

Endogeneity in Nonparametric and Semiparametric Regression Models

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1. INTRODUCTION

The analysis of data with endogenous regressors – that is, observable explanatory variables that are correlated with unobservable error terms – is arguably the main contribution of econometrics to statistical science. Although “endogeneity” can arise from a number of different sources, including mismeasured regressors, sample selection, heterogeneous treatment effects, and correlated random effects in panel data, the term originally arose in the context of “simultaneity,” in which the explanatory variables were, with the dependent variable, determined through a system of equations, so that their correlation with error terms arose from feedback from the dependent to the explanatory variables. Analysis of linear supply-and-demand systems (with normal errors) yielded the familiar rank and order conditions for identification, two- and three-stage estimation methods, and analysis of structural interventions. Although these multi-step estimation procedures have been extended to nonlinear parametric models with additive nonnormal errors (e.g., Amemiya, 1974 and Hansen 1982), extensions to nonparametric and semiparametric models have only recently been considered.

The aim of this chapter is to examine the existing literature on estimation of some “nonparametric” models with endogenous explanatory variables, and to compare the different identifying assumptions and estimation approaches for particular models and determine their applicability to others. To maintain a manageable scope for the chapter, we restrict our attention to nonparametric and semiparametric extensions of the usual simultaneous equations models (with endogenous regressors that are continuously distributed). We consider the identification and estimation of the “average structural function” and argue that this parameter is one parameter of central interest in the analysis of semiparametric and nonparametric models with endogenous regressors. The two leading cases we consider are additive nonparametric specifications in which the regression function is unknown, and nonadditive models in which there is some known transformation function that is monotone but not invertible. An important example of the latter, and one that we use as an empirical illustration, is

the binary response model with endogenous regressors. We do not explicitly consider the closely related problems of selectivity, heterogeneous treatment effects, correlated random effects, or measurement error (see Heckman et al., 1998, Angrist, Imbens, and Rubin, 1996, and Arellano and Honoré, 1999 for lucid treatments of these topics). Moreover, we consider only recent work on nonparametric and semiparametric variants of the two-stage least-squares (2SLS) estimation procedure; Matzkin (1994) gives a broader survey of identification and estimation of nonlinear models with endogenous variables. Also, for convenience, we restrict attention to randomly sampled data, though most of our discussion applies in non-*iid* contexts, provided the structural equations and stochastic restrictions involve only a finite number of observable random variables.

In the next subsections, a number of different generalizations of the linear structural equation are presented, and the objects of estimation (the parameters of interest) are defined and motivated. The sections that follow consider how two common interpretations of the 2SLS estimator for linear equations – the “instrumental variables” and “control function” approaches – may or may not be applicable to nonparametric generalizations of the linear model and to their semiparametric variants. The discussion then turns to a particular semiparametric model, the binary response model with linear index function and nonparametric error distribution, and describes in detail how estimation of the parameters of interest can be constructed by using the control function approach. This estimator is applied to the empirical problem of the relation of labor force participation to nonlabor income studied in Blundell and Powell (1999). The results point to the importance of the semiparametric approach developed here and the strong drawbacks of the linear probability model and other parametric specifications.

1.1. Structural Equations

A natural starting point for investigation of endogeneity is the classical linear structural equation

$$y = \mathbf{x}'\beta - u, \quad (1.1)$$

where (y, \mathbf{x}') represents a data point of dimension $[1 \times (k + 1)]$, β is a conformable parameter vector, and u is an unobservable disturbance term. The explanatory variables \mathbf{x} are assumed to include a subset of continuous *endogenous* variables, meaning that

$$E(\mathbf{x}u) \neq 0. \quad (1.2)$$

This is the standard single linear equation treated in the literature on simultaneous equations, and it is considered here as a base case for comparison with other nonlinear setups. To identify and estimate the coefficient vector β in this setup, the model must be completed by imposing restrictions on the unobservable

error terms u that are consistent with (1.2); the traditional approach assumes that some observable vector \mathbf{z} of instrumental variables is available, satisfying the moment condition

$$E(\mathbf{z}u) = 0, \quad (1.3)$$

which leads to the well-known 2SLS estimator of β (Basmann, 1959 and Theil, 1953). The algebraic form of the 2SLS estimator can be derived from a number of different estimation principles based on (1.3), or on stronger conditions that imply it. As we will see, some of these estimation approaches, under suitably strengthened stochastic restrictions, can be extended to the nonparametric and semiparametric generalizations of the linear model that are considered here, but in certain important cases this turns out not to be so.

At an opposite extreme from the standard linear model, the relation between y and its observable and unobservable determinants \mathbf{x} and u could be assumed to be of a general form

$$y = H(\mathbf{x}, u), \quad (1.4)$$

which may represent a single equation of a nonlinear simultaneous equation system, possibly with a limited or qualitative dependent variable. Of course, without further restrictions on H , this function would be unidentified even under strong conditions on the unobservables (like independence of \mathbf{x} and u), but it is useful to view the various nonparametric and semiparametric models that follow as special cases of this setup.

One important class of structural functions, treated in more detail in the paragraphs that follow, assumes H to be additively separable,

$$y = g(\mathbf{x}) + u, \quad (1.5)$$

which would be the nonparametric regression model if the expectation of u given \mathbf{x} could be assumed to be zero (i.e., if \mathbf{x} were exogenous). Identification and estimation of g when a subset of \mathbf{x} is endogenous and instrumental variables \mathbf{z} are available is the subject of a number of recent studies, including those by Newey and Powell (1989), Newey, Powell, and Vella (1999), Darolles, Florens, and Renault (2000), and Ng and Pinkse (1995).

A nonseparable variant of (1.4) would be Matzkin's (1991) nonparametric version of Han's (1987) generalized regression model,

$$y = t(g(\mathbf{x}), u), \quad (1.6)$$

in which h is a known function that is monotone, but not invertible, in its first argument (a "single index" if g is one dimensional), and g is an unknown function satisfying appropriate normalization and identification restrictions. A leading special case of this specification is the nonparametric binary choice model (Matzkin, 1992), in which H is an indicator function for positivity of the sum of $g(\mathbf{x})$ and u :

$$t(g(\mathbf{x}), u) = 1(g(\mathbf{x}) + u > 0).$$

We group all of the models (1.4)–(1.6) in the *nonparametric* category, where the term refers to the lack of parametric structure to the structural function H or h or the “regression” function g . *Semiparametric* models restrict H (and possibly the distribution of u) to have finite-dimensional parametric components; that is,

$$y = h(\mathbf{x}, \beta, u). \tag{1.7}$$

For example, another special case of (1.4) is Han’s (1987) original model, where the single-index function g is assumed to be linear in \mathbf{x} ,

$$y = t(\mathbf{x}'\beta, u); \tag{1.8}$$

estimation of this model when \mathbf{x} is endogenous is considered by, for example, Lewbel (1998) and Blundell and Powell (1999), which focuses on the binary response version of this linear index model. Yet another semiparametric special case,

$$y = s(\mathbf{x}, \beta, g(\cdot)) + u, \tag{1.9}$$

where β is a finite parameter vector and h is a known function, has been considered in the recent work by Ai and Chen (2000). Although estimation of the coefficient vector β is typically the main objective of a semiparametric analysis of such models, estimation of the distribution of u , or at least certain functionals of it, will also be needed to evaluate the response of the dependent variable y to possible exogenous movements in the explanatory variables.

1.2. Parameters of Interest

For a nonparametric model, the parameters of interest are actually unknown functions that summarize important characteristics of the structural function H and the distribution of the errors u ; these parameters will be identified if they can be extracted from the distributions of the observable random variables. From a random sample of observations on the dependent variable y , regressors \mathbf{x} , and instrumental variables \mathbf{z} , the joint distribution of $y, \mathbf{x}, \mathbf{z}$, is by definition identified, and conditional distributions and moments can be consistently estimated by using standard nonparametric methods. In particular, the conditional expectation of functions of y , given either \mathbf{x} or \mathbf{z} or both, can be estimated without imposing additional restrictions (besides, say, smoothness and finite moment restrictions) on the joint distribution of the observable data, and these conditional expectations clearly summarize key features of the structural function and error distribution. However, as is clear from the well-worn supply and demand examples, knowledge only of the conditional distributions of observables is insufficient for analysis of the results of certain types of structural interventions that affect the distribution of the regressors \mathbf{x} but not the structural error terms u . Thus, the expectation of y given the instruments \mathbf{z} , called the *reduced form for y* , may be of interest if the values of the instrumental variables are control variables for the policymaker, but for interventions that alter the explanatory

variables \mathbf{x} directly, independently of the error terms u , knowledge only of the joint distribution of the observables will be insufficient. Similarly, potential results of interventions that directly affect some components of a vector-valued structural function H – for example, a change in the supply function in a supply-and-demand system caused by rationing – clearly could not be analyzed solely from knowledge of the reduced form for y prior to the intervention, nor could the results of policies intended to change the distribution of the unobservable component u .

For an analysis of such policies, it would be most useful to know the form of the structural function $H(\mathbf{x}, u)$ from (1.4), along with the joint distribution of the errors u and \mathbf{x}, \mathbf{z} , but these may not be identifiable, at least for models without additive errors. An alternative summary version of the structural function H that can be more straightforward to estimate is the *average structural function (ASF)*, where the average is taken over the *marginal* distribution of the error terms u ,

$$G(\mathbf{x}) \equiv \int H(\mathbf{x}, u) dF_u, \quad (1.10)$$

for F_u , the marginal cumulative distribution function of u . In models with additively separable errors, that is,

$$H(\mathbf{x}, u) = g(\mathbf{x}) + u, \quad (1.11)$$

as in (1.5), the ASF $G(\mathbf{x})$ reduces to the usual regression function $g(\mathbf{x})$, which would correspond to $E[y|\mathbf{x}]$ if the error terms u had a conditional mean zero given \mathbf{x} . More generally, the ASF G would be the counterfactual conditional expectation of y given \mathbf{x} if the endogeneity of \mathbf{x} were absent, that is, if the regressors \mathbf{x} could be manipulated independently of the errors, which would be considered invariant to the structural change. For the heterogeneous treatment effect model (see Heckman and Robb, 1985 and Imbens and Angrist, 1994), the ASF is directly related to the average treatment effect recovered from an experimental design – specifically, the average treatment effect for a binary regressor would be $G(1) - G(0)$.

In some structural interventions, where the regressors \mathbf{x} can be manipulated directly, knowledge of the function G , or its derivatives, would be sufficient to assess the impact of the policy. For example, in the classical supply-and-demand example, with y corresponding to quantity demanded and \mathbf{x} representing price, the ASF would suffice to determine expected demand if the market supply function were replaced with a fixed price (by fiat or by the world market), and if the distribution of u were assumed to be invariant to this structural change. And, for the additively separable model (1.11), the ASF embodies the direct effect of the regressors for a particular observation, holding the error terms fixed; in this case the individual-specific, not just average, effects of changes in \mathbf{x} can be analyzed if G is known.

However, for interventions that do not directly determine the endogenous regressors, knowledge of the ASF is not enough; additional structural information about the structural function H and the distribution of \mathbf{x} (and possibly \mathbf{z}) would be required to evaluate the policy effect. For example, for the effects

of imposition of a sales tax to be analyzed, which would rescale \mathbf{x} in the structural function by a fixed amount exceeding unity, the “inverse supply function” relating price \mathbf{x} to quantity supplied y and other observable and unobservable covariates would have to be specified to account for the joint determination of price and quantity following the structural intervention. More generally, policies that alter some components of the function H rather than manipulating the argument \mathbf{x} require specification, identification, and consistent estimation of all components of H (including equations for all endogenous regressors) and the distribution of u . As shown by Roehrig (1988) and Imbens and Newey (2000), nonparametric identification of a fully specified system of simultaneous equations is possible under strong restrictions on the forms of the structural function – for example, invertibility of $H(\mathbf{x}, u)$ in the unobservable component u – but such restrictions may be untenable for limited dependent variable models such as the binary response model analyzed in the paragraphs that follow. Thus, the average structural function G may be the only feasible measure of the direct effect of \mathbf{x} on y for limited dependent variable models and for “limited information” settings in which the structural relations for the endogenous regressors in a single structural equation are incompletely specified.

Of course, the expected value of the dependent variable y need not be the only summary measure of interest; a complete evaluation of policy could require the entire distribution of y given an “exogenous” \mathbf{x} . Because that distribution can be equivalently characterized by the expectation of all measurable functions of y , a more ambitious objective is calculation of the ASF for any transformation $\tau(y)$ of y with finite first moment,

$$G_{\tau}(\mathbf{x}) \equiv \int \tau(H(\mathbf{x}, u))dF_u.$$

For those sets of stochastic restrictions on u that require additively separable form (1.11) for identification and estimation of the ASF, this collection of expectations can be evaluated directly from the marginal distribution of $u = y - g(\mathbf{x})$; for those stochastic restrictions that do require additivity of errors for identification of G , the collection of functions G_{τ} (and thus the structural distribution of y) can be derived by redefinition of the dependent variable to $\tau(y)$.

For semiparametric problems of the form (1.7), the finite-dimensional parameter vector β is typically of interest in its own right, because economic hypotheses can impose testable restrictions on the signs or magnitudes of its components. A primary goal of the statistical analysis of semiparametric models is to construct consistent estimators of β that converge at the parametric rate (the inverse of the square root of the sample size), with asymptotically normal distributions and consistently estimable asymptotic covariance matrices. For some sets of restrictions on error distributions, this objective may be feasible even when estimation of the ASF $G(\mathbf{x})$ is not. For example, for the semiparametric model (1.7), the form of the reduced form for y ,

$$E[y|\mathbf{z}] = E[h(\mathbf{x}'\beta, u)|\mathbf{z}],$$

can, under appropriate restrictions, be exploited to obtain estimates of the single-index coefficients β even if G is not identified, as noted as follows.

Even for a fully nonparametric model, some finite-dimensional summary measures of G may be of interest. In particular, the “average derivative” of $G(\mathbf{x})$ with respect to \mathbf{x} (Stoker, 1986) can be an important measure of marginal effects of an exogenous shift in the regressors. Altonji and Ichimura (2000) consider the estimation of the derivative y with respect to x for the case when y is censored. They are able to derive a consistent estimator for the nonadditive case. Unlike the estimation of single-index coefficients, which are generally identified only up to a scale factor, estimation of the average derivatives of the ASF G is problematic unless G itself is identifiable.

