A Global Model of International Yield Curves: 
No-Arbitrage Term Structure Approach

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Abstract

This paper extends the popular no-arbitrage affine term structure model of Duffee (2002) to model jointly bond markets and exchange rates across the UK and its two main trading partners. Using a monthly data set of forward rates for UK, US, and Germany (Euro) from 1992.10 to 2008.5, we have shown that two global factors account for a significant variation in bond yields of all countries. Also we have shown that to explain country specific movements in yield curves local factors are required. Although the factors are latent, we show that global factors are related to global inflation and global economic activity, while local factors are closely linked to monetary policy rates. The results are broadly similar to the conclusion of Diebold et al (2007), but the advantage of our joint international model is that we are able to decompose interest rates in risk-free rates and risk premia. Additionally, we are able to study the implications for foreign exchange rates. In particular, we show that while the differences in risk-free rates matters, to a large extent, changes in the exchange rate are determined by exchange rate risk premia.

JEL classification: C33, E43, F31

Very preliminary and incomplete draft. Please do not quote.

1 Introduction

Monetary policymakers routinely analyse financial market variables to extract information relevant for policy. Of particular interest are the term structure of nominal interest rates both in
the domestic economy and its main trading partners. Interest rates affect agents’ consumption, saving and investment decisions; and embody important information about investors’ expectations regarding the future course of monetary policy. Unfortunately, those expectations cannot simply be read directly from forward interest rates, since there is a substantial body of evidence indicating that there are significant (and time-varying) term premia in forward rates. In this paper, we present no-arbitrage models of the term structure of nominal interest rates that allow us to decompose forward interest rates into expectations of future short-term rates and term premia in the United Kingdom, United States and euro area within a consistent framework.

Co-movement between interest rates in different economies has increased since the 1970s (See Figures 1 and 2), which suggests that bond yields in different countries are not only driven by local, country-specific factors, but are also affected by global factors. Consistently modelling bond yields across different economies within the same framework allows us to shed light on these factors. It also allows us to derive implications for nominal exchange rates, which are other variables of great interest for policymakers. In theory, interest rates and exchange rates should be related by an Uncovered Interest rate Parity (UIP) condition, which states that expected changes in exchange rates should be equal to the differential between risk-free interest rates at home and abroad. However, one of the most puzzling aspects of exchange rate behaviour is the tendency for currencies with high interest rates to appreciate rather than depreciate as UIP would predict. One possible reason for the failure of UIP to hold in practice is the existence of risk premia; the model we present in this paper provides consistent estimates of foreign exchange risk premia for the UK and its main trading partners.

Figure 1. International 1-month forward rates, 5 years ahead

![One-month forward rates, five years ahead](image)

Figure 2. 5-year rolling correlations between changes in 5-year forwards 5-year ahead

![Five-year rolling correlation coefficient](image)

Previous research confirm that international yield curves have a lot in common. Indeed, an empirical analysis of yield curves for Germany, UK, US, and Japan by Diebold et al (2007) showed that ‘global yield factors do indeed exist and are economically important, explaining significant fractions of country yield curve dynamics, with interesting differences across countries.’
Following Diebold et al (2007), we propose to model international yield curves in a unified framework that allows for both global and local (country-specific) factors. However, Diebold et al (2007) model has no implications for foreign exchange movements. Additionally, the advantage of our approach is that we model international yield curves jointly in a no-arbitrage framework, which allows us to analyze whether yield curves move due to changes in expectations or due to term premia dynamics.

The class of no-arbitrage term structure models we shall use in this paper is sufficiently flexible to allow for time-varying risk premia: so-called ‘essentially affine term structure models’, as defined by Duffee (2002). In these models, all bonds in an economy can be priced using a single stochastic discount factor (SDF), for which a flexible functional form is assumed. Both bond yields and the price of risk (and therefore risk premia) are driven by a time-varying set of state variables, or ‘factors’, which follow some assumed time series process.

We estimate essentially affine models of the term structures in each of the three economies separately and jointly. Joint estimation across more than one economy requires us to rule out arbitrage opportunities not only across yields of different maturities but also across different economies. We assume the same functional form for SDFs in all economies, but allow its parameters to differ between economies. The state variables driving bond yields may be a mixture of ‘global’ factors, that affect bond yields in more than one economy, and some may be ‘local’ factors, that they only affect yields in a single economy.

