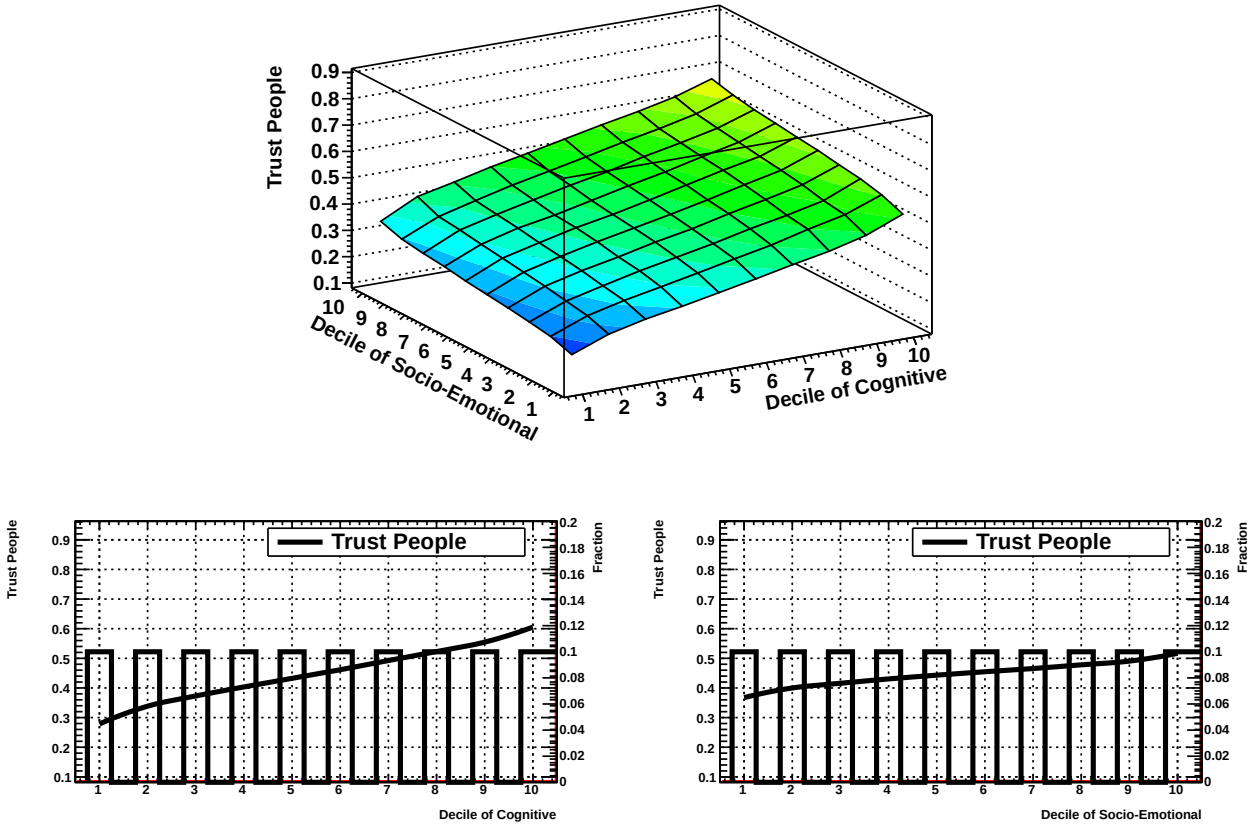
Source: Heckman, Humphries, and Veramendi (2016).

Figure 6: The Effect of Cognitive and Socio-emotional endowments on Ever Participated in Welfare (1996-2006)



Source: Heckman, Humphries, and Veramendi (2016).

Figure 7: The Effect of Cognitive and Socio-emotional endowments on Trusting People (2008)



Source: Heckman, Humphries, and Veramendi (2016).

Figure 8: The Probability and Returns of College Enrollment by Endowments Levels

Figure: Choice Probability, Early College Enrollment

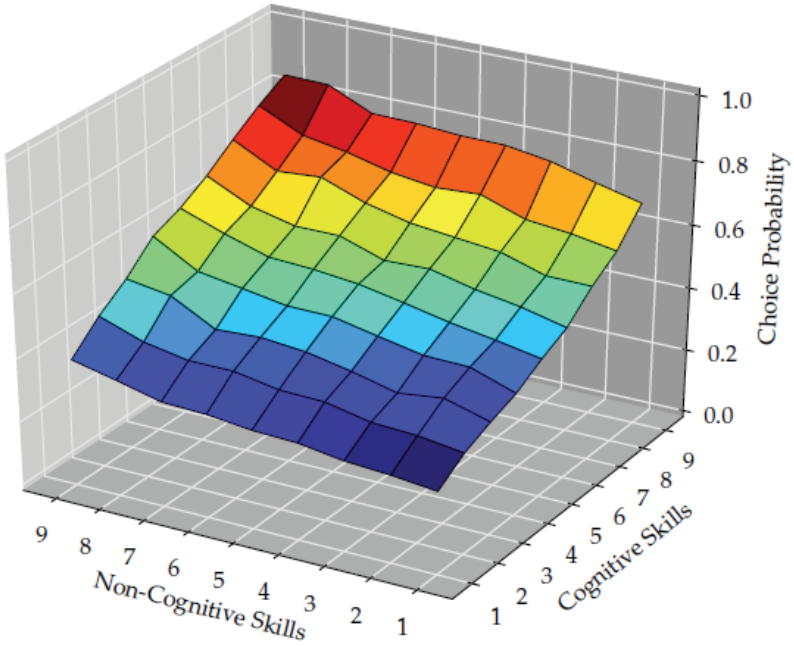
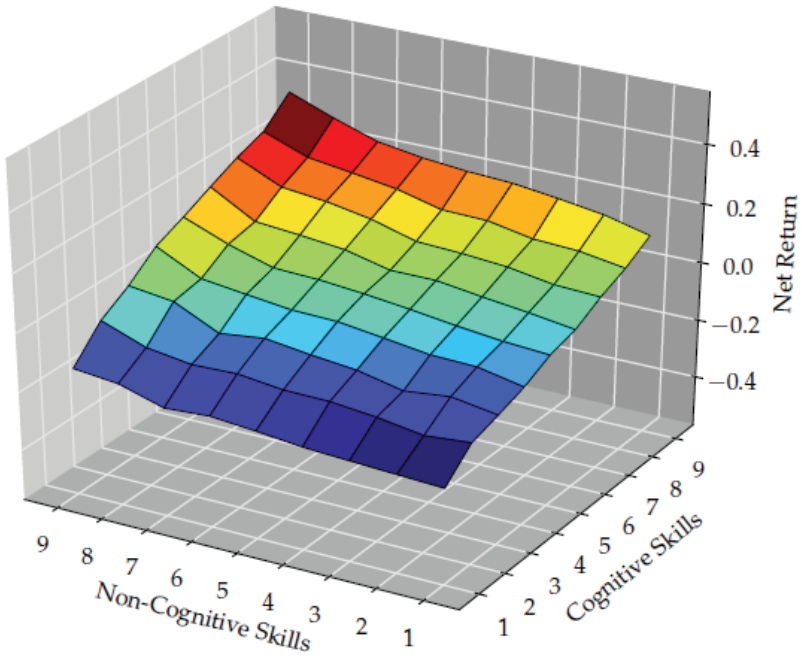
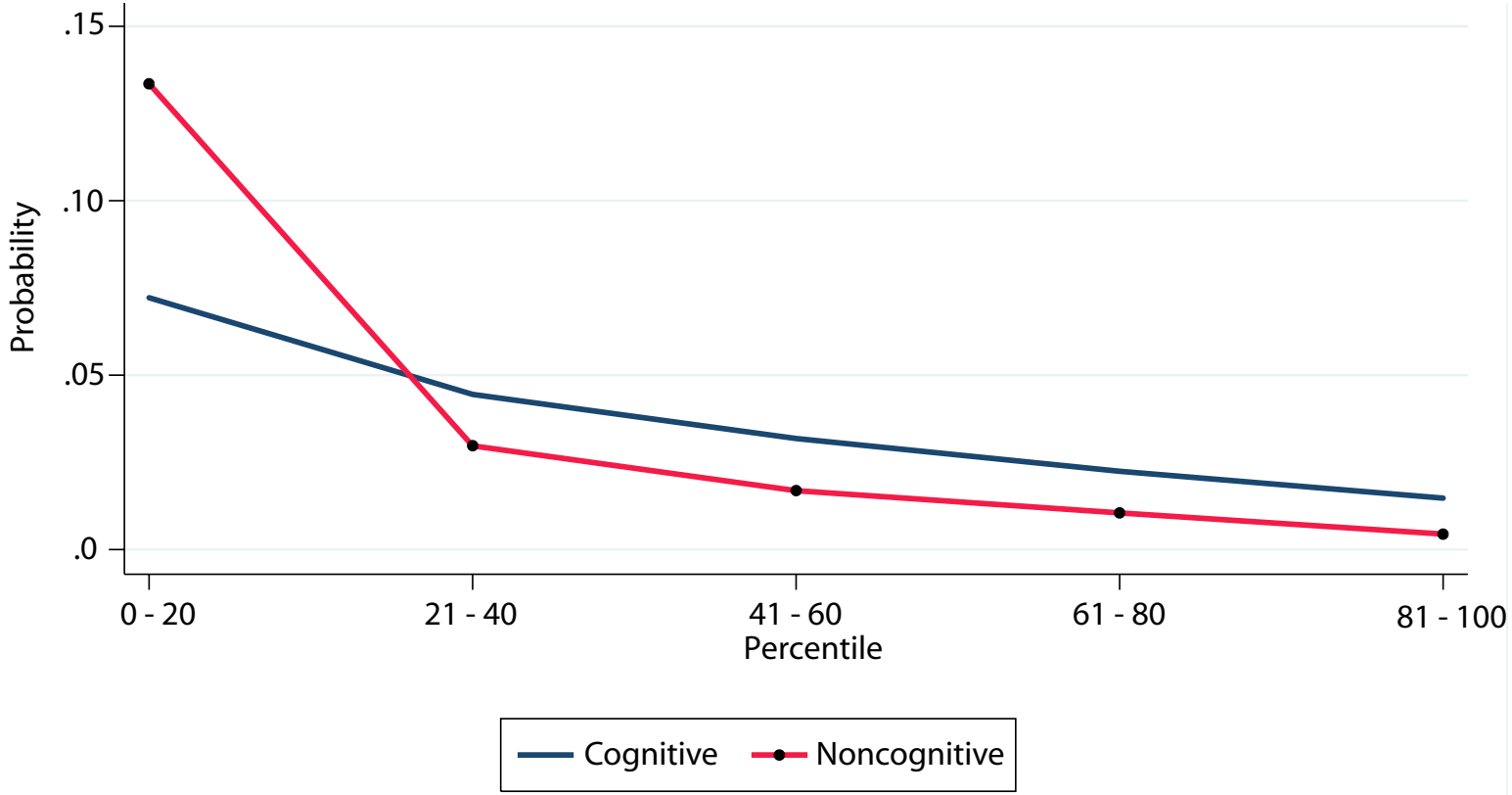