2. NONPARAMETRIC ESTIMATION UNDER ALTERNATIVE STOCHASTIC RESTRICTIONS

As in the traditional treatment of linear simultaneous equations, we will assume that there exists a $1 \times m$ vector \mathbf{z} of instrumental variables, typically with $m \geq k$. The particular stochastic restrictions on \mathbf{z} , \mathbf{x} , and u will determine what parameters are identified and what estimators are available in each of the model specifications (1.4)–(1.9). Each of the stochastic restrictions is a stronger form of the moment condition (1.3), and each can be used to motivate the familiar 2SLS estimator in the linear model with additive errors (1.1) under the usual rank condition, but their applicability to the nonparametric and semiparametric models varies according to the form of the structural function H .

2.1. Instrumental Variables Methods

2.1.1. The Linear Model

The instrumental variables (IV) version of the standard 2SLS estimator is the sample analog to the solution of a weaker implication of (1.3), namely,

$$0 = E(P[\mathbf{x}|\mathbf{z}]u) \equiv E(\Pi' \mathbf{z}u) = E(\Pi' \mathbf{z}(y - \mathbf{x}'\beta)), \tag{2.1}$$

where $P[x|z]$ is the population least-squares projection of \mathbf{x} on \mathbf{z} , with

$$\Pi \equiv \{E(\mathbf{z}\mathbf{z}')\}^{-1} E(\mathbf{z}'\mathbf{x}). \tag{2.2}$$

Replacing population expectations with sample averages in (2.1) yields the 2SLS estimator

$$\widehat{\beta}_{2SLS} = (\widehat{\mathbf{X}}'\widehat{\mathbf{X}})^{-1}\widehat{\mathbf{X}}'\mathbf{y}, \tag{2.3}$$

with

$$\widehat{\mathbf{X}} = \mathbf{Z}\widehat{\Pi} \quad \text{and} \quad \widehat{\Pi} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}, \tag{2.4}$$

and where \mathbf{X} , \mathbf{Z} , and \mathbf{y} are the $N \times k$, $N \times m$, and $N \times 1$ data arrays corresponding respectively to \mathbf{x} , \mathbf{z} , and y , for a sample of size N . When the linear form

of the residual $u = y - \mathbf{x}'\beta$ is replaced with a nonlinear, parametric version $u = m(y, \mathbf{x}, \beta)$, extension of this estimation approach yields the generalized IV estimator (GIVE) of Sargan (1958), the nonlinear two-stage least-squares (NLLS) estimator of Amemiya (1974), and the generalized method of moments (GMM) estimator (Hansen, 1982).¹

Another closely related formulation of 2SLS exploits a different implication of (1.3), namely,

$$0 = P[u|\mathbf{z}] = P[y|\mathbf{z}] - P[\mathbf{x}|\mathbf{z}]\beta, \tag{2.5}$$

where the population linear projection coefficients of u and y on \mathbf{z} are defined analogously to (2.2). Replacing $P[y|\mathbf{z}]$ and $P[\mathbf{x}|\mathbf{z}]$ with their sample counterparts and applying least squares yields Basman's (1959) version of 2SLS,

$$\hat{\beta}_{2SLS} = (\hat{\mathbf{X}}'\hat{\mathbf{X}})^{-1}\hat{\mathbf{X}}'\hat{\mathbf{y}}, \tag{2.6}$$

where now

$$\hat{\mathbf{y}} = \mathbf{Z}\hat{\pi} \equiv \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}.$$

Although logically distinct from the IV interpretation of 2SLS in (2.3), extension of the estimation approaches in either (2.3) or (2.6) yields the same NLLS and GMM estimators in the nonlinear parametric case, and we refer to generalization of either approach to the nonparametric or semiparametric structural equations as an *IV method*.

2.1.2. *Extensions to Additive Nonparametric Models*

To extend the IV methods to nonparametric settings, we find it natural to strengthen the unconditional moment restriction $E(\mathbf{z}u) = 0$ to a conditional mean restriction

$$E(u|\mathbf{z}) = 0, \tag{2.7}$$

just as the assumption of $E[\mathbf{x}u] = 0$ is strengthened to $E[u|\mathbf{x}] = 0$ for a nonparametric regression model. For the additive structural function (1.11), identification and estimation of $g(\mathbf{x})$ was considered by Newey and Powell (1989) and Darolles et al. (2000). Substitution of the error term $u = y - g(\mathbf{x})$ into condition (2.7) yields a relationship between the reduced form $E[y|\mathbf{z}]$ and the structural function g :

$$\begin{aligned} E[y|\mathbf{z}] &= E[g(\mathbf{x})|\mathbf{z}] \\ &= \int g(\mathbf{x})dF_{\mathbf{x}|\mathbf{z}}, \end{aligned} \tag{2.8}$$

¹ When β is overidentified, that is, the dimension m of the instruments \mathbf{x} exceeds the dimension k of β , asymptotically efficient estimation would be based on a different implication of (1.3), in which Π is replaced by a different $(m \times k)$ matrix, as noted by Hansen (1982).

where $F_{\mathbf{x}|\mathbf{z}}$ is the conditional cumulative distribution function (CDF) of \mathbf{x} given \mathbf{z} . The reduced form for y , $E[y|\mathbf{z}]$, and the conditional distribution of \mathbf{x} given \mathbf{z} are functionals of the joint distribution of the observable variables y , \mathbf{x} , and \mathbf{z} are identified; identifiability of the structural function g therefore reduces to the uniqueness of the solution of the integral equation (2.8). And, as noted in the Newey–Powell and Darolles–Florens–Renault manuscripts, this in turn reduces to the question of statistical completeness of the family of conditional distributions $F_{\mathbf{x}|\mathbf{z}}$ in the “parameter” \mathbf{z} . (See, e.g., Ferguson, 1967, Section 3.6, for a definition of completeness and its connection to minimum variance unbiased estimation in parametric problems.) Although conditions for completeness of $F_{\mathbf{x}|\mathbf{z}}$ are known for certain parametric classes of distributions (e.g., exponential families), and generally the “order condition” $\dim(\mathbf{z}) \geq \dim(\mathbf{x})$ must be satisfied, in a nonparametric estimation setting, uniqueness of the solution of (2.8) must be imposed as a primitive assumption. Darolles et al. (2000) give a more thorough discussion of the conditions for existence and uniqueness of the solution of (2.8) and its variants.

In the special case in which \mathbf{x} and \mathbf{z} have a joint distribution that is discrete with finite support, conditions for identification and consistent estimation of the ASF $g(\mathbf{x})$ are straightforward to derive. Suppose $\{\xi_j, j = 1, \dots, J\}$ are the set of possible values for \mathbf{x} and $\{\zeta_l, l = 1, \dots, L\}$ are the support points for \mathbf{z} , and let

$$\begin{aligned} \boldsymbol{\pi} &\equiv \text{vec}(E[y|\mathbf{z} = \zeta_l]), & (2.9) \\ \mathbf{P} &\equiv [P_{jl}] \equiv [\Pr\{\mathbf{x} = \xi_j | \mathbf{z}_l = \zeta_j\}] \end{aligned}$$

denote the vector of reduced-form values $E[y|\mathbf{z}]$ and the matrix of conditional probabilities that $\mathbf{x} = \xi_j$ given that $\mathbf{z} = \zeta_l$, respectively; these would clearly be identified, and consistently estimable, from a random sample of observations on y , \mathbf{x} , and \mathbf{z} . If $\mathbf{g} \equiv \text{vec}(g(\xi_j))$ denotes the vector of possible values of $g(\mathbf{x})$, the question of identifiability of \mathbf{g} using (2.8) is the question of uniqueness of the solution to the set of linear equations

$$\boldsymbol{\pi} = \mathbf{P}\mathbf{g}, \tag{2.10}$$

so that \mathbf{g} is identified if and only if $\text{rank}\{\mathbf{P}\} = J = \dim\{\mathbf{g}\}$, which requires the order condition $L = \dim\{\boldsymbol{\pi}\} \geq \dim\{\mathbf{g}\} = J$. When \mathbf{g} is identified, it may be consistently estimated (at a parametric rate) by replacing $\boldsymbol{\pi}$ and \mathbf{P} by estimators using the empirical CDF of the observed vectors in (2.10), and solving for $\hat{\mathbf{g}}$ in the just-identified case $J = L$, or using a minimum chi-square procedure when \mathbf{g} is overidentified ($J < L$).² More details for this finite-support case are given by Das (1999).

² Note that when $J = K = 2$, this is the “treatment effect” in the homogeneous treatment effect model case with additive errors. The heterogeneous treatment effect case is a specific form of the general nonadditive model.

2.1.3. *The Ill-Posed Inverse Problem*

Unfortunately, the simple structure of this finite-support example does not easily translate to the general case, in which \mathbf{x} and \mathbf{z} may have continuously distributed components. Unlike in typical nonparametric estimation problems, where identification results can be easily translated into consistent estimators of the identified functions under smoothness or monotonicity restrictions, identification of g and consistent estimators of the components $E[y|\mathbf{z}]$ and $F_{\mathbf{x}|\mathbf{z}}$ are not, by themselves, sufficient for a solution of a sample analogue of (2.8) to be a consistent estimator of g . First, it is clear that, unlike the standard nonparametric regression problem, the function $g(\mathbf{x})$ (and the reduced form and conditional distribution function) must be estimated for all values of \mathbf{x} in the support of the conditional distribution, and not just at a particular value \mathbf{x}_0 of interest; thus, consistency of g must be defined in terms of convergence of a suitable measure of distance between the functions $\hat{g}(\cdot)$ and $g(\cdot)$ (e.g., the maximum absolute difference over possible \mathbf{x} or the integrated squared difference) to zero in probability. Moreover, the integral equation (2.8), a generalization of the Fredholm integral equation of the first kind, is a notorious example of an ill-posed inverse problem: The integral $T_{\mathbf{z}}(g) \equiv \int g(\mathbf{x})dF_{\mathbf{x}|\mathbf{z}}$, although continuous in g for the standard functional distance measures, has an inverse that is not continuous in general, even if the inverse is well defined. That is, even if a unique solution \hat{g} of the sample version

$$\hat{E}[y|\mathbf{z}] = \int g(\mathbf{x})d\hat{F}_{\mathbf{x}|\mathbf{z}} \tag{2.11}$$

of (2.8) exists, that solution $\tilde{g} \equiv \hat{T}_{\mathbf{z}}^{-1}(\hat{E}[y|\mathbf{z}])$ is not continuous in the argument $\hat{E}[y|\mathbf{z}]$, so consistency of the reduced-form estimator (and the estimator of the conditional distribution of \mathbf{x} given \mathbf{z}) does not imply consistency of \hat{g} . Heuristically, the reduced form $E[y|\mathbf{z}]$ can be substantially smoother than the structural function $g(\mathbf{x})$, so that very different structural functions can yield very similar reduced forms; the ill-posed inverse problem is a functional analogue to the problem of multicollinearity in a classical linear regression model, where large differences in regression coefficients can correspond to small differences in fitted values of the regression function. Such ill-posed inverse problems are well known in applied mathematics and statistics, arising, for example, in the problem of estimation of the density of an unobservable variable x that measured with error; that is, observations are available only on $y = x + u$, where u is an unobservable error term with known density function (the deconvolution problem). O’Sullivan (1986) surveys the statistical literature on ill-posed inverse problems and describes the general “regularization” approaches to construction of consistent estimators for such problems. If the joint distribution of \mathbf{x} and \mathbf{z} is approximated by a distribution with a finite support (as common for “binning” approaches to nonparametric estimation of conditional distributions), then the ill-posed inverse problem would manifest itself as an extreme sensitivity of the

“transition” matrix \mathbf{P} to choice of J and L , and its near-singularity as J and L increase to infinity.

2.1.4. Consistent Estimation Methods

Newey and Powell (1989) impose further restrictions on the set of possible structural functions g to obtain a consistent estimator, exploiting the fact that the inverse of a bounded linear functional such as $T_x(g)$ will be continuous if the domain of the functional is compact. For this problem, compactness of the set of possible g functions with respect to, say, the “sup norm” measure of distance between functions, can be ensured by restricting the “Sobolev norm” of all possible g functions to be bounded above by a known constant. This Sobolev norm (denoted here as $\|g\|_S$) is a different but related distance measure on functions that involves a sum of the integrated squared values of $g(\mathbf{x})$ and a certain number of its derivatives. In effect, the requirement that $\|g\|_S$ is bounded, which ensures that g is sufficiently smooth, counteracts the ill-posed inverse problem by substantially restricting the possible candidates for the inverse function $\hat{T}_z^{-1}(\hat{E}[y|\mathbf{z}])$.