Given the no-arbitrage assumption, the exchange rate is endogenously determined by the dynamics of two stochastic discount factors and hence it is not required to incorporate this variable into the estimation. However, the estimation procedure is subject to several complicated issues: the likelihood function has an extremely complicated form and has to be optimized over 40-50 parameters, which makes it difficult to find a global maximum. Thus, including exchange rate data in the estimation might provide us with more precise estimates of unobservable market prices of risk and risk premia. The difficulty with incorporating the depreciation rate into the vector of observed variables is that the measurement equation becomes nonlinear. In this case we can still use maximum likelihood estimation method, based on the extended Kalman filter, linearizing the observation equation for the depreciation rate.

Although we employ a prevailing framework in a recent international term structure literature (see Dong (2007), Backus, Foresi and Telmer (2001), Chabi-Yo and Yang (2006)), our model provides several contributions. Primarily, we estimate three-country models, which provides us with UIP decomposition consistent across the UK, US and euro area. Second, instead of assuming one model specification, we estimate a number of models, allowing different combinations of local and global factors, which allows us to choose the most appropriate model specification.

The paper is organized as follows. In section 2 we describe our modelling framework, and in section 3 we discuss the data and the estimation process. In section 4 we provide preliminary results. Section 5 concludes.
2 The essentially affine framework

2.1 Theory

Our models belong to the ‘essentially affine’ class of models, as defined by Duffee (2002). In these models, both bond yields and the market prices of risk are affine functions of underlying state variables. The formulation of the stochastic discount factor is slightly more general than in the ‘completely affine’ models originally proposed by Duffee and Kan (1996), in that the price of risk is allowed to vary independently of interest rate volatility.

We estimate essentially affine models of the term structures in each of the three economies separately and jointly. Joint estimation across more than one economy requires us to rule out arbitrage opportunities not only across yields of different maturities but also across different economies. We assume the same functional form for the SDF but allow its parameters to differ between economies. The state variables driving bond yields may be a mixture of ‘global’ factors, that affect bond yields in more than one economy, and some may be ‘local’ factors, only affecting yields in a single economy.

2.2 The essentially affine term structure literature

Since Duffee (2002), an ever-increasing number of studies have estimated essentially affine term structure models for individual countries. For example, for the UK, Lildholdt et al (2007) estimate a ‘macro-factor’ model of the nominal term structure, Joyce et al (2006) estimate a latent factor model of the real term structure and Joyce et al (2007) jointly estimate a model of the nominal and real term structures using a combination of latent factors and observed inflation. Elsewhere, Kim and Orphanides (2005) estimate a [latent]-factor model of the US nominal term structure; and Durham (2006) and Hordahl and Tristani (2007) estimate joint models of nominal and real term structures for the US and the euro area respectively. For all countries, the papers found time varying term premia as crucial in explaining interest rate behavior.

While a number of studies have jointly estimated completely affine term structure models across multiple economies, relatively few have used essentially affine models. In one example, Dong (2006) jointly models the term structures in the US and Germany using a macro-factor model. Dong’s model attributes much of the movement in the foreign exchange risk premium to the macro factors. In this paper, we extend this to estimate joint latent factor essentially affine models for the all three economies (UK, US and euro area) together.

2.3 Model derivation

We derive the model below for the three-factor case, which is sufficiently general to nest all the specifications we shall consider. We derive the model in discrete time, following Backus et al (1998). We first derive the model for a single economy. This is derived from three basic elements: a process driving the unobservable factors; a formulation of the stochastic discount
factor that ensures the model is essentially affine; and the assumption that bond prices reflect
the fundamental asset pricing equation. We then generalize the model to cover the multiple-
economy case; show how we can decompose fitted yields into terms representing expectations of
future short-term rates, risk premia and convexity; and derive an expression for the expected
path of the exchange rate and the foreign exchange risk premium.

2.3.1 The single-country essentially affine term structure model

We first define a state vector relevant for pricing bonds, \( z_t \), which is a \((K \times 1)\) vector
containing as elements latent factors. The state vector is assumed to follow a first-order VAR
model:

\[
    z_{t+1} = \Phi z_t + \Omega^{1/2} \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim NID(0, I_K)
\]

where the \( \Omega \) and \( \Phi \) are \((K \times K)\) matrices. In our empirical analysis we allow the off-diagonal
elements of the \( \Phi \) matrix to be non-zero, which enables the factors to be correlated with one
another. Notice that we assume that the error terms are homoscedastic, so volatility in the
model is constant.