Figure: Net Return, Early College Enrollment



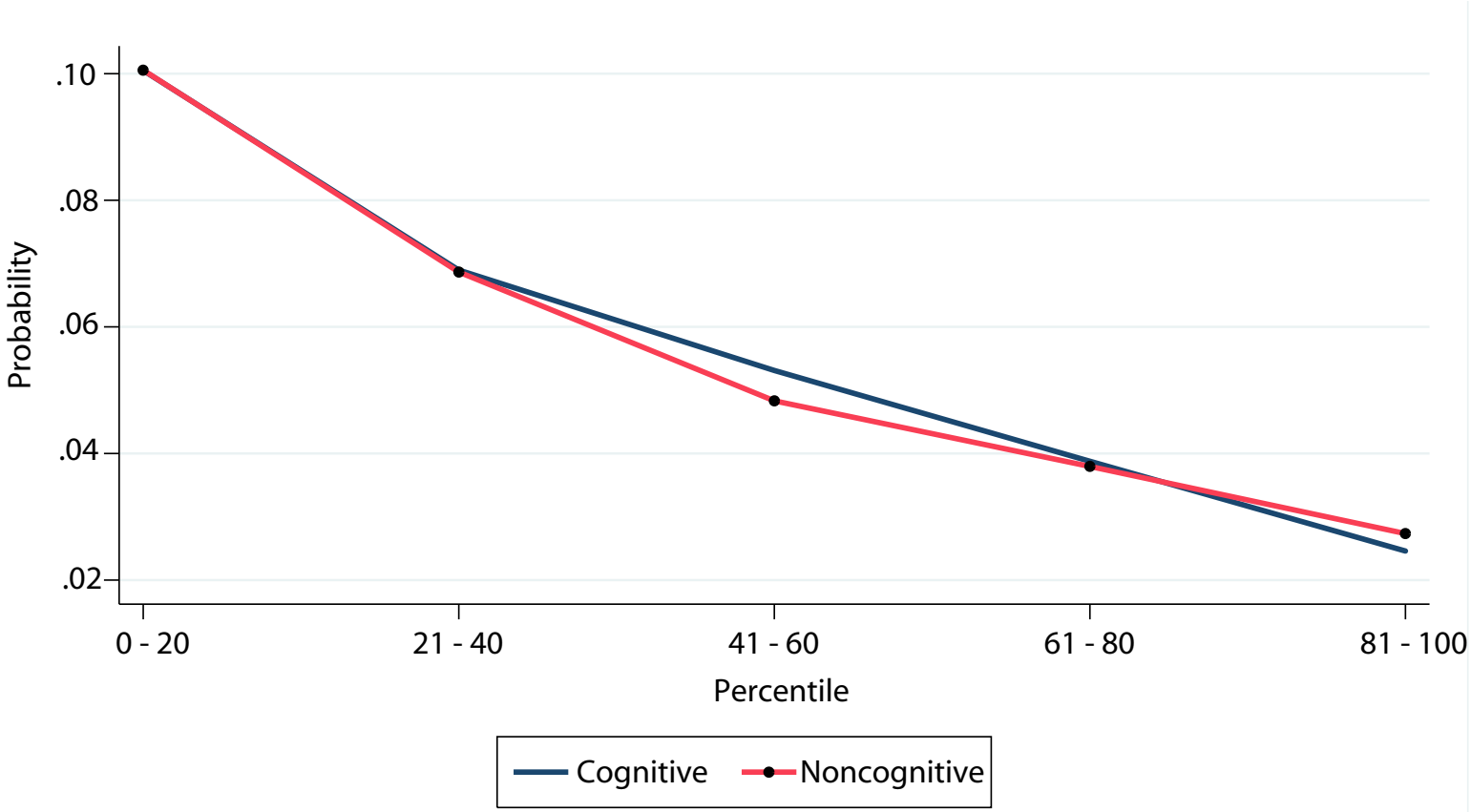
Source: Eisenhauer et al. (2014)
Note: Early college enrollment refer to the individuals who enroll in college immediately after having finished high school. Returns are expressed in units of millions of dollars.

Figure 9: Ever been in jail by age 30, by ability (males)



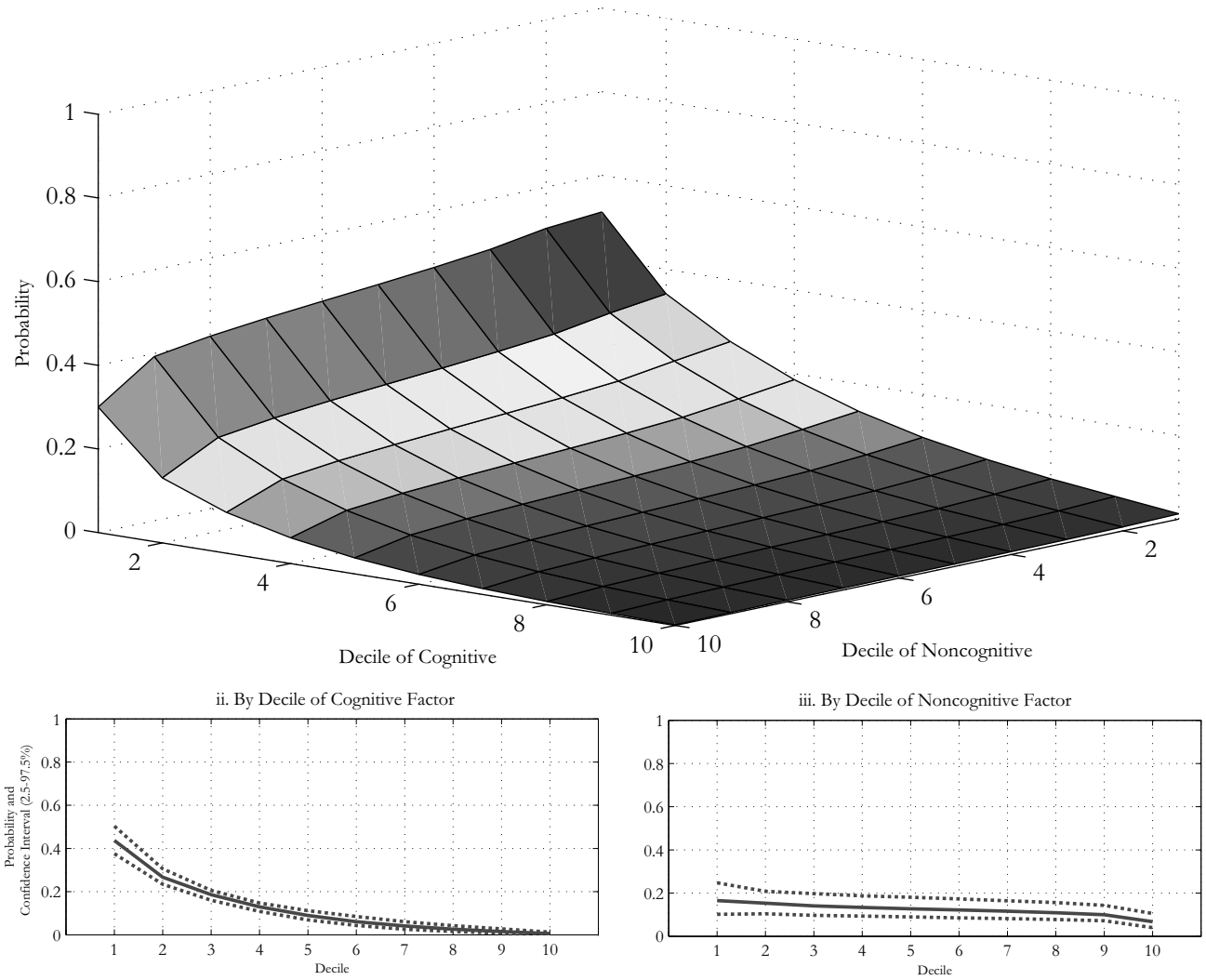
Note: This figure plots the probability of a given behavior associated with moving up in one ability distribution for someone after integrating out the other distribution. For example, the lines with markers show the effect of increasing noncognitive ability after integrating the cognitive ability.
Source: Heckman et al. (2006).

Figure 10: Probability of being single with children (females)



Note: This figure plots the probability of a given behavior associated with moving up in one ability distribution for someone after integrating out the other distribution. For example, the lines with markers show the effect of increasing noncognitive ability after integrating the cognitive ability.
Source: Heckman et al. (2006).

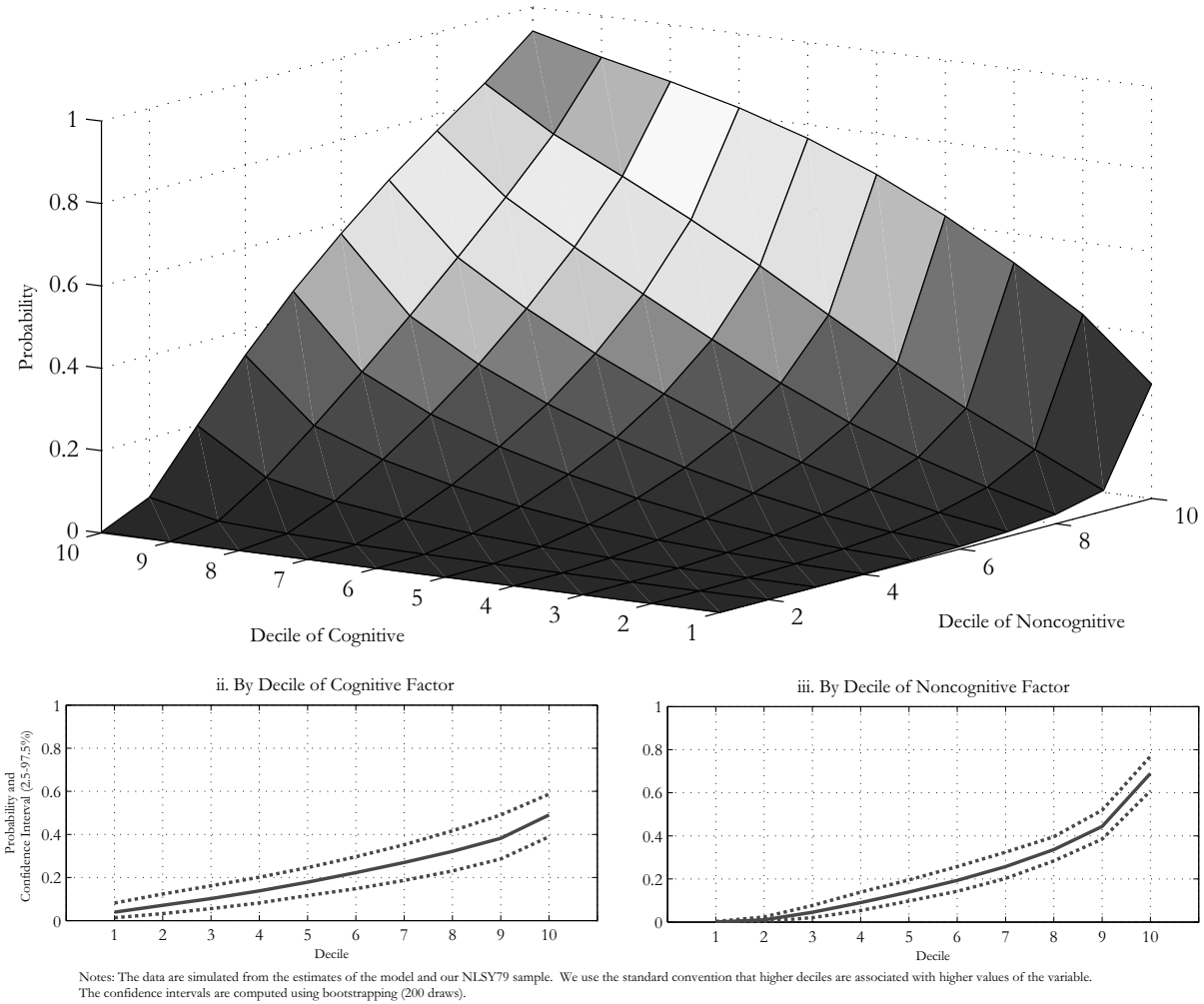
Figure 11: Probability of being a high school dropout by age 30 (males)