To obtain a computationally feasible estimation procedure, Newey and Powell assume that the structural function g can be well approximated by a function that is linear in parameters,

$$g(\mathbf{x}) \cong g_J(\mathbf{x}) \equiv \sum_{j=1}^J \alpha_j \rho_j(\mathbf{x}), \quad (2.12)$$

where the $\{\rho_j\}$ are known, suitably chosen “basis functions” (like polynomials or trigonometric functions) that yield an arbitrarily close approximation to g as the number of terms J in the sum is increased. For this approximation, the corresponding approximation to the reduced form $E[y|\mathbf{z}] = E[g(\mathbf{x})|\mathbf{z}]$ is

$$E[g_J(\mathbf{x})|\mathbf{z}] = \sum_{j=1}^J \alpha_j E[\rho_j(\mathbf{x})|\mathbf{z}], \quad (2.13)$$

which is itself linear in the same parameters $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_J)'$, so that constrained least-squares regression of y on nonparametric estimates $\hat{E}[\rho_j(\mathbf{x})|\mathbf{z}]$ of the conditional means of the basis functions $\rho_j(\mathbf{x})$ can be used to estimate the coefficients of the approximate structural function g_J under the compactness restriction. Furthermore, the square of the Sobolev norm $\|g_J\|_S$ of the linear approximating function g_J can be written as a quadratic form,

$$\|g_J\|_S^2 = \frac{1}{2} \boldsymbol{\alpha}' \mathbf{S}_J \boldsymbol{\alpha}, \quad (2.14)$$

where the matrix \mathbf{S}_J is a known matrix constructed by using integrals involving the basis functions $\rho_j(\mathbf{x})$ and their derivatives. Minimization of the sum of squared differences between observed values of y and the estimators

$\{\hat{E}[\rho_j(\mathbf{x})|\mathbf{z}]\}$, subject to the restriction that the quadratic form in (2.14) is bounded above by a known constant B , yields an estimator of the coefficient vector α that is of “penalized least squares” form:

$$\hat{\alpha} = (\hat{\mathbf{R}}'\hat{\mathbf{R}} + \hat{\lambda}\mathbf{S}_J)^{-1}\hat{\mathbf{R}}'\mathbf{y}, \tag{2.15}$$

where $\hat{\mathbf{R}}$ is the matrix of the first-stage estimators $\{\hat{E}[\rho_j(\mathbf{x})|\mathbf{z}]\}$ for the sample, and $\hat{\lambda}$ is a Lagrange multiplier for the constraint $\alpha'\mathbf{S}_J\alpha \leq 2B$. Imposition of the compactness condition thus introduces an adjustment for multicollinearity (through the term $\hat{\lambda}\mathbf{S}_J$) to the otherwise-familiar 2SLS formula, to account for the near-singularity of the fitted values in the first stage, which is at the heart of the ill-posed inverse problem.

Newey and Powell (1989) give conditions under which the resulting estimator of the structural function g ,

$$\hat{g}(\mathbf{x}) \equiv \sum_{j=1}^J \hat{\alpha}_j \rho_j(\mathbf{x}), \tag{2.16}$$

is consistent; in addition to the compactness restrictions on the set of possible structural functions, these conditions restrict the form of the basis functions ρ_j and require that the number of terms J in the approximating function increase to infinity with the sample size. However, unlike some other nonparametric regression methods based on series approximations, J can be arbitrarily large for finite samples, and its value need not be related to the sample size to ensure convergence of bias and variance to zero, but instead is governed by the trade-off between numerical precision of the series approximation to g and computational convenience. The Newey and Powell manuscript does not discuss the rate of convergence or asymptotic distribution of \hat{g} , nor appropriate choice of the constraint constant B or, equivalently, the Lagrange multiplier $\hat{\lambda} = \hat{\lambda}(B)$, which acts as a smoothing parameter in the second-stage estimator.

A conceptually simple variant of this estimation strategy can be based on a finite-support approximation to the joint distribution of \mathbf{x} and \mathbf{z} . Once the data are binned into partitions with representative values $\{\xi_j\}$ and $\{\zeta_l\}$, the linear relation (2.10) between the vector \mathbf{g} of structural function values and the reduced-form vector π and transition matrix \mathbf{P} will hold approximately (with the approximation improving as the number of bins increases), and the components π and \mathbf{P} can be estimated using bin averages and frequencies. Though the estimated transition matrix $\hat{\mathbf{P}}$ may be nearly singular even if the approximating bins are chosen with $L \gg J$ for J large, the structural function vector \mathbf{g} could be estimated by ridge regression, that is,

$$\hat{\mathbf{g}} = (\hat{\mathbf{P}}'\hat{\mathbf{P}} + \lambda\mathbf{S})^{-1}\hat{\mathbf{P}}'\hat{\pi}, \tag{2.17}$$

for some nonsingular matrix \mathbf{S} and smoothing parameter λ that shrinks to zero as the sample size increases. This can be viewed as a histogram version of the series estimator proposed by Newey and Powell, which uses bin indicators

as the basis functions and kernel regression (with uniform kernels) in the first stage.

Darolles et al. (2000) take a different approach to the estimation of the structural function g in (2.8). They embed the problem of the solution of (2.8) in the mean-squared error minimization problem, defining the structural function g as

$$g(\cdot) \equiv \underset{\phi(\cdot)}{\operatorname{argmin}} E \left[\left\| E[y|\mathbf{z}] - \int \phi(\mathbf{x}) dF_{\mathbf{x}|\mathbf{z}} \right\|^2 \right], \tag{2.18}$$

and note that the “normal equations” for this functional minimization problem are of the form

$$\begin{aligned} E[E[y|\mathbf{z}|\mathbf{x}]] &\equiv \tau(\mathbf{x}) \\ &= E[E[g(\mathbf{x})|\mathbf{z}|\mathbf{x}]] \\ &\equiv T^*(g)(\mathbf{x}). \end{aligned} \tag{2.19}$$

That is, they transform (2.8) into another integral equation by taking conditional expectations of the reduced form $E[y|\mathbf{z}]$ given the original explanatory variables \mathbf{x} ; an advantage of this formulation is that the transformation $T^*(g) = \tau$ has the same argument (\mathbf{x}) as the structural function g , as is standard for the literature on the solution of Fredholm integral equations. Although the ill-posed inverse problem persists, a standard solution method for this formulation is Tikhonov regularization, which replaces the integral equation (2.19) with the approximate problem

$$\tau(\mathbf{x}) = T^*(g^\lambda)(\mathbf{x}) + \lambda g^\lambda(\mathbf{x}), \tag{2.20}$$

for λ being a small, nonnegative smoothing parameter. Although (2.20) reduces to (2.19) as $\lambda \rightarrow 0$, it is a Fredholm integral equation of the *second* kind, which is free of the ill-posed inverse problem, when λ is nonzero. Again approximating the solution function g^λ as a linear combination of basis functions,

$$g^\lambda(\mathbf{x}) \cong g_j^\lambda(\mathbf{x}) \equiv \sum_{j=1}^J \alpha_j^\lambda \rho_j(\mathbf{x}), \tag{2.21}$$

as in (2.12), a further approximation to the equation (2.19) is

$$E[E[y|\mathbf{z}|\mathbf{x}]] \cong \sum_{j=1}^J \alpha_j^\lambda \{E[E[\rho_j(\mathbf{x})|\mathbf{z}|\mathbf{x}]] + \lambda \rho_j(\mathbf{x})\}. \tag{2.22}$$

This suggests a two-stage strategy for estimation of the α^λ coefficients: In the first stage, obtain nonparametric estimators of the components $\tau(\mathbf{x}) = E[E[y|\mathbf{z}|\mathbf{x}]]$ and the doubly averaged basis functions $\{T^*(\rho_j)(\mathbf{x}) = E[E[\rho_j(\mathbf{x})|\mathbf{z}|\mathbf{x}]]\}$ using standard nonparametric estimation methods; then, in the second stage, regress the fitted $\hat{\tau}(\mathbf{x})$ on the constructed regressors $\{\hat{T}^*(\rho_j)(\mathbf{x}) - \lambda \rho_j(\mathbf{x}), j = 1, \dots, J\}$. The terms $\lambda \rho_j(\mathbf{x})$ serve to attenuate

the severe multicollinearity of the doubly averaged basis functions in this second-stage regression.

Darolles, Florens, and Renault take the basis functions ρ_j to be the eigenfunctions of the estimated double-averaging operator \hat{T}^* , that is, the solutions to the functional equations $\hat{T}^*(\rho_j) = \nu_j \rho_j$ for scalar eigenvalues ν_j , which simplifies both computation of the estimator and derivation of the asymptotic theory. For this choice of basis function, they derive the rate of convergence and asymptotic normal distribution of the estimator $\hat{g}^\lambda(\mathbf{x}) = \sum_{j=1}^J \hat{\alpha}_j^\lambda \rho_j(\mathbf{x})$ under certain regularity conditions. The rate of convergence is comparable to, but slower than, the rate of convergence of standard nonparametric estimators of the reduced form $E[y|\mathbf{z}]$, as a result of the bias introduced by approximating (2.19) by (2.20) for nonzero λ . Their manuscript also proposes an alternative estimator of g based on regression of $\hat{\tau}(\mathbf{x})$ on a subset of the doubly averaged basis functions with eigenvalues bounded away from zero, that is, $\{\hat{T}^*(\rho_j)(\mathbf{x}) : |\nu_j| > b_n\}$ for some $b_n \rightarrow 0$, and extends the identification analysis to permit the structural function g to be additively separable in its endogenous and exogenous components.

2.1.5. *Nonadditive Models*

Both the Newey–Powell and Darolles–Florens–Renault approaches exploit the additive separability of the error terms u in the structural function for y ; for models with nonadditive errors, that is, $H(\mathbf{x}, u) \neq g(\mathbf{x}) + u$, the IV assumption imposed in these papers apparently does not suffice to identify the ASF G of (1.10). Of course, it is clear that, for a nonadditive model, the conditional mean assumption (2.7) would not suffice to yield identification even for parametric structural functions, but imposition of the still-stronger assumption of independence of u and \mathbf{z} , denoted here as

$$u \perp \mathbf{z} \tag{2.23}$$

[which implies (2.7), and thus (1.3), provided u has finite expectation which can be normalized to zero], will still not suffice in general for identification of the ASF G . This is evident from inspection of the reduced form $E[y|\mathbf{z}]$ in the nonadditive case:

$$\begin{aligned} E[y|\mathbf{z}] &= E[H(\mathbf{x}, u)|\mathbf{z}] \\ &= \int H(\mathbf{x}, u) dF_{u,\mathbf{x}|\mathbf{z}} \\ &= \int \left[\int H(\mathbf{x}, u) dF_{u|\mathbf{x},\mathbf{z}} \right] dF_{\mathbf{x}|\mathbf{z}} \\ &\neq \int \left[\int H(\mathbf{x}, u) dF_u \right] dF_{\mathbf{x}|\mathbf{z}} \\ &= E[G(\mathbf{x})|\mathbf{z}]. \end{aligned} \tag{2.24}$$

That is, independence of u and \mathbf{z} does not imply independence of u and \mathbf{x} , \mathbf{z} , or even conditional independence of u and \mathbf{z} given \mathbf{x} . In the additive case

$H(\mathbf{x}, u) = g(\mathbf{x}) + u$, conditional expectations of each component require only the conditional distributions of u given \mathbf{z} and of \mathbf{x} given \mathbf{z} , and not the joint distribution of u, \mathbf{x} given \mathbf{z} , which is not identified under (2.23) without further conditions on the relation of \mathbf{x} to \mathbf{z} . Furthermore, because (2.24) also holds in general for any function of y , restriction (2.23) does not yield restrictions on the conditional distribution of the observable y given \mathbf{z} that might be used to identify the ASF G for general nonseparable structural functions H .

Of course, failure of the reduced-form relation (2.24) to identify the ASF G does not directly imply that it could not be identified by using some other functionals of the joint distribution of the observables y, \mathbf{x} , and \mathbf{z} , and it is difficult to provide a constructive proof of nonidentification of the ASF under the independence restriction (2.23) at this level of generality (i.e., with structural function H and the joint distribution of u, \mathbf{x} , and \mathbf{z} otherwise unspecified). Still, the general nonidentification result can be illustrated by considering a simple (slightly pathological) binary response example in which the ASF is unidentified under (2.23). Suppose H is binary, with y generated as

$$y = 1(x + u \geq 0), \tag{2.25}$$

for a scalar regressor x generated by a multiplicative model

$$x = z \cdot e, \tag{2.26}$$

for some scalar instrumental variable z with $\Pr\{z \geq 0\} = 1$. For this example, the ASF is

$$G(x) \equiv 1 - F_u(x), \tag{2.27}$$

with F_u the marginal CDF of u . Now suppose the errors u and e are generated as

$$\begin{aligned} u &\equiv \varepsilon \cdot \text{sgn}(\eta), \\ e &\equiv \eta \cdot \text{sgn}(\varepsilon), \end{aligned} \tag{2.28}$$

with ε, η , and z being mutually independently distributed, and

$$\text{sgn}(\eta) \equiv 1 - 2 \cdot 1(\eta < 0).$$

This model is pathological because $\text{sgn}(u) = \text{sgn}(x)$ by construction, and thus $y = 1(x \geq 0)$, whenever $z \neq 0$. Still, the independence condition (2.23) is satisfied, the dependent variable y is nonconstant if the support of e includes positive and negative values, and the endogenous regressor x has a conditional expectation that is a nontrivial function of the instrument z when $E[e] = E[\eta] \cdot (1 - 2 \cdot \Pr\{\varepsilon < 0\}) \neq 0$. Nevertheless, the ASF $G(x)$ is identified only at $x = 0$, when zero is in the support of z (with $G(0) = E[y|z = 0]$) and is not identified elsewhere.

This example demonstrates that, without further restrictions on the form of the structural function H and/or the conditional distribution of u given \mathbf{x}

and \mathbf{z} , the assumption of independence of the structural error u and the instruments \mathbf{z} is insufficient to identify the ASF in nonadditive models even when the endogenous regressors \mathbf{x} are not independent of the instruments \mathbf{z} . The nonidentification of the ASF here is a consequence of the dependence of the support of the endogenous variable x (either zero or the positive or negative half-line) on the realized value of the error term e . In general, if the nature of the endogeneity of \mathbf{x} is restricted by assuming

$$\mathbf{x} = h(\mathbf{z}, \mathbf{e}) \tag{2.29}$$

for some function h that is invertible in the error terms \mathbf{e} , that is,

$$\mathbf{e} = k(\mathbf{z}, \mathbf{x}), \tag{2.30}$$

and if the support of \mathbf{x} given \mathbf{e} is independent of \mathbf{e} . Imbens (2000) has shown how the ASF G is identified under the independence restriction (2.23) and these additional restrictions on the nature of the endogeneity of \mathbf{x} . Imbens' identification argument is based on the control function approach described in more detail later.