In order to get an affine form for bond yields, we assume that the nominal stochastic discount
factor (SDF) takes the following form:

\[
    M_{t+1} = \exp \left[ -r_t - \frac{\Lambda_t' \Lambda_t}{2} - \Lambda_t' \varepsilon_{t+1} \right]
\]

where risk-free rate, \( r_t \), and market prices of risk, \( \Lambda_t \), are linearly related to the state vector:

\[
    r_t = r + \gamma' z_t
\]

\[
    \Lambda_t = \left( \lambda_{K \times 1} + \beta_{K \times K} z_t \right)
\]

The logarithm of the SDF is:

\[
    \ln M_{t+1} = m_{t+1} = -r - \gamma' z_t - \frac{\Lambda_t' \Lambda_t}{2} - \Lambda_t' \varepsilon_{t+1}
\]

The elements in the matrix \( \Lambda_t \) represent the 'market prices of risk' associated with shocks to the
SDF associated with each factor. The formulation in (2) is consistent with time-varying term
 premia in the case where some of the elements of \( \beta \) are non-zero. It also nests the case where
term premia are constant, where all the elements of \( \beta \) are zero, and the case where term premia
are zero, where all the elements of \( \lambda \) and \( \beta \) are zero.

The logs of bond prices in the model are affine functions of the state vector. To see this,
we first assume that the log price of a zero-coupon bond with \( n \) periods to maturity at time \( t \) is
given by:
\[ \ln P_t^n = p_t^n = A_n + B'_n z_t \]  
\( (4) \)

where:

\[ B_n = \begin{bmatrix} 
B_{n,1} & B_{n,2} & B_{n,3} 
\end{bmatrix} \]

Since \( P_t^0 = 1 \), it follows that:

\[ A_0 = 0, \quad B_0 = \begin{bmatrix} 
0 & 0 & 0 
\end{bmatrix} \]

The fundamental asset pricing equation states that:

\[ P_t^n = E_t \left[ M_{t+1} P_{t+1}^{n-1} \right] \]  
\( (5) \)

so that the price of an \( n \)-period bond today is equal to the expected value of the product of the price next period and the SDF next period. On the assumption that bond prices and the SDF are jointly lognormal, we can use the property of lognormality to expand out this expression to get:

\[ p_t^n = E_t \left[ m_{t+1} + p_{t+1}^{n-1} \right] + \frac{1}{2} \text{var} \left[ m_{t+1} + p_{t+1}^{n-1} \right] \]

If we now substitute in for next period’s SDF and for the bond price, it is possible after some algebraic manipulation (shown in Appendix A) to show that the linear expression for bond prices must satisfy the following two recursive equations:

\[\begin{align*}
A_n &= -r + A_{n-1} - B'_{n-1} \Omega^{1/2} \lambda + \frac{B'_{n-1} \Omega B_{n-1}}{2} \\
B'_n &= -\gamma' + B_{n-1}' \left( \Phi - \Omega^{1/2} \beta \right) \quad (6, 7)
\end{align*}\]

So, providing that the \( A_n \) and \( B_n \) parameters satisfy these restrictions, log bond prices are affine in the factors and the model satisfies no arbitrage. And since log bond prices are affine, it follows that yields are also affine in the factors, with continuously compounded yields given by:

\[ \hat{y}_t^n = \frac{p_t^n}{n} = \frac{A_n}{n} - \frac{B'_n}{n} z_t \]

The general relationship between spot and forward yields is:

\[ f_t^n = (n + 1) y_{t+1}^{n+1} - n y_t^n \]

so forward yields are therefore given by:

\[ f_t^n = -(A_{n+1} + B'_{n+1}) z_t + (A_n + B'_n) z_t \]
From a monetary policy perspective, the forward curve is of greater interest than the spot curve because it more directly conveys information about market expectations of future interest rates, although term premia and the convexity effect are likely to distort the observed forward curve away from pure market expectations of future interest rates. The affine yield curve model allows us to decompose the domestic and foreign forward curves into interest rate expectations, term premia and a convexity effect, such that:

\[ f^n_t = E_t [y^1_{t+n}] + \phi_{t,n} + \omega_{t,n} \]  

(8)

where \( E_t [y^1_{t+n}] \) is the expected future one-period (i.e. short) rate \( n \) periods ahead, \( \phi_{t,n} \) is the term premium in the forward curve at maturity \( n \), and \( \omega_{t,n} \) is the convexity effect at maturity \( n \).