Notes: The data are simulated from the estimates of the model and our NLSY79 sample. We use the standard convention that higher deciles are associated with higher values of the variable. The confidence intervals are computed using bootstrapping (200 draws).

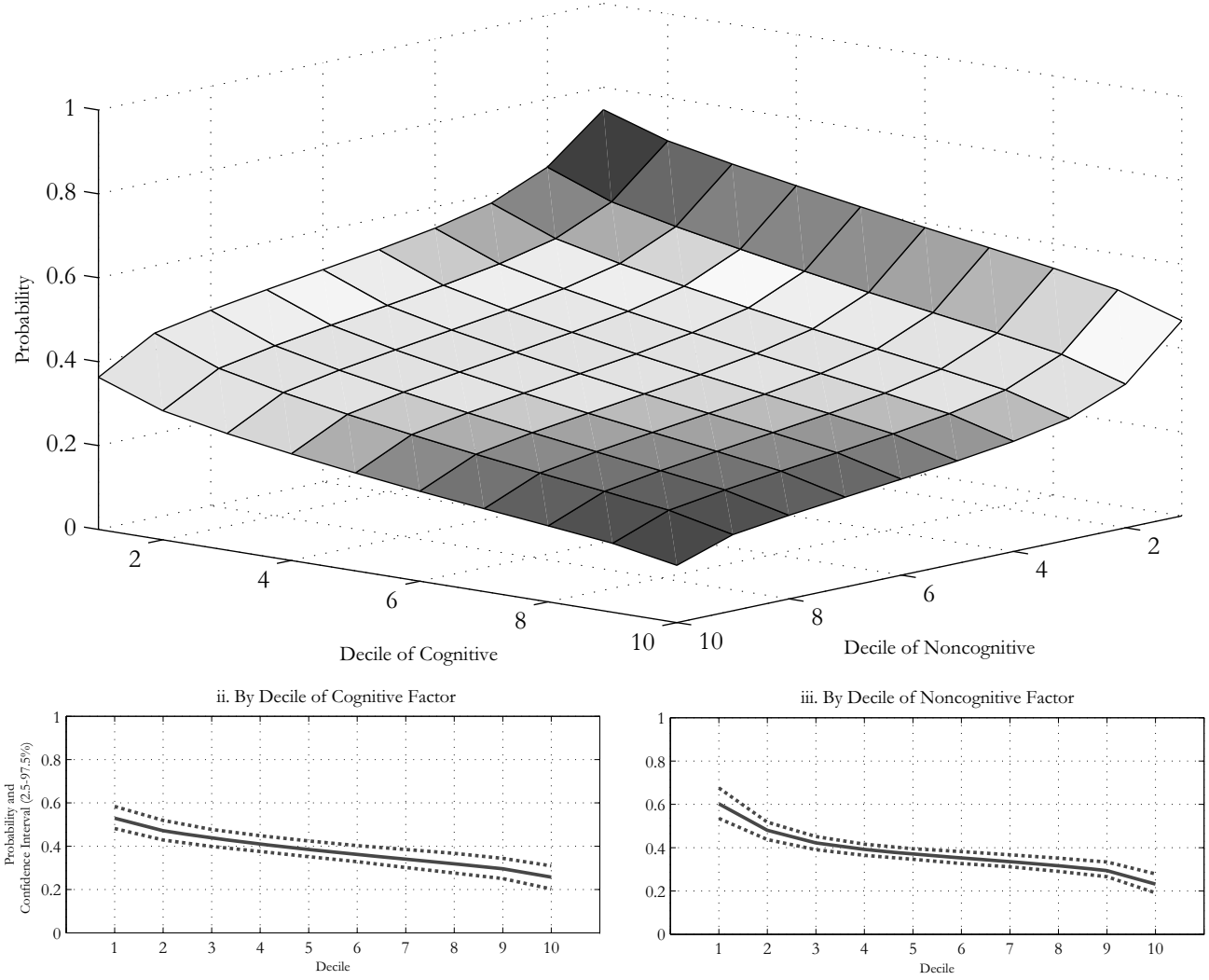
Source: Heckman et al. (2006).

Figure 12: Probability of being a 4-year college graduate by age 30 (males)



Source: Heckman et al. (2006).

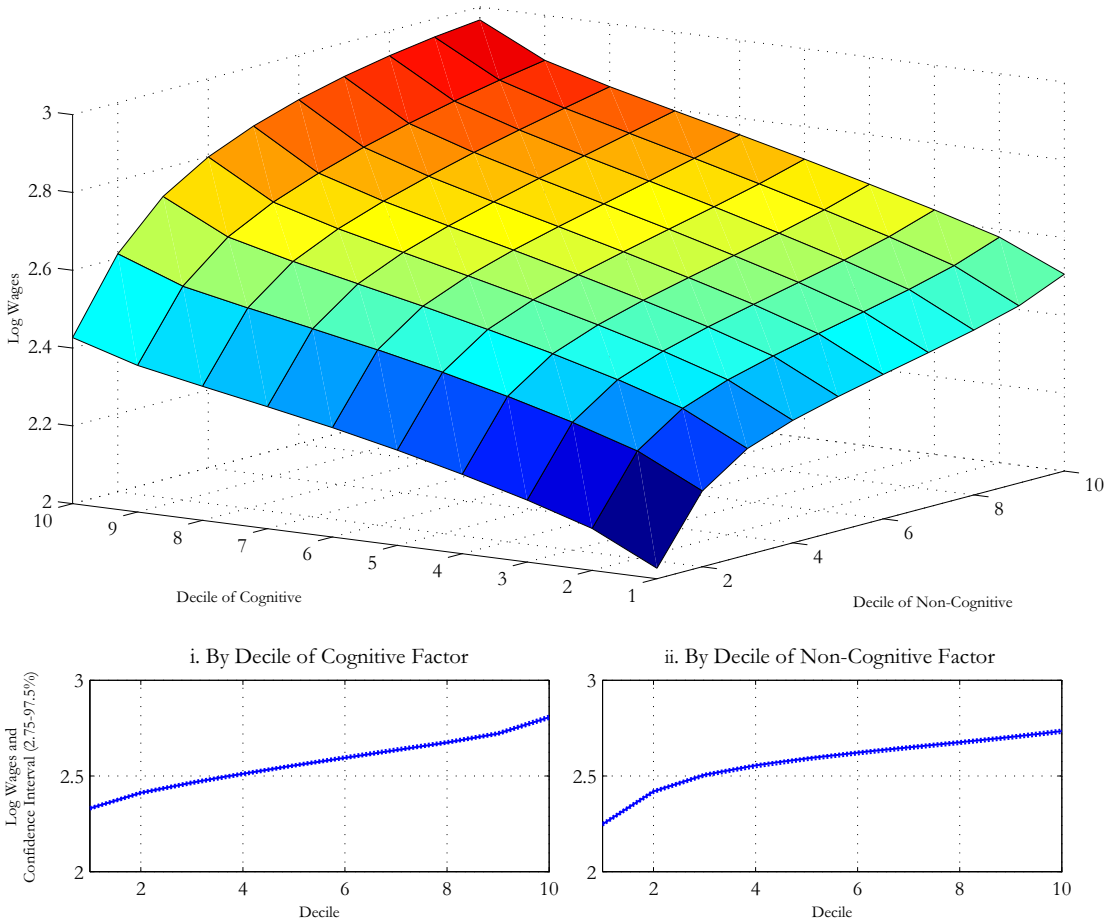
Figure 13: Probability of daily smoking by age 18 (males)



Notes: The data are simulated from the estimates of the model and our NLSY79 sample. We use the standard convention that higher deciles are associated with higher values of the variable. The confidence intervals are computed using bootstrapping (200 draws).

Source: Heckman et al. (2006).

Figure 14: Mean log wages by age 30 (males)

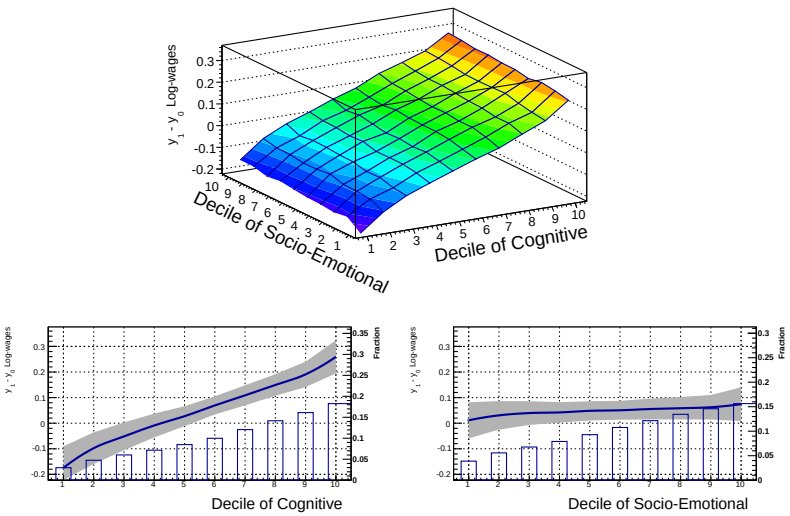


Notes: The data are simulated from the estimates of the model and our NLSY79 sample. We use the standard convention that higher deciles are associated with higher values of the variable. The confidence intervals are computed using bootstrapping (50 draws).