As an alternative to imposing such restrictions on the nature of the endogeneity of \mathbf{x} , additional structure on the form of the structural function H – such as invertibility of the structural function $H(\mathbf{x}, u)$ in the error term u – may yield more scope for identification of the ASF when the stochastic restrictions involve only the conditional distribution of u given \mathbf{z} . For example, suppose there is some invertible transformation $t(y)$ of y for which the additive form (1.11) holds:

$$t(y) = g(\mathbf{x}) + u, \tag{2.31}$$

where u satisfies the conditional mean restriction (2.7). If the transformation t were known, then estimation of g could proceed by using the Newey–Powell or Darolles–Florens–Renault approaches, and the ASF could be estimated by averaging the estimator of $H(\mathbf{x}, u) = t^{-1}(g(\mathbf{x}) + u)$ over the marginal empirical distribution of $u = t(y) - g(\mathbf{x})$. When t is unknown, the conditional mean restriction (2.7) yields an integral equation

$$\begin{aligned} 0 &= E[u|\mathbf{z}] \\ &= \int t(y)dF_{y|\mathbf{z}} - \int g(\mathbf{x})dF_{\mathbf{x}|\mathbf{z}}, \end{aligned} \tag{2.32}$$

which has multiple solutions, such as $t(y) \equiv g(\mathbf{x}) \equiv k$ for any constant k . Still, with appropriate normalizations on the unknown functions t and g , like $E[t(y)] = 0$ and $E[(t(y))^2] = 1$, it may be possible to extend the estimation approaches for the ill-posed inverse problem to joint estimation of t and g , though this may require overidentification, that is, $m = \dim(\mathbf{z}) > \dim(\mathbf{x}) = k$.

For the semiparametric problems (1.7), the parametric components β of the structural function may well be identified and consistently estimable, at the

parametric (root- N) rate, even if the ASF G is not identified. Ai and Chen (2000) propose sieve estimation of semiparametric models of the form (1.9) under the assumption that the instrumental variables \mathbf{z} are independent of the error terms u ; although the estimator of the infinite-dimensional nuisance function $h(\cdot)$ is not generally consistent with respect to the usual distance measures (such as integrated square differences), the corresponding estimator $\hat{\beta}$ of the parametric component β is root- N consistent and asymptotically normal under the regularity conditions they impose.

Lewbel (1998, 2000) considers the single-index generalized regression model (1.8), constructing consistent estimators of the index coefficients β under the assumption that one of the components of the explanatory variables \mathbf{x} , say x_1 , is continuously distributed and independent of the structural error u – and is thus a component of the set of instruments \mathbf{z} satisfying (2.23) given earlier. Provided there exists an exogenous variable x_1 that satisfies these conditions, Lewbel's approach permits a weaker stochastic restriction than independence of \mathbf{z} (including the special regressor \mathbf{x}_1) and u – namely, that u need only be independent of x_1 conditionally on the other components of \mathbf{x} and of the instrument vector \mathbf{z} . The conditional mean restriction $E[u|\mathbf{z}] = 0$ can also be weakened to the moment restriction $E[\mathbf{z}u] = 0$ in this setup. The conditional independence restriction is similar to the restrictions imposed for the control function methods described later. Nevertheless, even if the coefficient vector β were known a priori, the endogeneity of the remaining components of \mathbf{x} , and thus of the index $\mathbf{x}'\beta$, would yield the same difficulties in identification of the ASF G as in (2.24).

2.1.6. *Fitted-Value Methods*

When the conditional mean (2.7) or independence (2.23) restrictions of the IV approach does not suffice to identify the ASF G in a nonparametric model, the researcher can either abandon the ASF concept and focus on alternative summary measures that are identified, or impose stronger restrictions on the structural function or error distributions to achieve identification of the ASF. Though imposing additional restrictions on the structural function H (such as additivity of the error terms u) can clearly help achieve identifiability of G , such restrictions may be implausible when the range of y is restricted (e.g., when y is binary), and it is more customary to strengthen the restrictions on the conditional error distribution u given the instruments \mathbf{z} to identify the parameters of interest.

One alternative set of restrictions and estimation procedures are suggested by Theil's (1953) version of the 2SLS estimator for simultaneous equations. Defining the first-stage residuals \mathbf{v} as the difference between the regressors \mathbf{x} and their linear projections onto \mathbf{z} ,

$$\mathbf{v} \equiv \mathbf{x} - P[\mathbf{x}|\mathbf{z}] \equiv \mathbf{x} - \Pi'\mathbf{z}, \quad (2.33)$$

where Π is defined in (2.2), the condition (1.3), when combined with the

definition of \mathbf{v} and the linear structural function (1.1), yields the restriction

$$\begin{aligned} 0 &= E[P[\mathbf{x}|\mathbf{z}](u + \mathbf{v}'\beta)] \\ &= E[(\Pi'\mathbf{z})(y - (\mathbf{z}'\Pi)\beta), \end{aligned} \tag{2.34}$$

so that the structural coefficients β are the least-squares regression coefficients of the regression of the dependent variable y on the fitted values $\Pi'\mathbf{z}$ of the regressors \mathbf{x} . The sample analogue of the population regression coefficients of y on $\Pi'\mathbf{z}$ is Theil's version of 2SLS,

$$\widehat{\beta}_{2SLS} = (\widehat{\mathbf{X}}'\widehat{\mathbf{X}})^{-1}\widehat{\mathbf{X}}'\mathbf{y}, \tag{2.35}$$

where $\widehat{\mathbf{X}} = \mathbf{Z}\widehat{\Pi}$ is defined as in (2.4). The motivation for this form of 2SLS is the replacement of the endogenous regressors \mathbf{x} with that part of \mathbf{x} (its linear projection on \mathbf{z}) that is uncorrelated with the error u in the linear structural equation.

In a nonparametric setting, it is natural to define the first-stage residuals \mathbf{v} as deviations from conditional expectations, rather than linear projections:

$$\begin{aligned} \mathbf{v} &\equiv \mathbf{x} - E[\mathbf{x}|\mathbf{z}] \\ &\equiv \mathbf{x} - \Pi(\mathbf{z}). \end{aligned} \tag{2.36}$$

By construction, $E[\mathbf{v}|\mathbf{z}] = \mathbf{0}$, and, as for the IV approaches, the moment condition (1.3) would be replaced by the stronger conditional mean restriction (2.7), or the still-stronger assumption of independence of the errors and the instruments,

$$(u, \mathbf{v}) \perp\!\!\!\perp \mathbf{z}. \tag{2.37}$$

A nonparametric generalization of Theil's version of 2SLS would estimate $E[\mathbf{x}|\mathbf{z}] = \Pi(\mathbf{z})$ by a suitable nonparametric method in the first stage, and then substitute the fitted values $\widehat{\Pi}(\mathbf{z})$ into the structural function in a second-stage estimation procedure. As noted by Amemiya (1974), though, substitution of fitted values into nonlinear structural functions generally yields inconsistent estimates of the structural parameters, even in parametric problems; estimation methods that use substitution of fitted values into the structural function rely heavily on linearity of the regression function, so that the model can be written in terms of a composite error $u + \mathbf{v}'\beta$ with similar stochastic properties to the structural error u . For the general (nonadditive) structural function H of (1.4), substitution of the reduced form into the structural function yields $y = H(\Pi(\mathbf{z}) + \mathbf{v}, u)$, and, analogously to (2.24), the reduced form for y bears no obvious relation to the ASF G under condition (2.37). Even when the structural function H is additive, $H(\mathbf{x}, u) = g(\mathbf{x}) + u$, the reduced form for y can be written as

$$E[y|\mathbf{z}] = E[\mathbf{y}|\Pi(\mathbf{z})] = \int g(\Pi(\mathbf{z}) + \mathbf{v})dF_{\mathbf{v}},$$

so insertion of the first-stage equation for \mathbf{x} into the structural function yields an ill-posed inverse relation between the reduced form for y and the ASF g . Thus the fitted-value approach inherits similar limitations to the IV approach, though it may simplify the resulting integral equation, which depends on $\Pi(\mathbf{z})$ rather than \mathbf{z} and involves the marginal distribution of \mathbf{v} rather than the conditional distribution of \mathbf{x} given \mathbf{z} .

Of course, for structural equations in which the conditional expectations $E[\mathbf{x}|\mathbf{z}] = \Pi(\mathbf{z})$ are the “right” explanatory variables, the fitted-value estimation method, using nonparametric estimates of $\Pi(\mathbf{z})$, is an obvious way to proceed.³ And, as was true for the IV approach, for semiparametric problems, consistent estimation of the parametric component β may be feasible even when the ASF G is not identified. For example, for the generalized regression model $y = h(\mathbf{x}'\beta, u)$ of (1.8), the reduced form for y given \mathbf{z} is of single-index form when the errors u, \mathbf{v} are independent of the instrument vector \mathbf{z} : $E[y|\mathbf{z}] \equiv G^*(\Pi(\mathbf{z})'\beta)$ for some function G^* , so that a nonparametric estimator of the first-stage regression function $\Pi(\mathbf{z})$ can be combined with a standard estimation method for single-index regression models.⁴

2.2. Control Function Methods

2.2.1. The Linear Model

Although insertion of the fitted values from the first-stage nonparametric regression of \mathbf{x} on \mathbf{z} is not generally helpful in identification and estimation of the ASF G , alternative assumptions and procedures involving the use of the residuals \mathbf{v} from this first-stage regression to control for endogeneity of the regressors \mathbf{x} do yield identification of the ASF even for the general, nonadditive structural function (1.4). This control function approach has its antecedent in another algebraically equivalent interpretation of the 2SLS estimator $\hat{\beta}_{2SLS}$ as the coefficients on \mathbf{x} in a least-squares regression of y on \mathbf{x} and the residuals $\hat{\mathbf{v}}$ from a linear regression of \mathbf{x} on \mathbf{z} :

$$\begin{pmatrix} \hat{\beta}_{2SLS} \\ \hat{\rho}_{2SLS} \end{pmatrix} = (\widehat{\mathbf{W}}'\widehat{\mathbf{W}})^{-1}\widehat{\mathbf{W}}'\mathbf{y}, \tag{2.38}$$

where

$$\widehat{\mathbf{W}} = [\mathbf{X} \widehat{\mathbf{V}}] \text{ and } \widehat{\mathbf{V}} = \mathbf{X} - \widehat{\mathbf{X}} = \mathbf{X} - \mathbf{Z}\widehat{\Pi},$$

³ Some asymptotic results for such estimators were given by Ahn and Manski (1993) and by Ahn (1995), which showed, for example, that the rate of convergence of the restricted reduced-form estimator $\hat{E}[y|\hat{\Pi}(\mathbf{z})]$ is the smaller of the rate of convergence of $\hat{\Pi}$ to Π and of $\hat{E}[y|\Pi(\mathbf{z})]$ to $E[y|\Pi(\mathbf{z})]$ where Π is known.

⁴ See, for example, Ichimura (1993) and others described in Horowitz (1993) and Powell (1994).

and where $\hat{\rho}_{2SLS}$ are the coefficients on the first-stage residuals \hat{V} .⁵ This construction exploits another consequence of the moment condition $E(\mathbf{z}u) = \mathbf{0}$, that

$$\begin{aligned} P[u|\mathbf{x}, \mathbf{z}] &= P[u|\Pi'\mathbf{z} + \mathbf{v}, \mathbf{z}] \\ &= P[u|\mathbf{v}, \mathbf{z}] \\ &= P[u|\mathbf{v}] \\ &\equiv \mathbf{v}'\boldsymbol{\rho} \end{aligned} \tag{2.39}$$

for some coefficient vector $\boldsymbol{\rho}$; the third equality follows from the orthogonality of both error terms u and \mathbf{v} with \mathbf{z} . Thus, this particular linear combination of the first-stage errors \mathbf{v} is a function that controls for the endogeneity of the regressors \mathbf{x} . It also follows from this formulation that

$$P[u|\mathbf{x}, \mathbf{z}] = P[u|\mathbf{x}, \mathbf{v}], \tag{2.40}$$

which can be used as the basis for a test of overidentification if $\dim(\mathbf{z}) > \dim(\mathbf{v}) = \dim(\mathbf{x})$.

This approach treats endogeneity as an omitted variable problem, where the inclusion of estimates of the first-stage errors \mathbf{v} (the part of the regressors \mathbf{x} that is correlated with \mathbf{z}) as a covariate corrects the inconsistency of least-squares regression of y on \mathbf{x} , in the same way that the Heckman (1979) two-step estimator corrects for selectivity bias through introduction of an appropriately estimated regressor derived from a parametric form for the error distribution. The control function approach to correct for endogeneity has been extended to nonlinear parametric models by Blundell and Smith (1986, 1989), who show how introduction of first-stage residuals into single-equation Probit or Tobit procedures yields consistent estimators of the underlying regression coefficients when some of the regressors are endogenous.

2.2.2. *Extensions to Additive Nonparametric Models*

Application of the control function approach to nonparametric and semiparametric settings requires strengthening of the linear projection restrictions (2.39) and (2.40) to conditional mean restrictions,

$$\begin{aligned} E(u|\mathbf{x}, \mathbf{z}) &= E(u|\mathbf{x}, \mathbf{v}) \\ &= E(u|\mathbf{v}), \end{aligned} \tag{2.41}$$

⁵ It has been difficult to locate a definitive early reference to the control function version of 2SLS. Dhrymes (1970, equation 4.3.57) shows that the 2SLS coefficients can be obtained by a least-squares regression of y on $\hat{\mathbf{X}}$ and \hat{V} , while Telser (1964) shows how the seemingly unrelated regressions model can be estimated by using residuals from other equations as regressors in a particular equation of interest. Heckman (1978) references this paper in his comprehensive discussion of estimating simultaneous models with discrete endogenous variables.

or, for nonadditive models, the stronger conditional independence assumptions

$$\begin{aligned}
 u|\mathbf{x}, \mathbf{z} &\sim u|\mathbf{x}, \mathbf{v} \\
 &\sim u|\mathbf{v}.
 \end{aligned}
 \tag{2.42}$$

While the control variates⁶ \mathbf{v} are typically taken to be deviations of \mathbf{x} from its conditional mean $E[\mathbf{x}|\mathbf{z}]$, as in (2.36), this is not required; more generally, \mathbf{v} can be any function of the observable random vectors

$$\mathbf{v} = \nu(y, \mathbf{x}, \mathbf{z})
 \tag{2.43}$$

that is identified and consistently estimable, provided (2.41) or (2.42) holds for this formulation (and \mathbf{v} is not a nontrivial function of \mathbf{x}). This permits the control function approach to be applied to some systems of nonlinear simultaneous equations for which the appropriate reduced form is difficult or impossible to derive, such as the coherent simultaneous binary response models considered in the paragraphs that follow. In comparison to the identifying assumptions (2.7) or (2.23) for the IV approaches, the corresponding assumptions (2.41) or (2.42) for the control function approach are no more nor less general. Both sets of assumptions are implied by the independence restriction (2.37), which may be plausible for certain applications, and which permits a choice between the two estimation approaches.

Estimation of the ASF g in the additive model (1.11) under the conditional mean exclusion restriction (2.41) was considered by Newey, Powell, and Vella (1999) and by Ng and Pinkse (1995) and Pinkse (2000). Applications can be found in Blundell, Browning, and Crawford (2000) and Blundell and Duncan (1998), for example. When the errors are additive, substitution of $u = y - g(\mathbf{x})$ into (2.41) yields a generalized additive regression form for y :

$$\begin{aligned}
 E[y|\mathbf{x}, \mathbf{v}] &= E[(g(\mathbf{x}) + u)|\mathbf{x}, \mathbf{v}] \\
 &= g(\mathbf{x}) + \eta(\mathbf{v}),
 \end{aligned}
 \tag{2.44}$$

for some control function η . With a suitable normalization, say, $E[\eta(\mathbf{v})] = 0$, the ASF g can be estimated using standard additive nonparametric regression methods applied to the regression of y on \mathbf{x} and the first-stage residuals $\hat{\mathbf{v}}$. Both Newey et al. and Ng and Pinkse propose an estimation of (2.44) using a series approximation for the functions g and η :

$$g(\mathbf{x}) + \eta(\mathbf{v}) \cong \sum_{j=1}^J \alpha_j \rho_j(\mathbf{x}) + \sum_{l=1}^L \gamma_l \psi_l(\mathbf{v}),
 \tag{2.45}$$

where $\{\rho_j\}$ and $\{\psi_l\}$ are appropriate basis functions, and where the number of terms J and L for each approximating series increases to infinity as the sample size increases. The second stage using this series approximation is a least-squares regression of y on the basis functions $\{\rho_j(\mathbf{x})\}$ and $\{\psi_l(\hat{\mathbf{v}})\}$, and

⁶ This use of the term “control variate” is logically distinct from, but similar in spirit to, its use in the literature on Monte Carlo methods; see, for example, Hammersley and Handscomb (1964), Section 5.5.

the estimator of g is given by (2.16), assuming $\rho_1(\mathbf{x}) \equiv 1$ – which enforces the normalization $E[\eta(\mathbf{v})] = 0$. The cited manuscripts give regularity conditions for consistency of the estimator \hat{g} , and derive its rate of convergence, which is the same as the rate for a direct nonparametric regression of y on the regressors \mathbf{x} ; Newey et al. also give conditions under which the estimator $\hat{g}(\mathbf{x})$ is asymptotically normal.

2.2.3. *Nonadditive Models*

Unlike the IV approach, a stronger independence condition (2.42) of the conditional mean exclusion restrictions (2.41) for the control function approach does lead to a consistent estimator of the ASF G when the structural function H is nonadditive, as in (1.4). Blundell and Powell (1999) point out how averaging the conditional mean of y given \mathbf{x} and \mathbf{v} over the marginal distribution of the first-stage errors \mathbf{v} gives the ASF G for the nonadditive model. Because

$$\begin{aligned}
 E[y|\mathbf{x}, \mathbf{v}] &= E[H(\mathbf{x}, u)|\mathbf{x}, \mathbf{v}] \\
 &= \int H(\mathbf{x}, u) dF_{u|\mathbf{x}, \mathbf{v}} \\
 &= \int H(\mathbf{x}, u) dF_{u|\mathbf{v}} \\
 &\equiv H^*(\mathbf{x}, \mathbf{v}),
 \end{aligned}
 \tag{2.46}$$

under the strong exclusion restriction (2.42), it follows that the generalized control function H^* can be integrated over the marginal distribution of the (observable) reduced-form errors to obtain the ASF:

$$\begin{aligned}
 \int H^*(\mathbf{x}, \mathbf{v}) dF_{\mathbf{v}} &= \int \left[\int H(\mathbf{x}, u) dF_{u|\mathbf{v}} \right] dF_{\mathbf{v}} \\
 &= \int H(\mathbf{x}, u) dF_u \\
 &\equiv G(\mathbf{x}).
 \end{aligned}
 \tag{2.47}$$

In a sense, the control function exclusion restriction (2.42) permits replacement of the unidentified structural errors u with the identified control variable \mathbf{v} through iterated expectations, so that averaging the structural function H over the marginal distribution of the structural errors u is equivalent to averaging the (identified) intermediate regression function H^* over the marginal distribution of \mathbf{v} . The intermediate structural function H^* is a nonadditive generalization of the previous control function $\eta(\mathbf{v}) = E[u|\mathbf{v}]$ for additive models.

For the binary response example described in (2.25) through (2.28) herein, the first-stage residuals are of the form

$$\begin{aligned}
 v &= z \cdot (e - E[e]) \\
 &= z \cdot [\eta \cdot \text{sgn}(\varepsilon) - E[\eta] \cdot (\Pr\{\varepsilon \geq 0\} - \Pr\{\varepsilon < 0\})],
 \end{aligned}
 \tag{2.48}$$

and the conditional exclusion restriction (2.42) holds only if the structural error u is degenerate, that is, $u = 0$ with probability one. In this case, x is exogenous, and the intermediate structural function reduces to

$$\begin{aligned} H^*(x, v) &= E[1(x \geq 0)|x, v] \\ &= 1(x \geq 0), \end{aligned} \quad (2.49)$$

which trivially integrates to the true ASF $G(x) = 1(x \geq 0)$. Thus, imposition of the additional restriction (2.42) serves to identify the ASF here. An alternative control variate to the first-stage errors v would be

$$v^* \equiv \text{sgn}(x) = \text{sgn}(u), \quad (2.50)$$

which satisfies (2.42) when the structural error u is nondegenerate. Because v^* is functionally related to x , the intermediate structural function $H^*(x, v^*) = E[y|x, v^*]$ is not identified with this control variate.

Translating the theoretical formulation (2.47) to a sampling context leads to a “partial mean” (Newey, 1994b) or “marginal integration” (Linton and Nielson, 1995 and Tjostheim and Auestad, 1996) estimator for the ASF G under the conditional independence restrictions (2.42) of the control function approach. That is, after obtaining a first-stage estimator \hat{v} of the control variate v , which would be the residual from a nonparametric regression of \mathbf{x} on \mathbf{z} when v is defined by (2.36), one can obtain an estimator \hat{H}^* of the function H^* in (2.46) by a nonparametric regression of y on \mathbf{x} and \hat{v} . A final estimation step would average H^* over the observed values of \hat{v} ,

$$\hat{G}(\mathbf{x}) = \int \hat{E}(y|\mathbf{x}, v) d\hat{F}_{\hat{v}} \equiv \int \hat{H}^*(\mathbf{x}, v) d\hat{F}_{\hat{v}}, \quad (2.51)$$

where $\hat{F}_{\hat{v}}$ is the empirical CDF of the residuals \hat{v} . Alternatively, if v were assumed to be continuously distributed with density f_v , the ASF G could be estimated by integrating \hat{H}^* over a nonparametric estimator $\hat{f}_{\hat{v}}$ of f_v ,

$$\tilde{G}(\mathbf{x}) = \int \hat{E}(y|\mathbf{x}, v) \hat{f}_{\hat{v}} d\mathbf{v}. \quad (2.52)$$

Either the partial mean (2.51) or marginal integration (2.52) is an alternative to the series estimators based on (2.45) for the additive structural function (1.11), but this latter approach is not applicable to general structural functions, because the intermediate regression function H^* need not be additive in its \mathbf{x} and v components.

2.2.4. Support Restrictions

The identification requirements for the ASF G are simpler to interpret for the control function approach than the corresponding conditions for identification using the IV approaches, because they are conditions for identification of the nonparametric regression function H^* , which is based on observable random vectors. For example, in order for the ASF $G(\mathbf{x})$ to be identified from the

partial-mean formulation (2.47) for a particular value \mathbf{x}_0 of \mathbf{x} , the support of the conditional distribution of \mathbf{v} given $\mathbf{x} = \mathbf{x}_0$ must be the same as the support of the marginal distribution of \mathbf{v} ; otherwise, the regression function $H^*(\mathbf{x}_0, \mathbf{v})$ will not be well defined for all \mathbf{v} , nor will the integral of H^* over the marginal distribution of \mathbf{v} . For those components of \mathbf{x} that are exogenous, that is, those components of \mathbf{x} that are also components of the instrument vector \mathbf{z} , the corresponding components of \mathbf{v} are identically zero, both conditionally on \mathbf{x} and marginally, so this support requirement is automatically satisfied. However, for the endogenous components of \mathbf{x} , the fact that \mathbf{x} and \mathbf{v} are functionally related through the first-stage relation $\mathbf{v} = \mathbf{x} - \Pi(\mathbf{z})$, or the more general form (2.43), means that the support condition generally requires that \mathbf{v} , and thus \mathbf{x} , must be continuously distributed, with unbounded support (conditionally on the instruments \mathbf{z}) if its marginal distribution is nondegenerate. Similar reasoning, imposing the requirement that $H^*(\mathbf{x}, \mathbf{v})$ be well defined on the support of \mathbf{x} for all possible \mathbf{v} , and noting that

$$E[y|\mathbf{x}, \mathbf{v}] = E[y|\Pi(\mathbf{z}), \mathbf{v}] \tag{2.53}$$

implies that the first-stage regression function $\Pi(\mathbf{z}) = E[\mathbf{x}|\mathbf{z}]$ must also be continuously distributed, with full-dimensional support, for the nondegenerate components of $\mathbf{v} = \mathbf{x} - E[\mathbf{x}|\mathbf{z}]$.

The requirement that the endogenous components of \mathbf{x} be continuously distributed is the most important limitation of the applicability of the control function approach to estimation of nonparametric and semiparametric models with endogenous regressors. When the structural equations for the endogenous regressors have limited or qualitative dependent variables, the lack of an invertible representation (2.43) of the underlying error terms for such models generally makes it impossible to construct a control variate \mathbf{v} for which the conditional independence restrictions (2.41) or (2.42) are plausible. The requirement of an additive (or invertible) first-stage relation for the regressors \mathbf{x} in the control function approach is comparable with the requirement of an additive (or invertible) structural function H for the identification of the ASF G using the IV approach.

In contrast, when the errors are invertible and the support of \mathbf{x} does not depend on them, Imbens (2000) has shown how a particular control variate – the conditional cumulative distribution of \mathbf{x} given \mathbf{z} , evaluated at the observed values of those random variables – can be used to identify the ASF G under the independence restriction (2.23), because it satisfies the conditional independence restriction (2.42). Also, when it is applicable, the control function approach to estimation with endogenous regressors is compatible with other estimation strategies that use control function methods to adjust for different sources of specification bias, such as selection bias (e.g., Ahn and Powell, 1993, Das, Newey, and Vella, 1998, Heckman, 1978, Heckman and Robb, 1985, Honoré and Powell, 1997, and Vytlačil, 1999) or correlated random effects in panel data models (Altonji and Matzkin, 1997).

3. BINARY RESPONSE LINEAR INDEX MODELS

3.1. Model Specification and Estimation Approach

Although the control function approach adopted by Blundell and Powell (1999) applies to fully nonparametric problems (as described herein), their discussion focuses attention on estimation of the parametric and nonparametric components of a particular semiparametric single-index model (1.8), the binary response model with linear index,

$$y = 1\{\mathbf{x}'\beta + u > 0\}, \quad (3.1)$$

where the conditional independence assumption (2.42) is assumed to hold for $\mathbf{v} \equiv \mathbf{x} - E[\mathbf{x}|\mathbf{z}]$. For this linear binary response model, the ASF G is the marginal CDF of $-u$ evaluated at the linear index $\mathbf{x}'\beta$,

$$G(\mathbf{x}) = F_{-u}(\mathbf{x}'\beta), \quad (3.2)$$

which is interpreted as the counterfactual conditional probability that $y = 1$ given \mathbf{x} , if \mathbf{x} were exogenous, that is, if the conditional distribution of u given \mathbf{x} were assumed to be identical to its true marginal distribution. Under the conditional independence restriction (2.42), the intermediate regression function H^* is the conditional CDF of $-u$ given \mathbf{v} , again evaluated at $\mathbf{x}'\beta$:

$$H^*(\mathbf{x}, \mathbf{v}) = F_{-u|\mathbf{v}}(\mathbf{x}'\beta|\mathbf{v}). \quad (3.3)$$

Given a random sample of observations on y , \mathbf{x} , and \mathbf{z} from the model (3.1) under the exclusion restriction (2.42), the estimation approach proposed by Blundell and Powell for the parameters of interest in this model follows three main steps. The first step uses nonparametric regression methods – specifically, the Nadaraya–Watson kernel regression estimator – to estimate the error term \mathbf{v} in the reduced form, as well as the unrestricted conditional mean of y given \mathbf{x} and \mathbf{v} ,

$$E[y|\mathbf{x}, \mathbf{v}] \equiv H^*(\mathbf{w}), \quad (3.4)$$

where \mathbf{w} is the $1 \times (k + q)$ vector

$$\mathbf{w} = (\mathbf{x}', \mathbf{v}')'. \quad (3.5)$$

This step can be viewed as an intermediate structural estimation step, which imposes the first exclusion restriction of (2.42) but not the second. The remaining estimation steps use semiparametric “pairwise differencing” or “matching” methods to obtain an estimator of the index coefficients β , followed by partial-mean estimation of the ASF G .