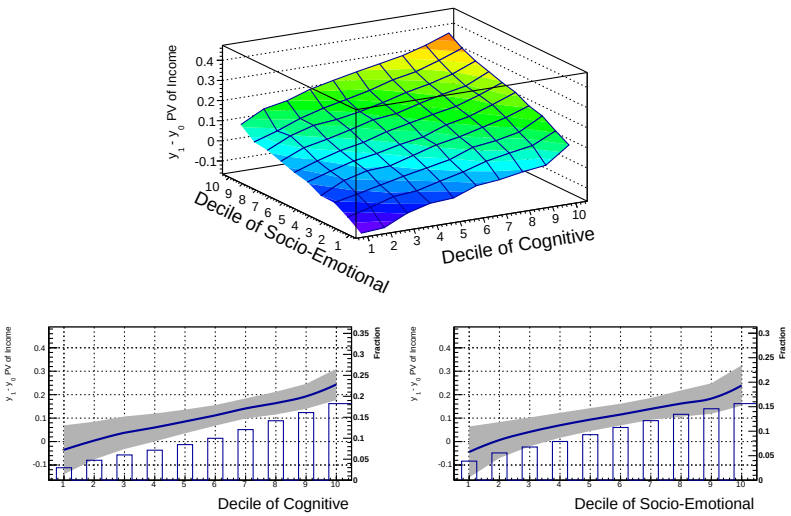
Source: Heckman et al. (2006).

Figure 15: Average Treatment Effect of Graduating from a Four-Year College by Outcome

A. (log)Wages



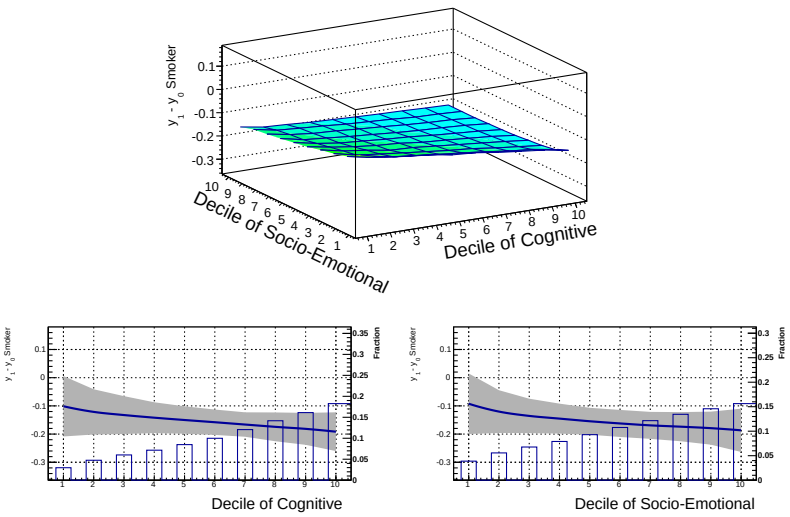
B. PV Wages



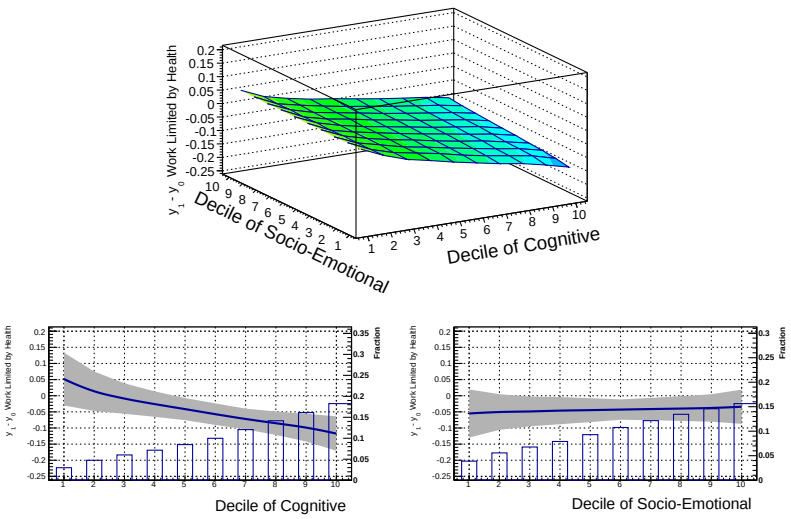
Source: Heckman, Humphries, and Veramendi (2016).

Figure 15: Average Treatment Effect of Graduating from a Four-Year College by Outcome, Cont'd

C. Smoking



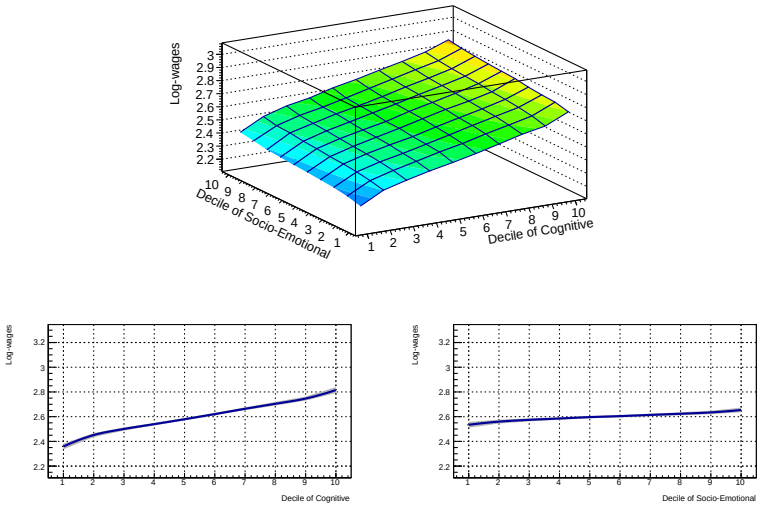
D. Health Limits Work



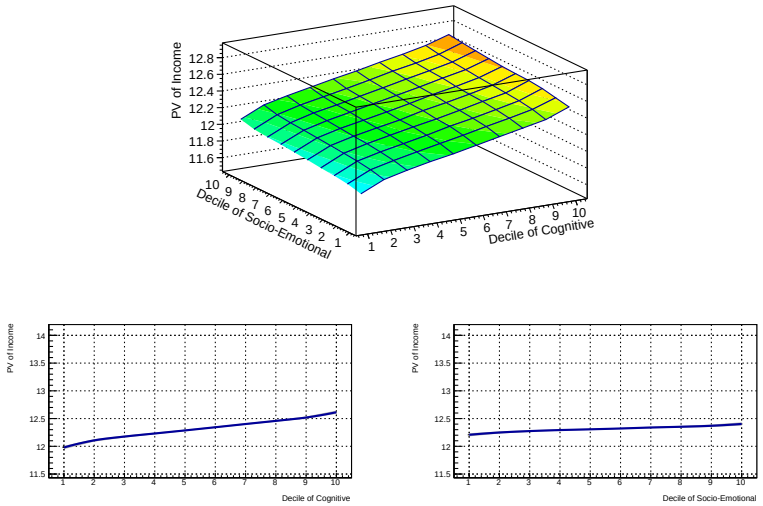
Source: Heckman, Humphries, and Veramendi (2016).

Figure 16: The Effect of Cognitive and Socioemotional Endowments

A. (log)Wages



B. PV Wages

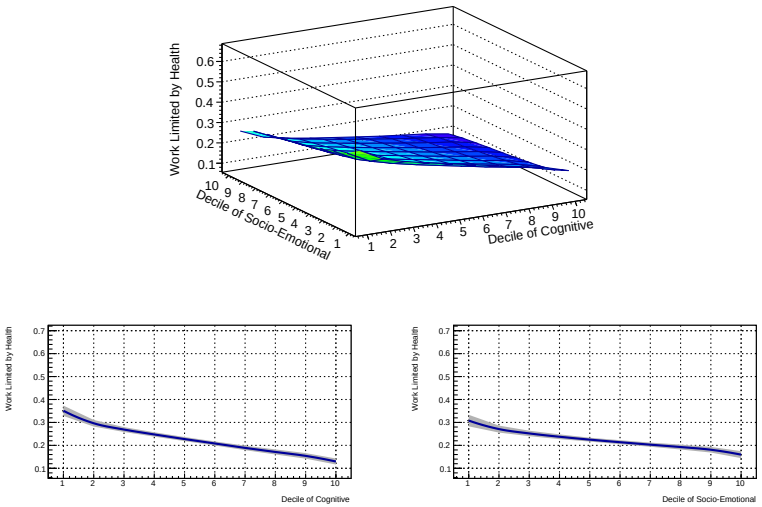


Notes: For each of the four outcomes, we present three figures that study the impact of cognitive and socioemotional endowments. The top figure in each panel displays the levels of the outcome as a function of cognitive and socioemotional endowments. In particular, we present the average level of outcomes for different deciles of cognitive and socioemotional endowments. Notice that we define as "decile 1" the decile with the lowest values of endowments and "decile 10" as the decile with the highest levels of endowments. The bottom left figure displays the average levels of endowment across deciles of cognitive endowments. The bottom right figure mimics the structure of the left-hand side figure but now for the socioemotional endowment.

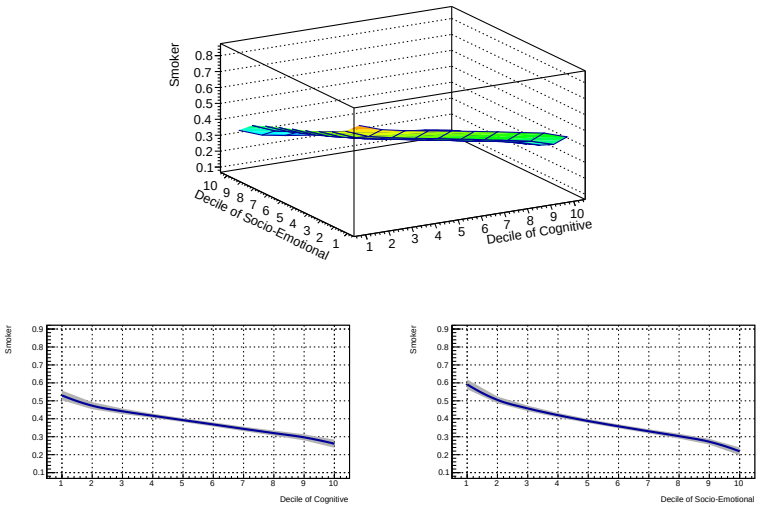
Source: Heckman, Humphries, and Veramendi (2016)