3.1.1. *The Semiparametric Estimator of the Index Coefficients*

After using kernel regression methods to obtain an estimator $\hat{H}^*(\mathbf{x}, \hat{\mathbf{v}}) = \hat{E}[y|\mathbf{x}, \hat{\mathbf{v}}]$ of the intermediate regression function H^* , Blundell and Powell use a semiparametric estimation method to extract an estimator of β from the relation

$$\begin{aligned} H^*(\mathbf{w}) &= E[y|\mathbf{x}, \mathbf{v}] \\ &= E[y|\mathbf{x}'\beta, \mathbf{v}] \\ &\equiv \Gamma(\mathbf{x}'\beta, \mathbf{v}), \end{aligned} \tag{3.6}$$

which is a consequence of the single-index form of the binary response model (3.1). Although a number of standard methods for estimation of the coefficients of the single-index regression model $E[y|\mathbf{x}] = \Gamma(\mathbf{x}'\beta)$ could be extended to the multi-index model⁷ (3.6), the particular estimator of β adopted by Blundell and Powell (1999) is an adaptation of a method proposed by Ahn, Ichimura, and Powell (1996), which imposes additional regularity conditions of both continuity and monotonicity of $H^*(\lambda, \mathbf{v}) = E[y|\mathbf{x}'\beta = \lambda, \mathbf{v}]$ in its first argument. These conditions follow from the assumption that u is continuously distributed, with support on the entire real line, conditional on \mathbf{v} , because of (3.3).

Because the structural index model is related to the conditional mean of y given $\mathbf{w}' = (\mathbf{x}', \mathbf{v}')$ by the relation

$$H^*(\mathbf{w}) = \Gamma(\mathbf{x}'\beta_0, \mathbf{v}), \tag{3.7}$$

invertibility of $\Gamma(\cdot)$ in its first argument implies

$$\mathbf{x}'\beta_0 - \psi(g(\mathbf{w}), \mathbf{v}) = 0, \text{ (w.p.1),} \tag{3.8}$$

where

$$\psi(\cdot, \mathbf{v}) \equiv \Gamma^{-1}(\cdot, \mathbf{v}), \tag{3.9}$$

that is, $\Gamma(\psi(g, \mathbf{v}), \mathbf{v}) \equiv g$. So, if two observations (with subscripts i and j) have identical conditional means, that is, $H^*(\mathbf{w}_i) = H^*(\mathbf{w}_j)$, and identical reduced-form error terms ($\mathbf{v}_i = \mathbf{v}_j$), it follows from the assumed invertibility of Γ that their indices $\mathbf{x}_i\beta_0$ and $\mathbf{x}_j\beta_0$ are also identical:

$$\begin{aligned} (\mathbf{x}_i - \mathbf{x}_j)\beta_0 &= \psi(g(\mathbf{w}_i), \mathbf{v}_i) - \psi(g(\mathbf{w}_j), \mathbf{v}_j) = \mathbf{0} \\ \text{if } g(\mathbf{w}_i) &= g(\mathbf{w}_j), \quad \mathbf{v}_i = \mathbf{v}_j. \end{aligned} \tag{3.10}$$

For any nonnegative function $\omega(\mathbf{w}_i, \mathbf{w}_j)$ of the conditioning variables \mathbf{w}_i and \mathbf{w}_j , it follows that

$$\begin{aligned} 0 &= E[\omega(\mathbf{w}_i, \mathbf{w}_j) \cdot ((\mathbf{x}_i - \mathbf{x}_j)\beta_0)^2 \mid g(\mathbf{w}_i) = g(\mathbf{w}_j), \mathbf{v}_i = \mathbf{v}_j] \\ &\equiv \beta_0' \Sigma_w \beta_0, \end{aligned} \tag{3.11}$$

⁷ See Horowitz (1993) and Powell (1994).

where

$$\Sigma_w \equiv E[\omega(\mathbf{w}_i, \mathbf{w}_j) \cdot (\mathbf{x}_i - \mathbf{x}_j)'(\mathbf{x}_i - \mathbf{x}_j) \mid g(\mathbf{w}_i) = g(\mathbf{w}_j), \mathbf{v}_i = \mathbf{v}_j]. \tag{3.12}$$

That is, the nonnegative-definite matrix Σ_w is singular, and, under the identifying assumption that Σ_w has rank $k - 1 = \dim(\mathbf{w}) - 1$ – which requires that any nontrivial linear combination $(\mathbf{x}_i - \mathbf{x}_j)a$ of the difference in regressors has nonzero variance when $a \neq 0, a \neq \beta_0, \mathbf{v}_i = \mathbf{v}_j$, and $(\mathbf{x}_i - \mathbf{x}_j)\beta_0 = 0$ – the unknown parameter vector β_0 is the eigenvector (with an appropriate normalization) corresponding to the unique zero eigenvalue of Σ_w .

Given the preliminary nonparametric estimators $\hat{\mathbf{v}}_i$ and $\hat{g}(\tilde{\mathbf{w}}_i)$ of \mathbf{v}_i and $g(\mathbf{w}_i)$ defined herein, and assuming smoothness (continuity and differentiability) of the inverse function $\psi(\cdot)$ in (3.9), one can obtain a consistent estimator of Σ_w for a particular weighting function $\omega(\mathbf{w}_i, \mathbf{w}_j)$ by a pairwise differencing or matching approach, which takes a weighted average of outer products of the differences $(\mathbf{x}_i - \mathbf{x}_j)$ in the $\binom{n}{2}$ distinct pairs of regressors, with weights that tend to zero as the magnitudes of the differences $|\hat{g}(\tilde{\mathbf{w}}_i) - \hat{g}(\tilde{\mathbf{w}}_j)|$ and $|\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_j|$ increase. The details of this semiparametric estimation procedure for β are developed in Blundell and Powell (1999), who demonstrate consistency of the resulting estimator $\hat{\beta}$ and characterize the form of its asymptotic (normal) distribution.

3.1.2. *The Partial-Mean Estimator of the ASF*

Once the consistent estimator $\hat{\beta}$ of β is obtained, the remaining parameter of interest for this model is $G(\mathbf{x}'\beta)$, the marginal probability that $y_{1i} = 1$ given an exogenous \mathbf{x} . The conditional cumulative distribution function $F_{-u|\mathbf{v}}(\mathbf{x}'\beta \mid \mathbf{v}) \equiv \Gamma(\mathbf{x}'\beta, \mathbf{v})$ is first estimated by using a kernel regression estimator $\hat{E}[y|\mathbf{x}'\hat{\beta}, \hat{\mathbf{v}}]$; the Blundell–Powell approach then estimates $G(\bar{\lambda})$ from the sample average of $\hat{\Gamma}(\bar{\lambda}, \hat{\mathbf{v}}_i)$,

$$\hat{G}(\bar{\lambda}) = \sum_{i=1}^n \hat{\Gamma}(\bar{\lambda}, \hat{\mathbf{v}}_i)\tau_i, \tag{3.13}$$

where τ_i is some “trimming” term that downweights observations for which Γ is imprecisely estimated. Consistency of this approach requires adapting the arguments in Newey (1994b) and Linton and Nielson (1995) for the case of the averaging over the estimated residual $\hat{\mathbf{v}}_i$.

4. COHERENCY AND ALTERNATIVE SIMULTANEOUS REPRESENTATIONS

One interpretation of the linear index binary response model described in Section 3 is as the “triangular form” of some underlying joint decision problem. For simplicity, suppose the explanatory variables \mathbf{x} can be partitioned as

$$\mathbf{x}' = (\mathbf{z}'_1, y_2), \tag{4.1}$$

where y_2 is a single continuously distributed endogenous regressor; also, suppose the instrument vector \mathbf{z} is also partitioned into subvectors corresponding to the “included” and “excluded” components of \mathbf{x} ,

$$\mathbf{z}' = (\mathbf{z}'_1, \mathbf{z}'_2). \tag{4.2}$$

Then, for a random sample $\{y_i, \mathbf{x}_i, \mathbf{z}_i\}_{i=1}^n$ of observations on $y \equiv y_1$, \mathbf{x} , and \mathbf{z} , we can express the model (3.1) as

$$y_{1i} = 1\{y_{1i}^* > 0\}, \tag{4.3}$$

for a latent dependent variable y_{1i}^* of the form

$$y_{1i}^* = \mathbf{z}'_{1i}\beta_1 + y_{2i}\beta_2 + u_i. \tag{4.4}$$

If the simultaneity between y_{2i} and y_{1i} can be written in terms of a structural equation for y_{2i} in terms of the *latent* variable y_{1i}^* , that is,

$$y_{2i} = \mathbf{z}'_{2i}\gamma_1 + y_{1i}^*\gamma_2 + \varepsilon_i, \tag{4.5}$$

for some error term ε_i , then substitution of (4.4) in (4.5) delivers the first-stage regression model

$$y_{2i} = \mathbf{z}'_i\mathbf{\Pi} + \mathbf{v}_i \tag{4.6}$$

for some coefficient matrix $\mathbf{\Pi}$. This triangular structure has y_2 first being determined by \mathbf{z} and the error terms \mathbf{v} , while y_1 is then determined by y_2 , \mathbf{z} , and the structural error u .

In some economic applications, however, joint decision making may be in terms of the observed outcomes rather than latent outcomes implicit in (4.4) and (4.5). For example, consider the joint determination of savings (or consumption) and labor market participation. Let y_1 denote the discrete work decision and let y_2 denote other income including savings. Suppose that work involves a fixed cost α_1 . In this case the structural relationship for other income (y_{2i}) will depend on the discrete employment decision (y_{1i}), *not* the latent variable (y_{1i}^*). It will not be possible, therefore, to solve explicitly the reduced form for y_2 . Note that, for theoretical consistency, the fixed cost α_2 will also have to be subtracted from the income (or consumption) variable in the participation equation for those in work. As a result, for those who are employed, other income is defined net of fixed costs,

$$\tilde{y}_{2i} \equiv y_{2i} - \alpha_2 y_{1i}. \tag{4.7}$$

We may therefore wish to replace (4.5) with a model incorporating feedback between the *observed* dependent variables y_1 and y_2 ,

$$y_{2i} = \mathbf{z}'_{2i}\gamma_1 + y_{1i}\alpha_2 + \varepsilon_i; \tag{4.8}$$

that is, the realization $y_1 = 1$ results in a discrete shift $y_{1i}\alpha_2$ in other income. Because of the nonlinearity in the binary response rule (4.3), there is no explicit reduced form for this system. Indeed, Heckman (1978), in his extensive analysis

of simultaneous models with dummy endogenous variables, shows that (4.3), (4.4), and (4.8) are only a statistically “coherent” system, that is, one that processes a unique (if not explicit) reduced form, when $\gamma_2 = 0$.⁸

To provide a fully simultaneous system in terms of observed outcomes, and one that is also coherent, Heckman (1978) further shows that there must be a structural jump in the equation for y_{1i}^* ,

$$y_{1i}^* = y_{1i}\alpha_1 + \mathbf{z}'_{1i}\beta_1 + y_{2i}\beta_2 + u_i, \tag{4.9}$$

with the added restriction

$$\alpha_1 + \alpha_2\beta_2 = 0. \tag{4.10}$$

Heckman (1978) labels this the “principle assumption.” To derive this condition, notice that from (4.3), (4.8), and (4.9), we can write

$$y_{1i}^* = 1\{y_{1i}^* > 0\}(\alpha_1 + \alpha_2\beta_2) + \mathbf{z}'_{1i}\beta_1 + \mathbf{z}'_{2i}\gamma_1\beta_2 + u_i + \varepsilon_i\beta_2, \tag{4.11}$$

or

$$y_{1i}^* \leq 0 \Leftrightarrow 1\{y_{1i}^* > 0\}(\alpha_1 + \alpha_2\beta_2) + \mathbf{z}'_{1i}\beta_1 + \mathbf{z}'_{2i}\gamma_1\beta_2 + u_i + \varepsilon_i\beta_2 \leq 0. \tag{4.12}$$

Thus, for a consistent probability model with general distributions for the unobservables and exogenous covariates, we require the coherency condition (4.10).

Substituting for α_1 from (4.10) into (4.9), we have

$$y_{1i}^* = (y_{2i} - y_{1i}\alpha_2)\beta_2 + \mathbf{z}'_{1i}\beta_1 + u_i. \tag{4.13}$$

Note that this adjustment to y_{2i} , which guarantees statistical coherency, is *identical* to the condition for theoretical consistency in the fixed-cost model in which fixed cost α_2 is removed from other income for those who participate, as in (4.7).

Blundell and Smith (1994) derive a control function like the estimator for this setup under joint normality assumptions.⁹ However, the semiparametric approach developed in the previous section naturally extends to this case. Noting that $\tilde{y}_{2i} \equiv y_{2i} - \alpha_2 y_{1i}$, we find that the coherency condition implies that the model can be rewritten as

$$y_{1i} = 1\{\mathbf{z}'_{1i}\beta_1 + \tilde{y}_{2i}\beta_2 + u_i > 0\}, \tag{4.14}$$

⁸ See Gourieroux, Laffont, and Monfort (1980) for further discussion of coherency conditions, and see Lewbel (1999) for a recent statement of this result.

⁹ Blundell and Smith (1986) also develop an exogeneity test based on this estimator and consider results for the equivalent Tobit model. Rivers and Vuong (1988) provide a comprehensive treatment of limited information estimators for this class of parametric limited dependent variable models. They label the Blundell–Smith estimator the “two-stage conditional maximum likelihood” (2SCML) estimator and consider alternative limited information maximum likelihood (LIML) estimators. The efficiency and small sample properties of the 2SCML estimator are also considered. These are further refined in Blundell and Smith (1989). See also the important earlier related work of Amemiya (1978) and Lee (1981, 1993), which builds on the Heckman (1978) estimator.

and

$$\tilde{y}_{2i} = \mathbf{z}'_{2i}\gamma_2 + \varepsilon_i. \tag{4.15}$$

This specification could easily be generalized to allow for a more complex relationship in more complicated models of nonseparable decision making.¹⁰

If α_2 were known, then Equations (4.15) and (4.14) are analogous to (4.3), (4.4), and (4.6). Consequently, a semiparametric estimator using the control function approach would simply apply the estimation approach described in this chapter to the conditional model. Following the previous discussion, assumption (2.42) would be replaced by the modified conditional independence restrictions

$$\begin{aligned} u|\mathbf{z}_1, y_2, \mathbf{z}_2 &\sim u|\mathbf{z}_1, \tilde{y}_2, \varepsilon \\ &\sim u|\varepsilon. \end{aligned} \tag{4.16}$$

The conditional expectation of the binary variable y_1 given the regressors \mathbf{z}_1 , \tilde{y}_2 and errors ε would then take the form

$$\begin{aligned} E[y_1|\mathbf{z}_1, \tilde{y}_2, \varepsilon] &= \Pr[-u \leq \mathbf{z}'_{1i}\beta_1 + \tilde{y}_2\beta_2|\mathbf{z}_1, \tilde{y}_2, \varepsilon] \\ &= F_{-u|\varepsilon}(\mathbf{z}'_{1i}\beta_1 + \tilde{y}_2\beta_2|\varepsilon) \\ &\equiv \Gamma(\mathbf{z}'_{1i}\beta_1 + \tilde{y}_2\beta_2, \varepsilon). \end{aligned} \tag{4.17}$$

Finally, note that although α_2 is unknown, given sufficient exclusion restrictions on \mathbf{z}_{2i} , a root- n consistent estimator for α_2 can be recovered from (linear) 2SLS estimation of (4.8). More generally, if the linear form $\mathbf{z}'_{2i}\gamma_1$ of the regression function for y_2 is replaced by a nonparametric form $\gamma(\mathbf{z}_{2i})$ for some unknown (smooth) function γ , then a \sqrt{n} -consistent estimator of α_2 in the resulting partially linear specification for y_{2i} could be based on the estimation approach proposed by Robinson (1988), using nonparametric estimators of instruments ($\mathbf{z}_{1i} - E[\mathbf{z}_{1i}|\mathbf{z}_{2i}]$) in an IV regression of y_{2i} on y_{1i} .

5. AN APPLICATION

The empirical application presented here is taken from the Blundell and Powell (1999) study. In that paper we considered the participation in work by men without college education in a sample of British families with children. Employment in this group in Britain is surprisingly low. More than 12 percent of these men do not work, and this number approaches 20 percent for those men with lower levels of education. Largely as a consequence of the low participation rate, this

¹⁰ Note that to test this alternative specification against the triangular specification (4.3), (4.4), and (4.5), one may estimate

$$y_{2i} = \mathbf{z}_{2i}\gamma_1 + y_{1i}\alpha_2 + \widehat{y}_{2i}\delta_2 + w_i$$

by instrumental variables using \mathbf{z}_i as instruments, and then test the null hypothesis $\delta_2 = 0$, where \widehat{y}_{2i} is the prediction of y_{2i} under reduced-form specification (4.6).

group is subject to much policy discussion. We model the participation decision (y_1) in terms of a simple structural binary response framework that controls for market wage opportunities and the level of other income sources in the family. Educational level (z_1) is used as a proxy for market opportunities and is treated as exogenous for participation. However, other income (y_2), which includes the earned income of the spouse, is allowed to be endogenous for the participation decision.

As an instrument (z_{21}) for other family income, we use a welfare benefit entitlement variable. This instrument measures the transfer income the family would receive if neither spouse was working and is computed by using a benefit simulation routine designed for the evaluation of welfare benefits for households in the British data used here. The value of this variable depends on the local benefit rules, the demographic structure of the family, the geographic location, and housing costs. As there are no earnings-related benefits in operation in Britain over the period under study, we may be willing to assume it is exogenous for the participation decision. Moreover, although it will be a determinant of the reduced form for participation and other income, for the structural model herein, it should not enter the participation decision conditional on the inclusion of other income variables.

5.1. The Data

The sample consists of married couples drawn from the British Family Expenditure Survey (FES). The FES is a repeated continuous cross-sectional survey of households that provides consistently defined micro data on family incomes, employment status and education, consumption, and demographic structure. We consider the period 1985–1990. The sample is further selected according to the gender, educational attainment, and date of birth cohort of the head of household. We choose male heads of households, born between 1945 and 1954, who did not receive college education. We also choose a sample from the Northwest region of Britain. These selections are primarily to focus on the income and education variables.

For the purposes of modeling, the participating group consists of employees; the nonparticipating group includes individuals categorized as searching for work as well as the unoccupied. The measure of education used in our study is the age at which the individual left full-time education. Individuals in our sample are classified in two groups: those who left full-time education at age 16 or lower (the lower education base group), and those who left at age 17 or 18. Those who left at age 19 or older are excluded from this sample.

Our measure of exogenous benefit income is constructed for each family as follows: a tax and benefit simulation model¹¹ is used to construct a simulated

¹¹ The Institute for Fiscal Studies (IFS) tax and benefit simulation model is TAXBEN (www.ifs.org.uk), designed for the British FES data used in this paper. For an extensive discussion of the use of this data source in the study of male participation, see Blundell, Reed, and Stoker (1999).

Table 8.1. *Descriptive statistics*

Variable	Mean	Std. Dev.
Work (y_1)	0.871	0.387
Education > 16 (z_1)	0.196	0.396
ln (other inc.) (y_2)	5.016	0.434
ln (benefit inc.) (z_{21})	3.314	0.289
Education (sp.) (z_{22})	0.204	0.403
Age	39.191	10.256

Note: The number of observations is 1,606.

budget constraint for each individual family given information about age, location, benefit eligibility, and so on. The measure of out-of-work income is largely composed of income from state benefits; only small amounts of investment income are recorded. State benefits include eligible unemployment benefits,¹² housing benefits, child benefits, and certain other allowances. Because our measure of out-of-work income will serve to identify the structural participation equation, it is important that variation in the components of out-of-work income over the sample is exogenous for the decision to work. In the UK, the level of benefits that individuals receive out of work varies with age, time, and household size, and (in the case of housing benefit) by region. The housing benefit varies systematically with time, location, and cohort.

After making the sample selections described herein, our sample contains 1,606 observations. A brief summary of the data is provided in Table 8.1.¹³ The 87.1 percent employment figure for men in this sample is reduced to less than 82 percent for the lower education group that makes up more than 75 percent of our sample. As mentioned earlier, this lower education group refers to those who left formal schooling at 16 years of age or younger and will be the group on which we focus in much of this empirical application. The kernel density estimate of log other income for the low education subsample is given in Figure 8.1.

5.2. A Model of Participation in Work and Other Family Income

To motivate the specification, suppose that observed participation is described by a simple threshold model of labor supply. In this model, the desired supply of hours of work for individual i can be written as

$$h_i^* = \delta_0 + \delta_1' \mathbf{z}_{1i} + \delta_2 \ln w_i + \delta_3 \ln \mu_i + \zeta_i, \quad (5.1)$$

where \mathbf{z}_{1i} includes various observable social demographic variables, $\ln w_i$ is the log hourly wage, $\ln \mu_i$ is the log of “virtual” other income, and ζ_i is some

¹² The unemployment benefit included an earnings-related supplement in 1979, but this was abolished in 1980.

¹³ See Blundell and Powell (1999) for further details.

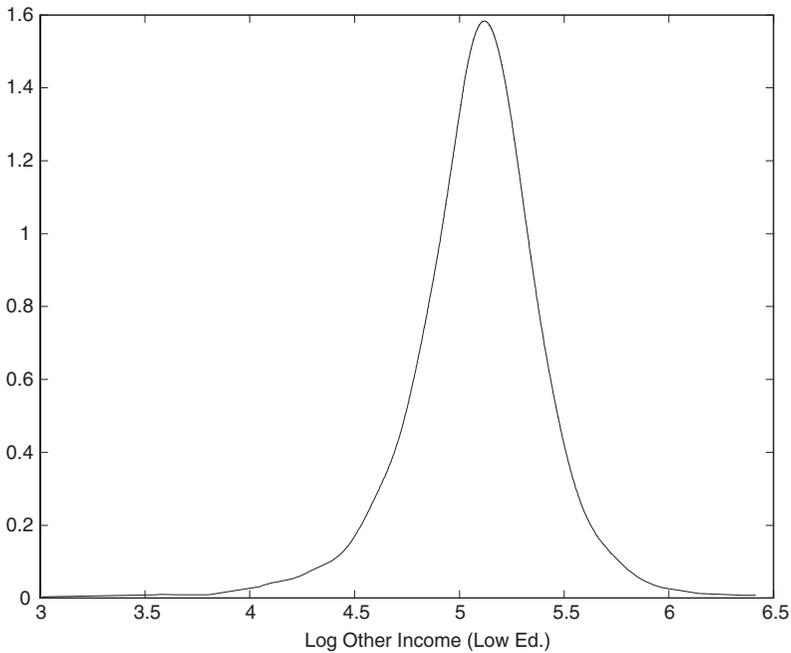


Figure 8.1. Density of Log Other Income for the Low Education Subsample.

unobservable heterogeneity. As $\ln w_i$ is unobserved for nonparticipants, we replace it in (5.1) by the wage equation

$$\ln w_i = \theta_0 + \theta_1' \mathbf{z}_{1i} + \omega_i, \tag{5.2}$$

where \mathbf{z}_{1i} now contains the education level for individual i as well as other determinants of the market wage. Labor supply (5.1) becomes

$$h_i^* = \phi_0 + \phi_1' \mathbf{z}_{1i} + \phi_2 \ln \mu_i + v_i. \tag{5.3}$$

Participation in work occurs according to the binary indicator

$$y_{1i} = 1\{h_i^* > h_i^0\}, \tag{5.4}$$

where

$$h_i^0 = \gamma_0 + \gamma_1' \mathbf{z}_{1i} + \xi_i \tag{5.5}$$

is some measure of reservation hours.

Combining these equations, we find that participation is now described by

$$y_{1i} = 1\{\phi_0 + \phi_1' \mathbf{z}_{1i} + \phi_2 \ln \mu_i + v_i > \gamma_0 + \gamma_1' \mathbf{z}_{1i} + \xi_i\} \tag{5.6}$$

$$= 1\{\beta_0 + \beta_1' \mathbf{z}_{1i} + \beta_2' y_{2i} + u_i > 0\}, \tag{5.7}$$

where y_{2i} is the log other income variable ($\ln \mu_i$). This other income variable is assumed to be determined by the reduced form

$$\begin{aligned} y_{2i} &= E[y_{2i}|\mathbf{z}_i] + v_i \\ &= \Pi(\mathbf{z}_i) + v_i, \end{aligned} \tag{5.8}$$

and $\mathbf{z}'_i = [\mathbf{z}'_{1i}, \mathbf{z}'_{2i}]$.

In the empirical application, we have already selected households by cohort, region, and demographic structure. Consequently, we are able to work with a fairly parsimonious specification in which \mathbf{z}_{1i} simply contains the education level indicator. The excluded variables \mathbf{z}_{2i} contain the log benefit income variable (denoted z_{21i}) and the education level of the spouse (z_{22i}).

5.3. Empirical Results

In Table 8.2 we present the empirical results for the joint normal–simultaneous Probit model. This consists of a linear reduced form for the log other income variable and a conditional Probit specification for the participation decision. Given the selection by region, cohort, demographic structure, and time period, the reduced form simply contains the education variables and the log exogenous benefit income variable. The results show a strong role for the benefit income variable in the determination of other income.

The first column of Probit results refers to the model without adjustment for the endogeneity of other income. These results show a positive and significant coefficient estimate for the education dummy variable and a small but significantly negative estimated coefficient on other income. The other income coefficient in Table 8.2 is the coefficient normalized by the education coefficient for comparability with the results from the semiparametric specification to be presented later. The impact of adjusting for endogeneity is quite dramatic. The income coefficient is now considerably larger and quite significant. The estimated education coefficient remains positive and significant.

Table 8.2. *Results for the simultaneous Probit specification*

Variable	Reduced Form		Standard Probit		Simult. Probit	
	y_2 Coeff.	Std. Error	Pr [Work] Coeff.	Std. Error	Pr [Work v] Coeff.	Std. Error
Education (z_1)	0.0603	0.0224	1.007	0.1474	1.4166	0.1677
ln (other inc.) (y_2)	—	—	−0.3364	0.1293	−2.8376	0.5124
ln (benefit inc.) (z_{21})	0.0867	0.0093	—	—	—	—
Education (sp.) (z_{22})	0.0799	0.0219	—	—	—	—
Exog. test					5.896 (t)	
R^2	0.0708		0.0550		0.0885	
F	30.69(3)		67.84 ($\chi^2_{(2)}$)		109.29 ($\chi^2_{(3)}$)	

Table 8.3. *Semiparametric results, parametric results, and bootstrap distributions*

Specification	$\hat{\beta}_2$	$\sigma_{\hat{\beta}_2}$	10%	25%	50%	75%	90%
Semi-P (with \hat{v})	-2.2590	0.5621	-4.3299	-3.6879	-2.3275	-1.4643	-1.0101
Semi-P (without \hat{v})	-0.1871	0.0812	-0.2768	-0.2291	-0.1728	-0.1027	-0.0675
Probit (with \hat{v})	-2.8376	0.5124	-3.8124	-3.3304	-2.9167	-2.4451	-1.8487
Probit (without \hat{v})	-0.3364	0.1293	-0.4989	-0.4045	-0.3354	-0.2672	-0.1991
Lin. prob. (with \hat{v})	-3.1241	0.4679	-3.8451	-3.3811	-3.1422	-2.8998	-2.5425
Lin. prob. (without \hat{v})	-0.4199	0.1486	-0.6898	-0.5643	-0.4012	-0.3132	-0.2412

Table 8.3 presents the semiparametric estimation results for the linear index coefficients. Bandwidths were chosen according to the $1.06\sigma_z n^{-1/5}$ rule (see Silverman, 1986).¹⁴ The education coefficient in the binary response specification is normalized to unity and so the β_1 estimates in Table 8.3 correspond to the ratio of the other income to the education coefficients. We present the standard Probit results in Table 8.2 for comparison (the mean of the bootstrap estimates for the education coefficient was 0.989, and for the income coefficient it was -0.327). The bootstrap figures relate to 500 bootstrap samples of size $n = 1,606$; the standard errors for the semiparametric methods are computed from a standardized interquartile range for the bootstrap distribution, and they are calculated using the usual asymptotic formulas for the Probit and linear probability estimators.

Figure 8.2 graphs the estimate of the ASF, $G(\mathbf{x}'\beta)$, derived from the semiparametric estimation with and without controls for the endogeneity of log other income. These plots cover the 5 percent to 95 percent range of the log other income distribution for the lower education group.