Figure 16: The Effect of Cognitive and Socioemotional Endowments, Cont'd

C. Health Limits Work



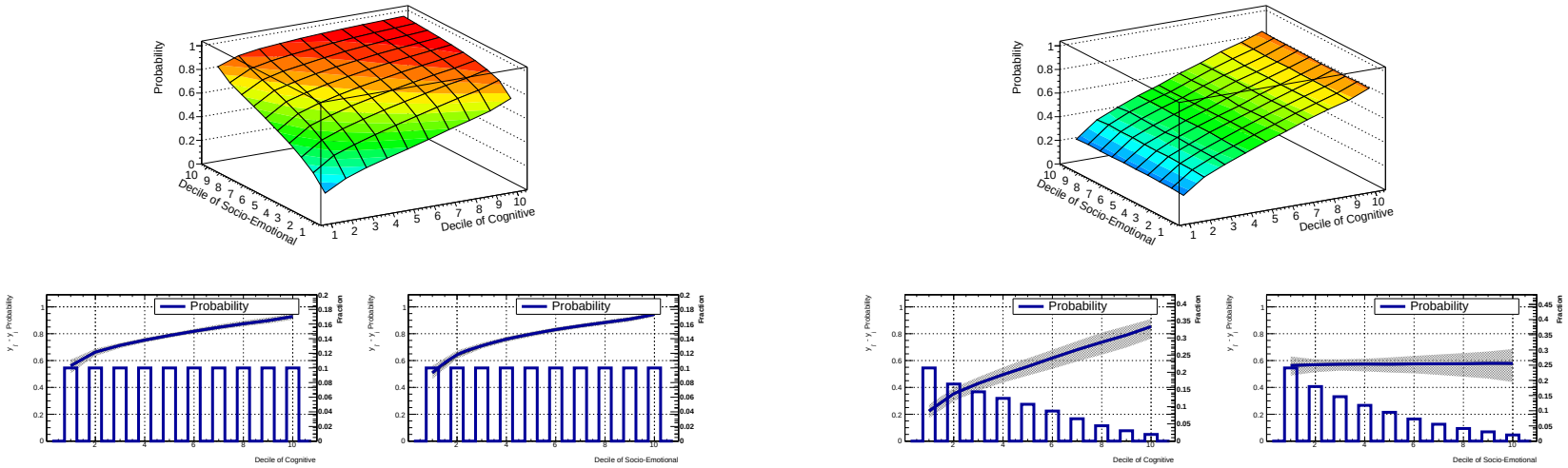
D. Smoking



Notes: For each of the four outcomes, we present three figures that study the impact of cognitive and socioemotional endowments. The top figure in each panel displays the levels of the outcome as a function of cognitive and socioemotional endowments. In particular, we present the average level of outcomes for different deciles of cognitive and socioemotional endowments. Notice that we define as "decile 1" the decile with the lowest values of endowments and "decile 10" as the decile with the highest levels of endowments. The bottom left figure displays the average levels of endowment across deciles of cognitive endowments. The bottom right figure mimics the structure of the left-hand side figure but now for the socioemotional endowment.

Figure 17: The Probability of Educational Decisions, by Endowment Levels (Final Schooling Levels are Highlighted Using Bold Letters)

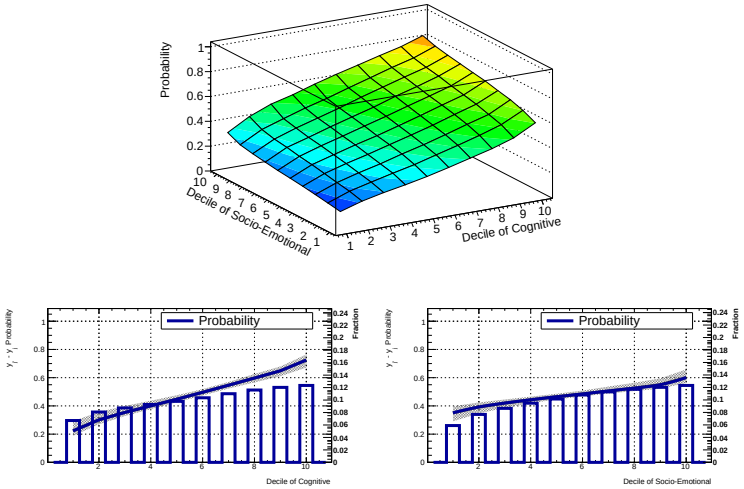
A. Dropping from HS vs. Graduating from HS B. **HS Dropout** vs. **Getting a GED**



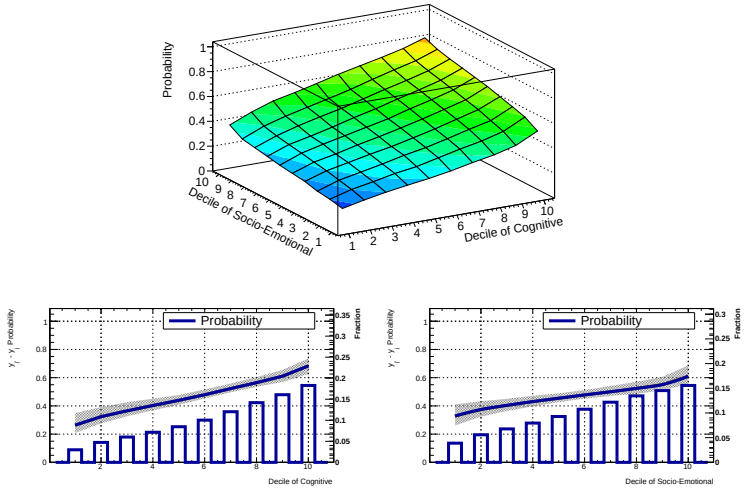
Notes: For each of the four educational choices, we present three figures that study the probability of that specific educational choice. Final schooling levels do not allow for further options. For each pair of schooling levels j and $j+1$, the first subfigure (top) presents $Prob(D_j = 0|d^C, d^{SE})$ where d^C and d^{SE} denote the cognitive and socioemotional deciles computed from the marginal distributions of cognitive and socioemotional endowments. $Prob(D_j = 0|d^C, d^{SE})$ is computed for those who reach the decision node involving a decision between levels j and $j+1$. The bottom left subfigures present $Prob(D_j = 0|d^C)$ where the socioemotional factor is integrated out. The bars in these figures display, for a given decile of cognitive endowment, the fraction of individuals visiting the node leading to the educational decision involving levels j and $j+1$. The bottom right subfigures present $Prob(D_j = 0|d^{SE})$ for a given decile of socioemotional endowment, as well as the fraction of individuals visiting the node leading to the educational decision involving levels j and $j+1$.
Source: Heckman, Humphries, and Veramendi (2016)

Figure 17: The Probability of Educational Decisions, by Endowment Levels (Final Schooling Levels are Highlighted Using Bold Letters), Cont'd

C. **HS Graduate** vs. College Enrollment

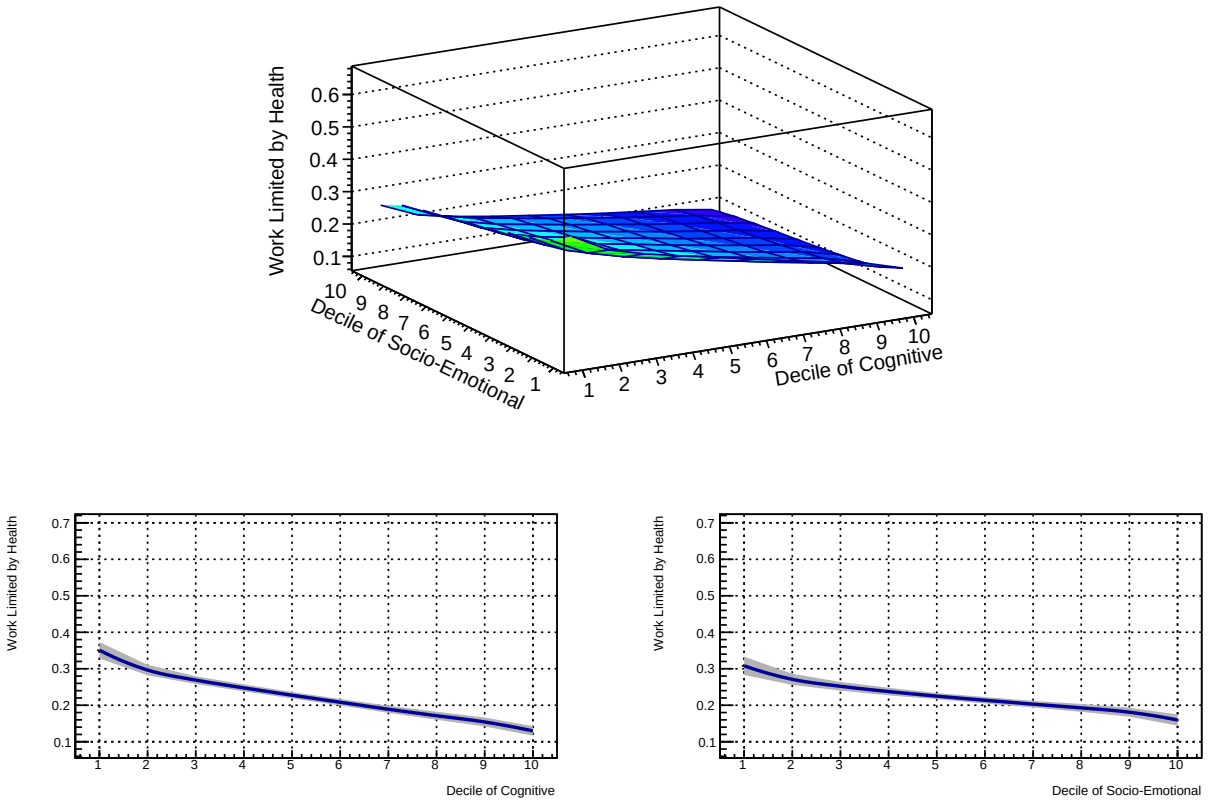


D. **Some College** vs. **4-year college degree**



Notes: For each of the four educational choices, we present three figures that study the probability of that specific educational choice. Final schooling levels do not allow for further options. For each pair of schooling levels j and $j+1$, the first subfigure (top) presents $Prob(D_j = 0|d^C, d^{SE})$ where d^C and d^{SE} denote the cognitive and socioemotional deciles computed from the marginal distributions of cognitive and socioemotional endowments. $Prob(D_j = 0|d^C, d^{SE})$ is computed for those who reach the decision node involving a decision between levels j and $j+1$. The bottom left subfigures present $Prob(D_j = 0|d^C)$ where the socioemotional factor is integrated out. The bars in these figures display, for a given decile of cognitive endowment, the fraction of individuals visiting the node leading to the educational decision involving levels j and $j+1$. The bottom right subfigures present $Prob(D_j = 0|d^{SE})$ for a given decile of socioemotional endowment, as well as the fraction of individuals visiting the node leading to the educational decision involving levels j and $j+1$.

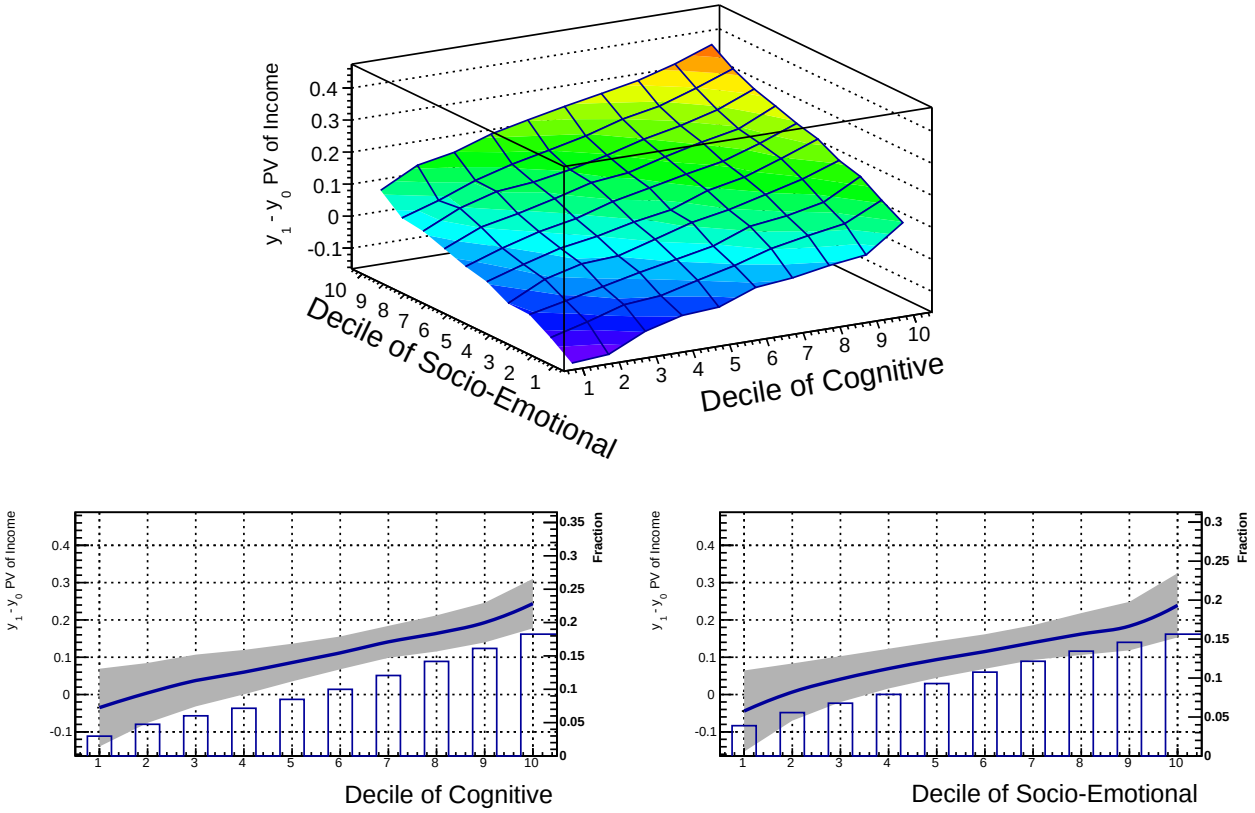
Figure 17: The Effect of Cognitive and Socioemotional Endowments on Work Limited



Source: Heckman et al. (2016).

Figure 17: Average Treatment Effect of Education on Present Value of Wages, by Decision Node and Endowment Levels (cont.)

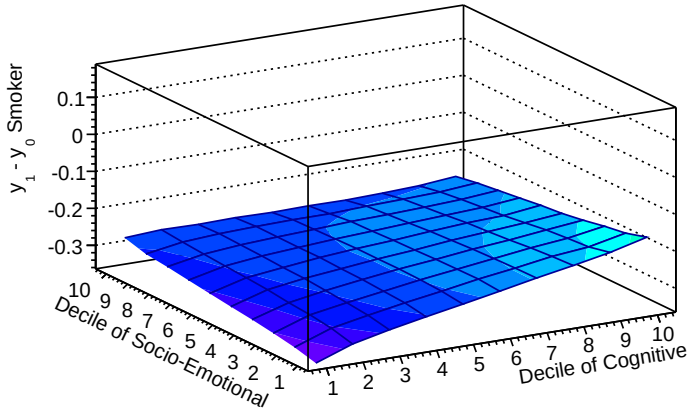
D. **Some College** vs. **4-year college degree**

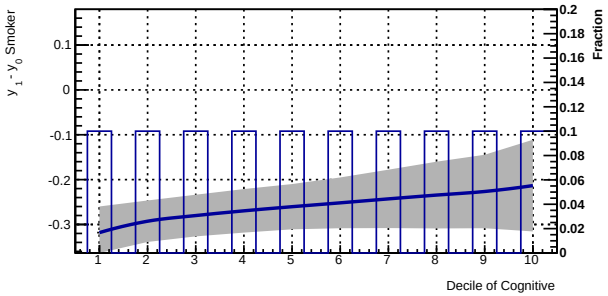


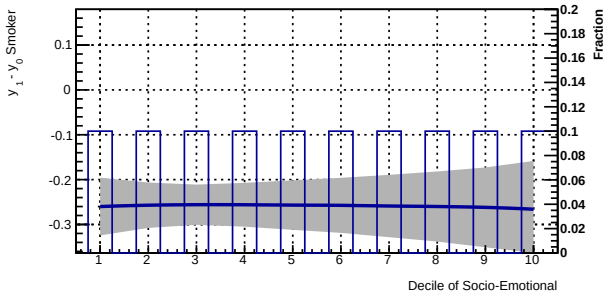
Source: Heckman et al. (2016).

Figure 18: Average Treatment Effect of Education on Smoking, by Decision Node and Endowment Levels

A. Dropping from HS vs. Graduating from HS







Source: Heckman et al. (2016).

Figure 19: Cognitive and Socio-Emotional Factors: Physical Health, Males

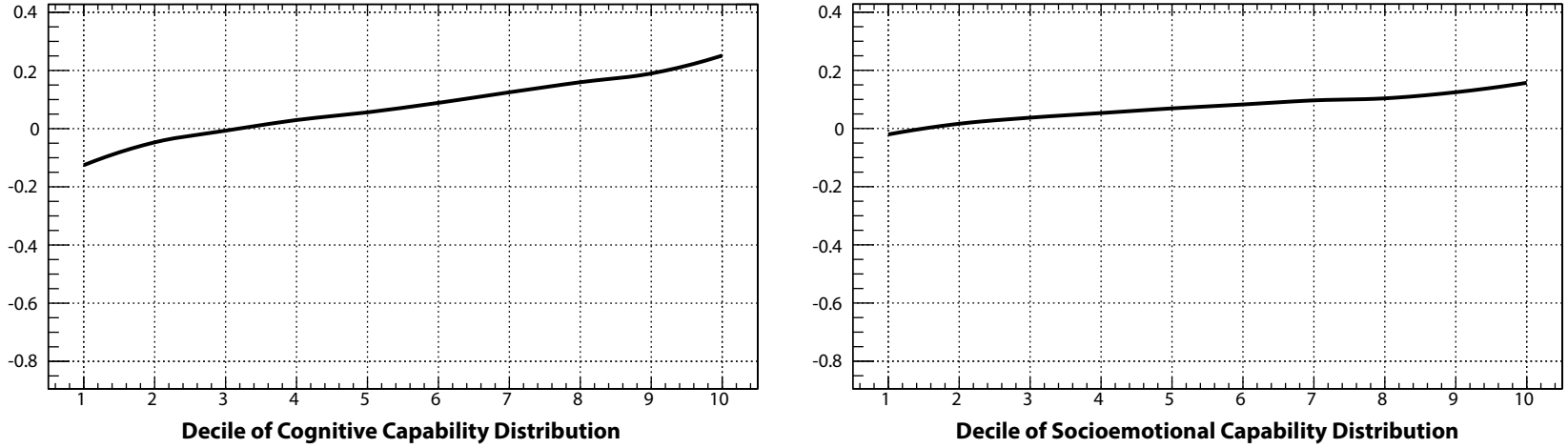


Figure 20: The Effect of Cognitive and Socio-Emotional Endowments on Mental Health at Age 40