In Figure 8.3, we compare these semiparametric results with the results of estimating $G(\mathbf{x}'\beta)$ using the Probit and linear probability models. This data set is likely to be a particularly good source on which to carry out this evaluation. First, we know that the correction for endogeneity induces a large change in the β coefficients. Second, the proportion participating in the sample is around 85 percent, which suggests that the choice of probability model should matter as the tail probabilities in the Probit and linear probability models will behave quite differently. They show considerable sensitivity of the estimated $G(\mathbf{x}'\beta)$, after allowing for endogeneity, across these alternative parametric models. Both the linear probability model and the Probit model estimates result in estimated probabilities that are very different from those implied by the semiparametric approach. Figure 8.3 shows this most dramatically. For example, the linear probability model estimates a probability that is more than ten percentage points higher at the 20th percentile point of the log other income distribution.

¹⁴ Blundell and Powell (1999) provide results for a similar model specification and also present sensitivity results for this bandwidth choice. In particular, sensitivity to the choice of a smaller bandwidth is investigated and found not to change the overall results.

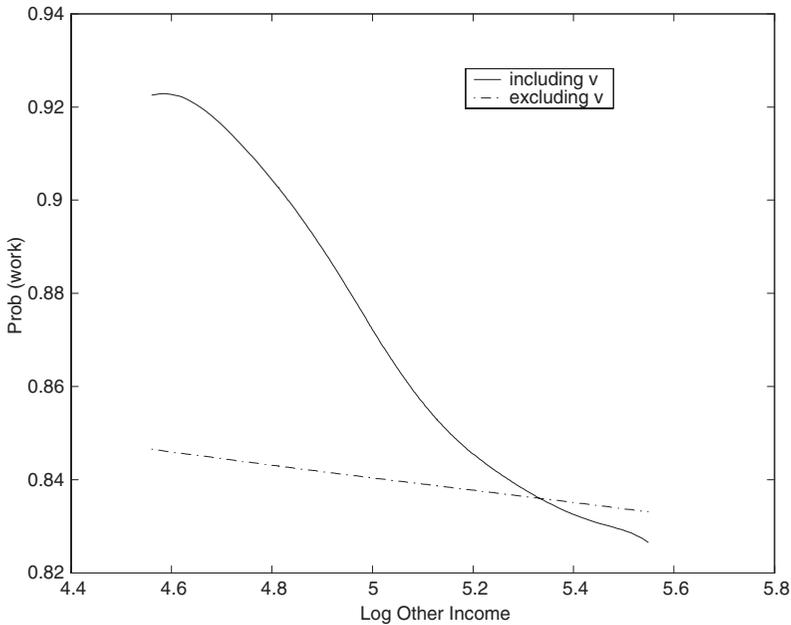


Figure 8.2. Semiparametric Model with and without Controls for Endogeneity.

This example points to the attractiveness of the semiparametric approach developed in this chapter. For this data set, we have found relatively small reductions in precision from adopting the semiparametric control function approach while finding quite different estimated responses in comparison with those from the parametric counterparts.¹⁵ The results show a strong effect of correcting for endogeneity and indicate that adjusting for endogeneity using the standard parametric models, the Probit and linear probability models, can give a highly misleading picture of the impact on participation of an exogenous change in other income. This is highlighted in Figure 8.3, where it is shown that the bias correction for endogeneity in the linear probability model was sufficient to produce predicted probabilities larger than unity over a large range of the income distribution. The Probit model did not fare much better. The semiparametric approach showed a strong downward bias in the estimated income responses when endogeneity of other income was ignored. The corrected semiparametric estimates appear plausible, and, although there were no shape restrictions imposed, the estimated ASF was monotonically declining in other income over the large range of the income distribution.

¹⁵ In Blundell and Powell (1999), we present the analogous analysis using the low-education subsample only. For this sample, the education dummy is equal to zero for all observations and is therefore excluded. Because x is now simply the log other income variable, this analysis is purely *nonparametric*. The results show a slightly shallower slope.

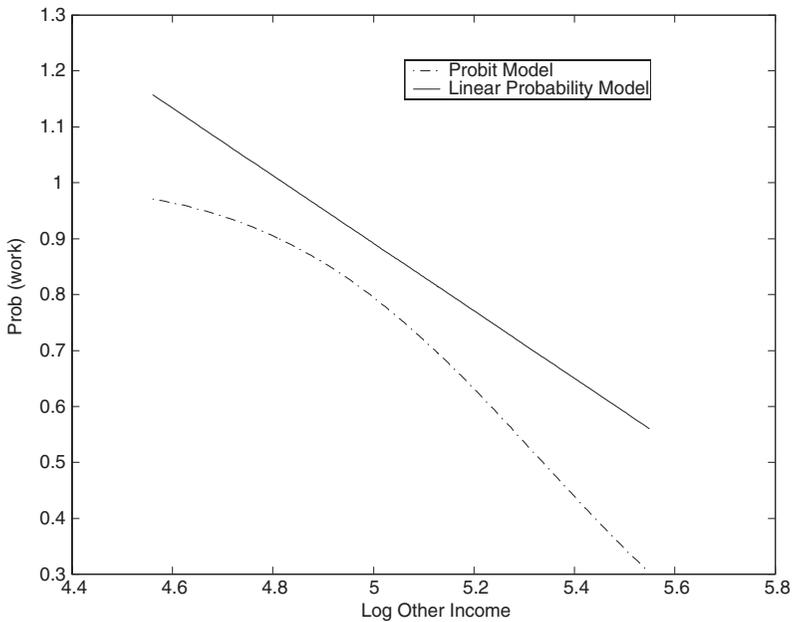


Figure 8.3. Linear Probability and Probit Models with Controls for Endogeneity.

5.4. The Coherency Model

In Blundell and Powell (1999), we also use this application to assess the alternative “coherency” model of Section 4, in which participation itself directly enters the equation determining other income. We interpret this as a fixed-cost model in which other income is dependent on whether or not fixed costs are paid, which in turn depends on participation. In this case, no explicit reduced form for y_2 , other income, exists. The model for other income may be written as

$$y_{2i} = \gamma_0 + y_{1i}\alpha_2 + \mathbf{z}'_{2i}\gamma_1 + \varepsilon_i, \tag{5.9}$$

where we are assuming that y_2 relates to the *level* of other income. Participation is now described by

$$y_{1i} = 1\{\beta_0 + y_{1i}\alpha_1 + \mathbf{z}'_{1i}\beta_1 + y_{2i}\beta_2 + u_i > 0\},$$

with the added coherency restriction

$$\alpha_1 + \beta_2\alpha_2 = 0. \tag{5.10}$$

Together these imply

$$y_{1i} = 1\{\beta_0 + \tilde{\gamma}'_{2i}\beta_2 + \mathbf{z}'_{1i}\beta_1 + u_i > 0\}, \tag{5.11}$$

Table 8.4. Results for the coherency specification

Variable	y_2 Coeff.	Std. Error	Probit		Probit	
			Pr[Work] Coeff.	Std. Error	Pr[Work ε] Coeff.	Std. Error
Work (y_1)	58.034	8.732	—	—	—	—
Education (z_1)	—	—	1.6357	0.2989	1.6553	0.3012
Adjusted income (\tilde{y}_2)	—	—	-0.7371	0.0643	-0.5568	0.1433
Benefit inc. (z_{21})	0.4692	0.1453	—	—	—	—
Education (sp.) (z_{22})	0.1604	0.0421	—	—	—	—
$\sigma_{u\varepsilon} = 0$ (t test)	—	—	—	—	2.556	—

with

$$\tilde{y}_{2i} = (y_{2i} - y_{1i}\alpha_2),$$

where we note that this fixed-cost adjustment to other income y_{2i} guarantees statistical coherency.

From the discussion in Section 4, we note that the conditional expectation of the binary response variable y_{1i} , given the regressors \mathbf{z}_1 , \tilde{y}_{2i} and errors ε , may be expressed as

$$E[y_1|\mathbf{z}_1, \tilde{y}_{2i}, \varepsilon] = \Pr[-u \leq \mathbf{z}'_1\beta_1 + \tilde{y}_{2i}\beta_2|\mathbf{z}_1, \tilde{y}_{2i}, \varepsilon] = F(\mathbf{z}'_1\beta_1 + \tilde{y}_{2i}\beta_2, \varepsilon). \tag{5.12}$$

Provided \tilde{y}_{2i} and ε_i can be measured, estimation follows the same procedure as in the triangular case.

The first column of Table 8.4 presents the estimates of the parameters of the structural equation for y_2 (5.9) in this coherency specification. These are recovered from IV estimation using the education of the husband as an excluded variable. The estimated “fixed cost of work” parameter seems reasonable; recall that the income variable has a mean of approximately £165 per week. The two sets of Probit results differ according to whether or not they control for ε . Notice that, having removed the direct simultaneity of y_1 on y_2 through the adjustment \tilde{y}_2 , we find much less evidence of endogeneity bias. Indeed, the coefficients on the adjusted other income variable in the two columns are quite similar (these are normalized relative to the education coefficient). If anything, after adjusting for fixed costs, we find that controlling for ε leads to a downward correction to the income coefficient.

The comparable results for the semiparametric specification are presented in Table 8.5. In these, the linear structural model estimates for the y_2 equation are used exactly as in Table 8.4. They show a very similar pattern with only a small difference in the other income coefficient between the specification that controls for ε and the one that does not. Again, the \tilde{y}_2 adjustment

Table 8.5. *Semiparametric results for the coherency specification*

Variable	Semi-P		Semi-P	
	Pr[Work] Coeff.	Std. Error	Pr[Work ε] Coeff.	Std. Error
Adjusted income (\tilde{y}_2)	-1.009	0.0689	-0.82256	0.2592

seems to capture much of the endogeneity between work and income in this coherency specification.

6. SUMMARY AND CONCLUSIONS

This chapter has considered nonparametric and semiparametric methods for estimating regression models of the form $y = H(\mathbf{x}, u)$, where the \mathbf{x} contains continuous endogenous regressors and where u represents unobserved heterogeneity. It has been assumed that there exists a set of instrumental variables \mathbf{z} with $\dim(\mathbf{z}) \geq \dim(\mathbf{x})$. This general specification was shown to cover a number of nonlinear models of interest in econometrics. The leading cases we considered were additive nonparametric specifications $y = g(\mathbf{x}) + u$, in which $g(\mathbf{x})$ is unknown, and nonadditive models $y = H(g(\mathbf{x}), u)$, in which $g(\mathbf{x})$ is unknown but H is a known function that is monotone but not invertible. An important example of the latter, and one that we used as an empirical illustration, is the binary response model with endogenous regressors. We have focused on identification and estimation in these leading nonparametric regression models and have defined the parameter of interest to be the ASF, $G(\mathbf{x}) \equiv \int H(\mathbf{x}, u) dF_u$, where the average is taken over the *marginal* distribution of the error terms u and where F_u denotes the marginal CDF of u .

In each of these leading cases, and their semiparametric variants, we have considered how three common estimation approaches for linear equations – the instrumental variables, fitted value, and control function approaches – may or may not be applicable. In the case where H and g are linear, *iid* distributed errors the covariance restriction $E(\mathbf{z}u) = 0$ and the rank condition are sufficient to guarantee identification and generate consistent and analytically identical estimators from each of these approaches. In the nonlinear models considered here, this is no longer the case.

In additive nonparametric specifications, we have considered restrictions on the model specification that are sufficient to identify $g(\mathbf{x})$, the ASF in this case. The relationship between the reduced form $E[y|\mathbf{z}]$ and the structural function g is given by $E[y|\mathbf{z}] = \int g(\mathbf{x}) dF_{\mathbf{x}|\mathbf{z}}$, where $F_{\mathbf{x}|\mathbf{z}}$ is the conditional CDF of \mathbf{x} given \mathbf{z} . Unlike in typical nonparametric estimation problems, identification of g faces an ill-posed inverse problem and consistent estimators of the components

$E[y|\mathbf{z}]$ and $F_{\mathbf{x}|\mathbf{z}}$ are not, by themselves, sufficient for consistent IV estimation of g . We have reviewed and assessed a number of approaches to IV estimation that have been proposed in the literature to overcome this problem. For the nonadditive case, IV estimation faces more severe difficulties. Without some further specific structure on H , such as invertibility, estimation by IV does not look promising. For our leading case in this nonadditive setting, the binary response model, H is not invertible.

Apart from some very specific cases, we have argued that the fitted-value approach is not well suited to estimation of parameters of interest in these nonlinear models. However, the control function approach has been shown to provide an attractive solution. This approach treats endogeneity as an omitted variable problem, in which the inclusion of estimates of the first-stage errors \mathbf{v} as a covariate corrects the inconsistency in $E(y|\mathbf{x})$. It has been shown to extend naturally under the conditional independence assumption that the distribution of u given \mathbf{x} and \mathbf{z} is the same as the conditional distribution of u given \mathbf{v} . This exclusion restriction permits replacement of the unidentified structural errors u with the identified control function \mathbf{v} through iterated expectations, so that averaging the structural function H over the marginal distribution of the structural errors u is equivalent to averaging the (identified) intermediate regression function of y on \mathbf{x} and \mathbf{v} over the marginal distribution of \mathbf{v} . We have derived a general approach to identification and estimation of the ASF $G(\mathbf{x})$ by this control function approach and have highlighted the importance of support restrictions on the distribution of the endogenous components of \mathbf{x} and \mathbf{z} .

We then considered the particular case of the linear index binary response model. In this semiparametric model, we have described in detail how estimation of the parameters of interest can be constructed using the control function approach. We considered a specific semiparametric matching estimator of the index coefficients that exploits both continuity and monotonicity implicit in the binary response model formulation. We have also shown how the partial-mean estimator from the nonparametric regression literature can be used to estimate the ASF directly. The control function estimator, for this semiparametric model, can easily be adapted to the case in which the model specification is not triangular and certain coherency conditions are required to be satisfied.

Finally, we have studied the response of labor force participation to nonlabor income, viewed as an endogenous regressor, using these techniques. The procedures we have developed appear to work well and suggest that the usual distributional assumptions underlying Probit and linear probability specifications could be highly misleading in binary response models with endogenous regressors. The application found relatively small reductions in precision from adopting the semiparametric approach. The semiparametric approach showed a strong downward bias in the estimated income responses when endogeneity of other income was ignored. The corrected semiparametric estimates appeared plausible, and, although there were no shape restrictions imposed, the estimated ASF was monotonically declining in other income over the large range of the income distribution.

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