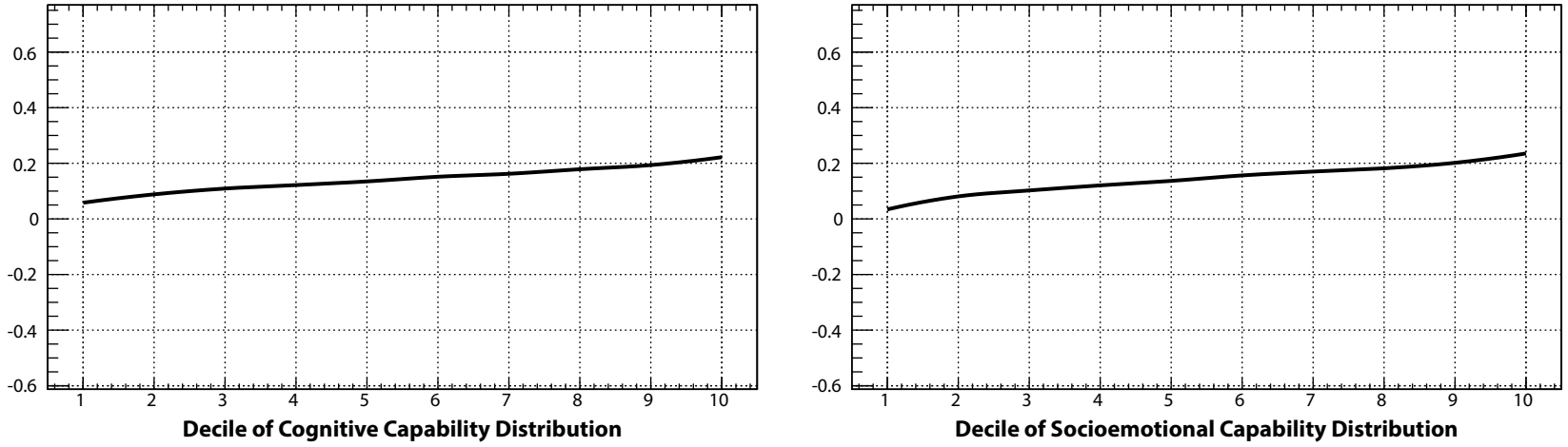


Figure 21: Probability of Being a Four-Year College Graduate by Age 30

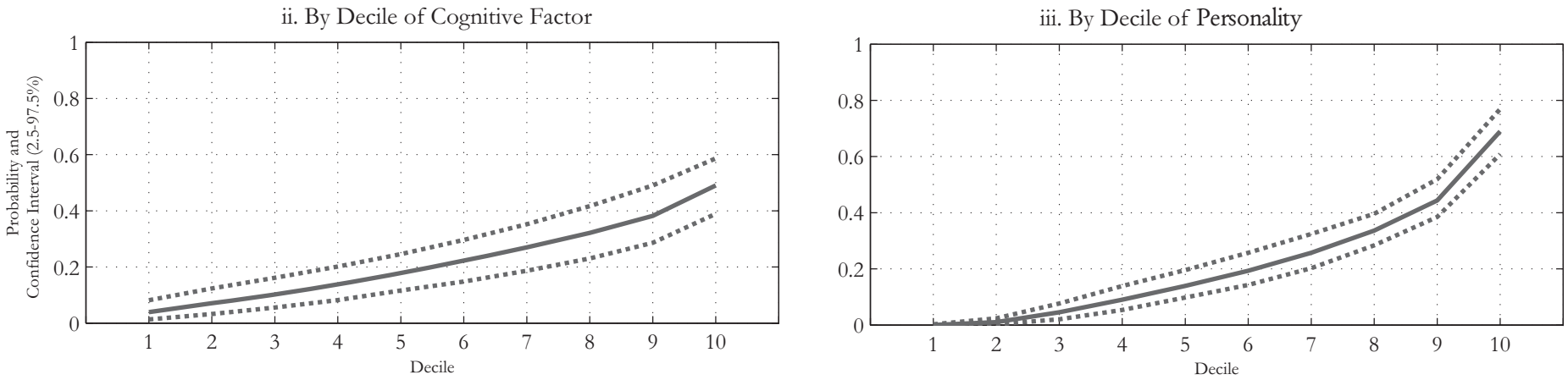
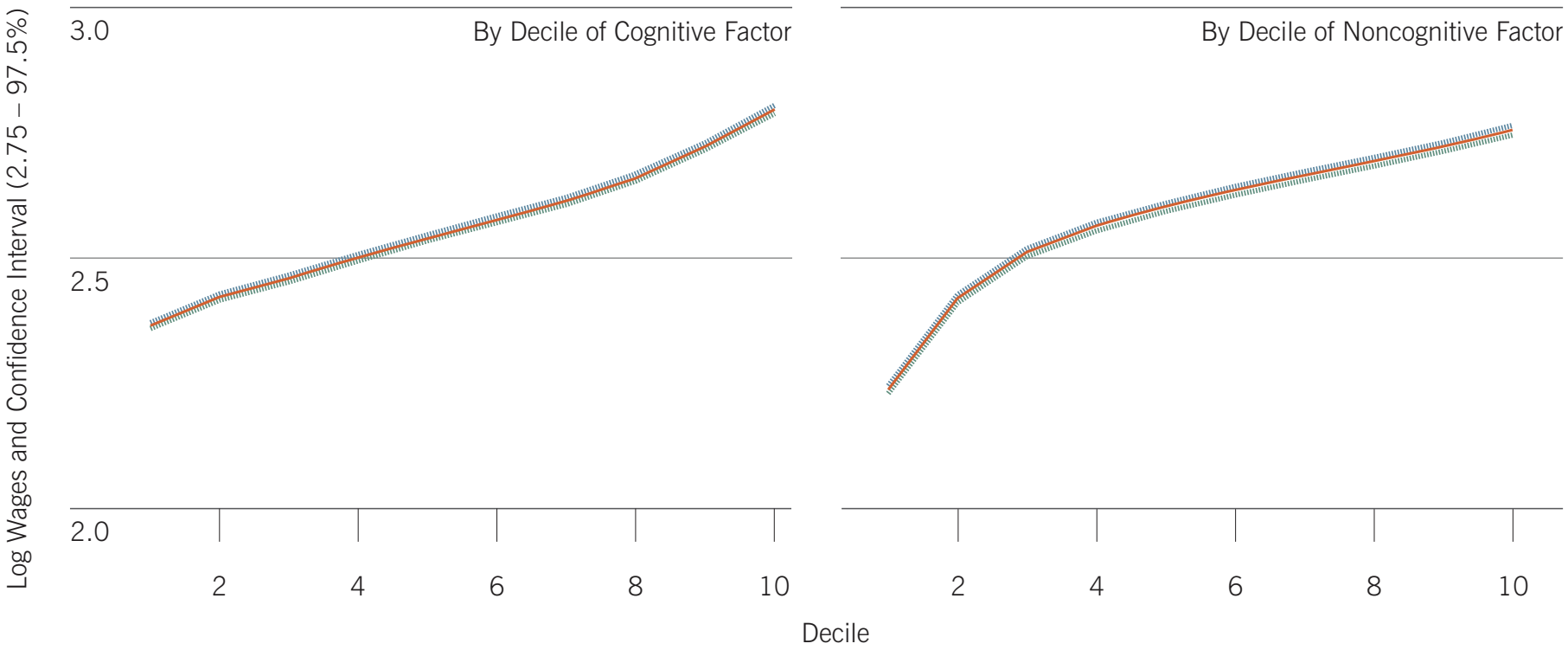
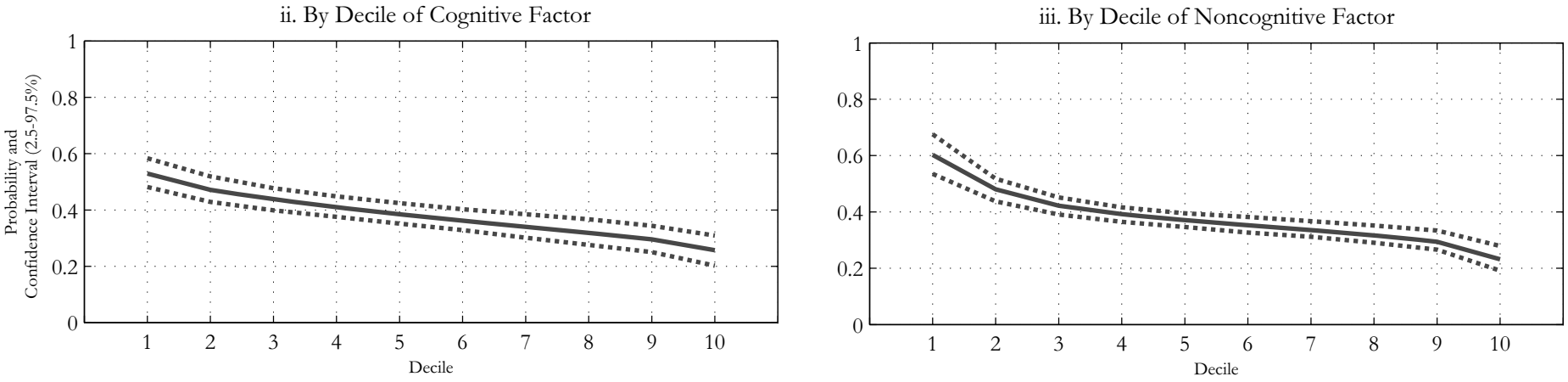


Figure 22: Mean Log Wages by Age 30 (males)



Notes: The data are simulated from the estimates of the model and our NLSY79 sample. We use the standard convention that higher deciles are associated with higher values of the variable. The confidence intervals are computed using bootstrapping (50 draws).

Figure 23: Probability of Daily Smoking by Age 18 (Males)



Notes: The data are simulated from the estimates of the model and our NLSY79 sample. We use the standard convention that higher deciles are associated with higher values of the variable. The confidence intervals are computed using bootstrapping (200 draws).

Figure 24: Cognitive and Socio-Emotional Factors: Probability of Graduating from Secondary School, Males

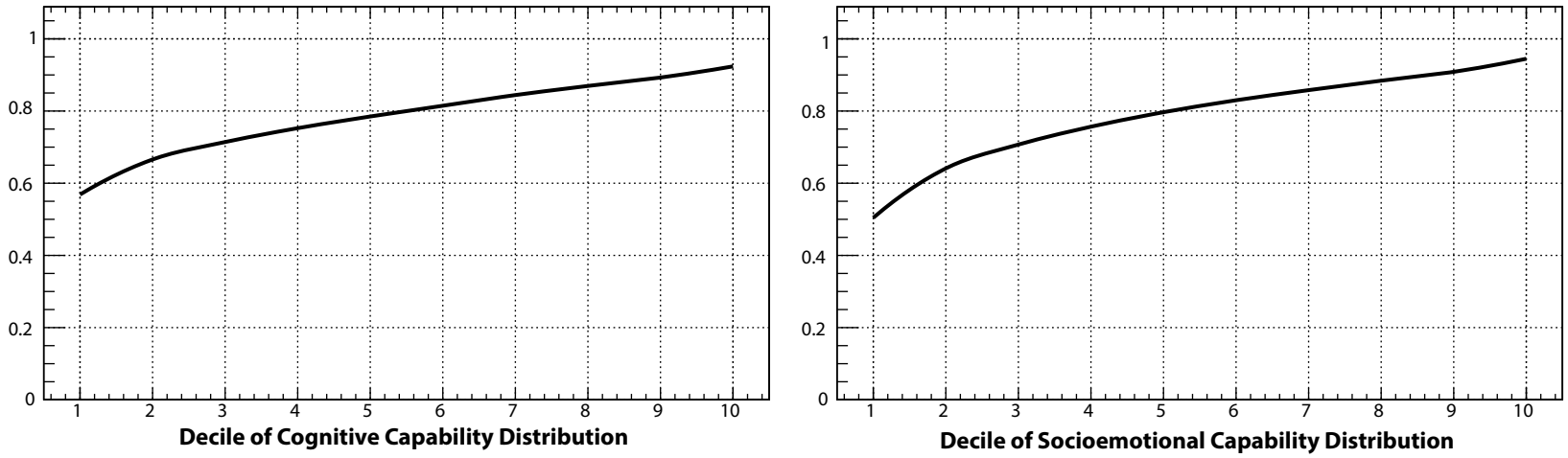


Figure 25: The Effect of Cognitive and Socio-Emotional Endowments on Probability of White-Collar Occupation

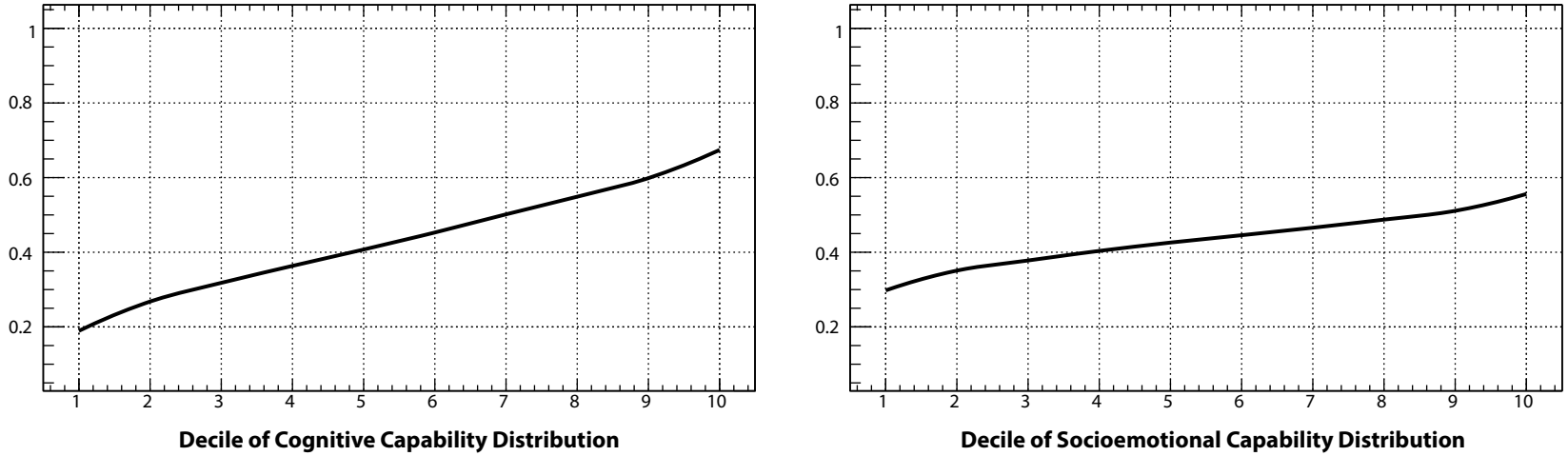
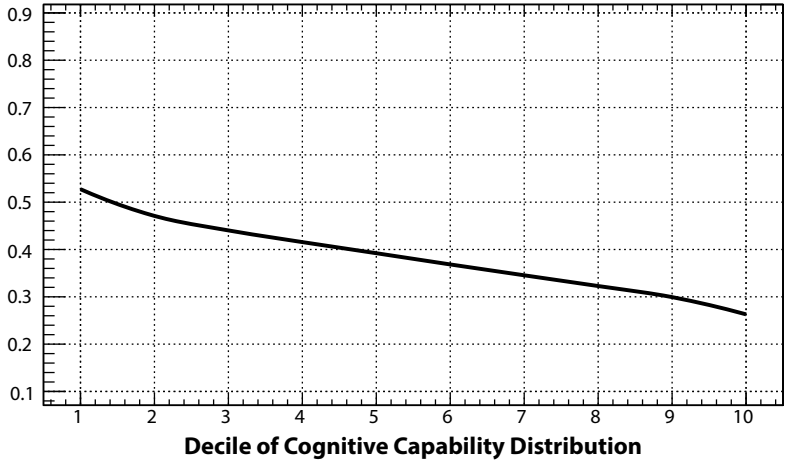


Figure 26: The Effect of Cognitive and Socio-Emotional Endowments on Smoking



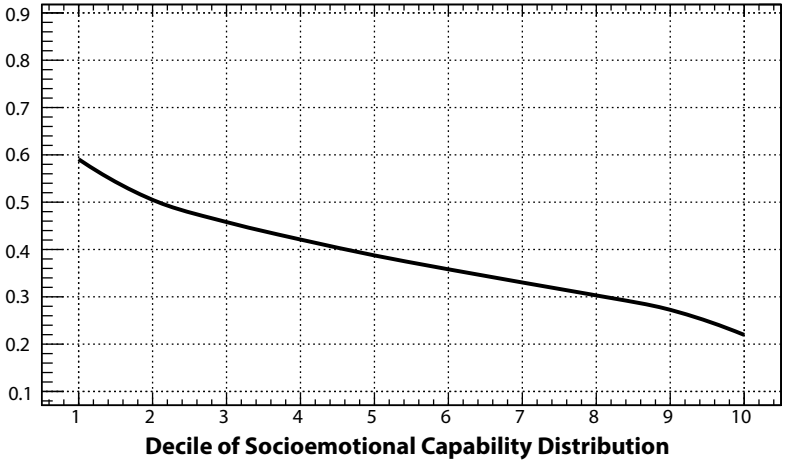


Figure 27: The Effect of Cognitive and Socio-Emotional Endowments on Heavy Drinking During Adulthood

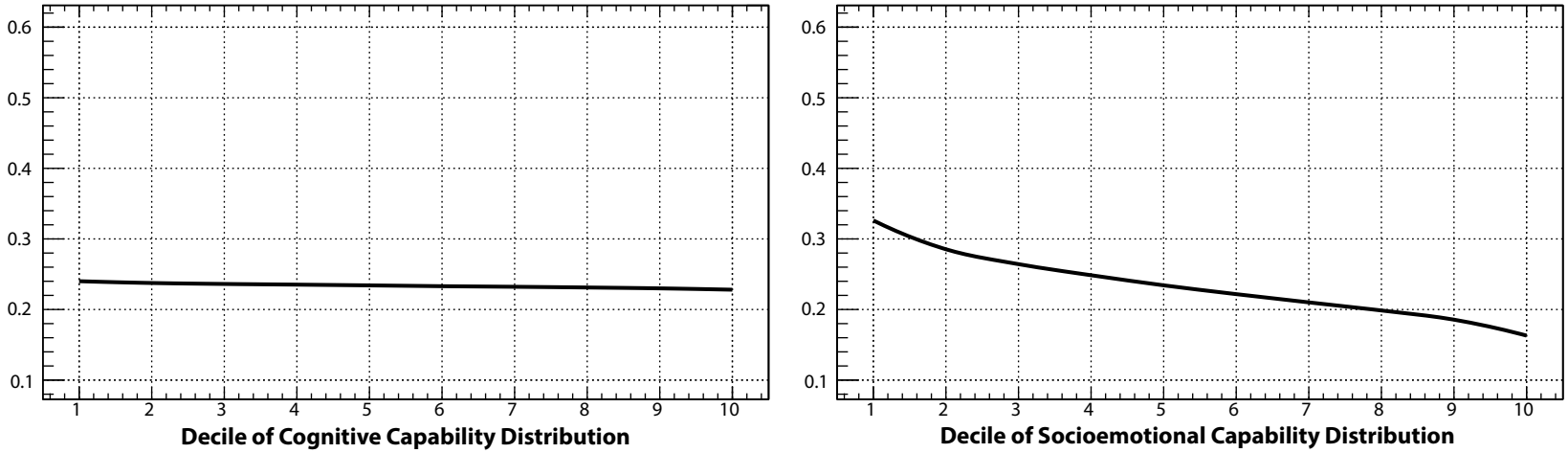


Figure 28: The Effect of Cognitive and Socio-Emotional Endowments on Pearlin's "Personal Mastery Scale": Sense of Self-Mastery

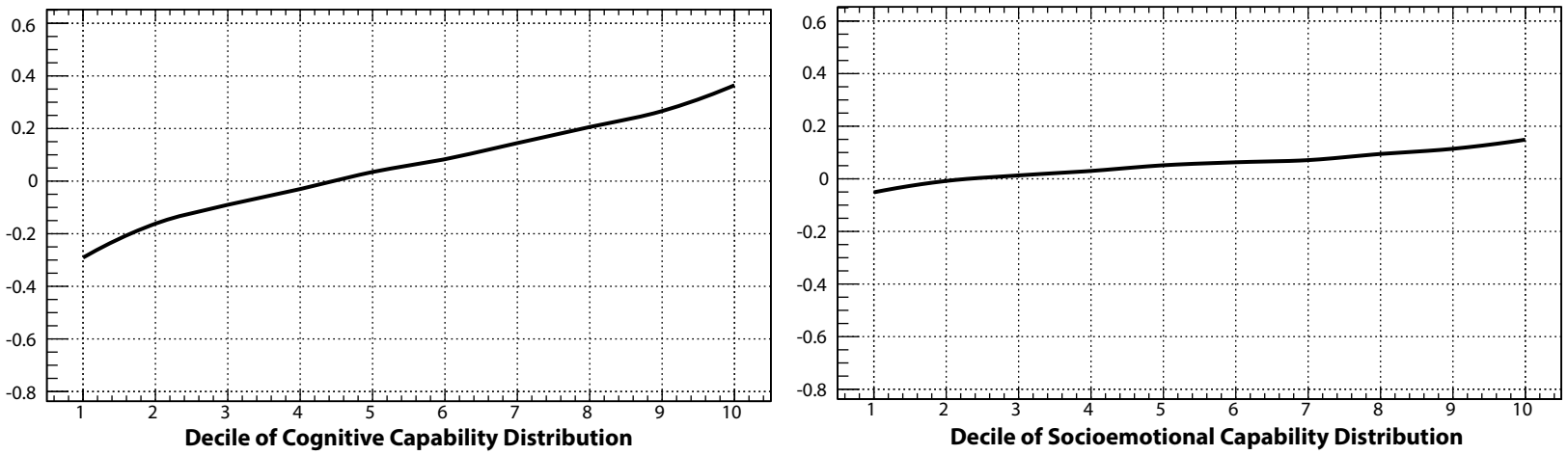


Figure 29: The Effect of Cognitive and Socio-Emotional Endowments on Trusting People (2008)

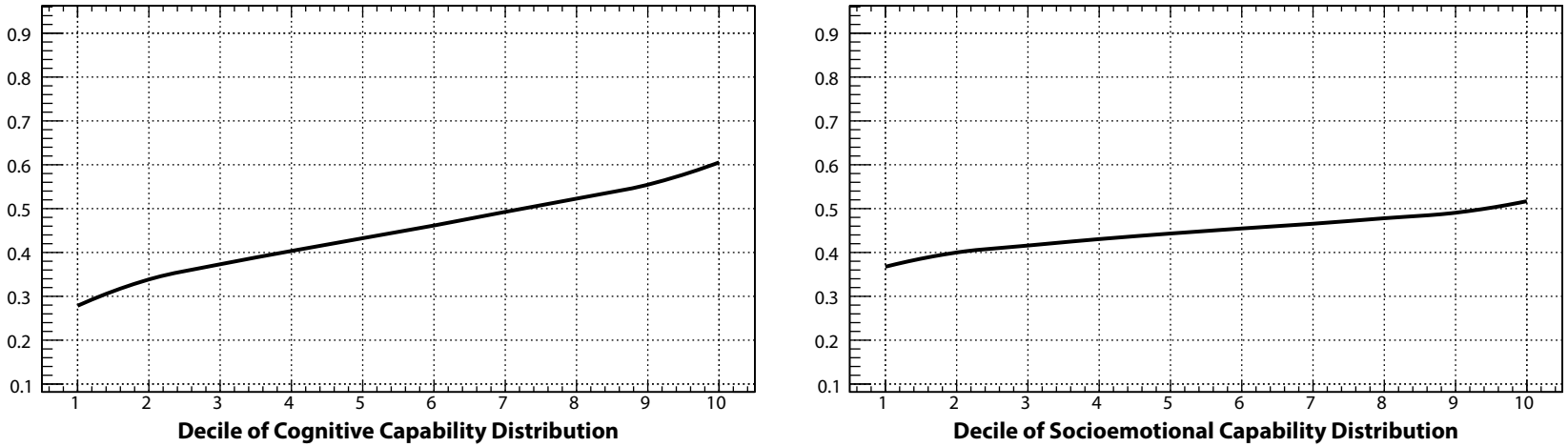


Figure 30: The Effect of Cognitive and Socio-Emotional Endowments on Ever Divorced