2.3.2 The extension to multi-country model

The fundamental asset pricing equation states that, for a foreign bond:

\[ P_t^n = E_t [M_{t+1}^* P_{t+1}^{n-1*}] \]

We assume that foreign bonds are priced using a foreign nominal SDF:

\[ \ln M_{t+1}^* = m_{t+1}^* = -r^* - \gamma^* z_t - \frac{\Lambda_t^* A_t^*}{2} - \Lambda_t^* \xi_{t+1} \]

This specification of the foreign SDF is directly analogous to that of the domestic SDF 5, with parameters specific to the foreign economy denoted with an asterisk. This specification nests all combinations of global and local factors. A factor can be specified as local by restricting the appropriate elements of \( \gamma^* \), \( \Lambda^* \), \( \lambda^* \), \( \beta^* \) and \( \beta^* \) to zero.

The derivation of the analogous expression for the yield of foreign bonds then proceeds in an analogous way to that for domestic bonds. The yield on an \( n \)-period foreign bond is therefore given by:

\[ y^n_t = \frac{P_t^n}{n} = \frac{A_n^*}{n} - \frac{B_n^{n*}}{n} z_t \]

where

\[ A_n^* = -r^* + A_{n-1}^* - B_{n-1}^{n*} \Omega^{1/2} \lambda^* + \frac{B_{n-1}^{n+1} \Omega B_{n-1}^{n*}}{2} \]  

(9)

\[ B_n^* = -\gamma^* + B_{n-1}^{n*} \left( \Phi - \Omega^{1/2} \beta^* \right) \]  

(10)

and

\[ A_0^* = 0, \quad B_0^* = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \]
Forward yields on foreign bonds are given by:

\[ f_t^{n*} = - (A^{*+1} + B^{*+1} z_t) + (A^{*+1} + B^{*+1} z_t) \]  

(11)

Given an estimate of the SDF estimated from bond prices, it is technically straightforward to estimate changes in exchange rates in the model without any additional information. As explained above, the fundamental asset pricing equation can be written from the perspective of a foreign investor in foreign bonds as:

\[ 1 = E_t \left[ M_{t+1} \right] \left[ R_{t+1} \right] \]  

(12)

where \( R_{t+1} \), the one-period return on an \( n \)-period foreign bond is:

\[ R_{t+1} = \frac{P_{t+1}^{n-1}}{P_{t+1}} \]

Domestic investor can invest in foreign bonds. Assuming that both markets are frictionless and arbitrage free for foreign and domestic investors, the one-period return on domestic and foreign bonds must be equal, once we have adjusted for the change in the exchange rate, \( S_t \), which is expressed as the foreign currency price of a unit of domestic currency:

\[ R_{t+1} = \frac{S_{t+1}}{S_t} R_{t+1} \]

Combining this with equations 5 and 12 gives:

\[ E_t \left[ M_{t+1} \right] \left[ R_{t+1} \right] = E_t \left[ M_{t+1} \frac{S_{t+1}}{S_t} R_{t+1} \right] \]

Under complete markets, \( M_{t+1} \) and \( M_{t+1} \) are unique, so that the foreign SDF equals to the domestic SDF multiplied by the ratio of the exchange rates at \( t+1 \) and \( t \):

\[ M_{t+1} = M_{t+1} \frac{S_{t+1}}{S_t} \]  

(13)

Taking logs and combining equations above, we get the following expression:

\[ \Delta s_t \equiv s_{t+1} - s_t = m_{t+1} - m_{t+1} \]

\[ \equiv (r_t - r_t^*) - \left( \frac{A_t^* A_t}{2} - \frac{\Lambda_t^* \Lambda_t}{2} \right) + (\Lambda_t^{*'} - \Lambda_t) \varepsilon_{t+1} \]  

(15)

where lower case letters denote log variables. The depreciation in the exchange rate (in logs) is
equal to the difference between the foreign and domestic log SDFs. Interestingly, the third term in the above equation shows that even if we assume homoschedastic state, the depreciation rate is heteroschedastic when the prices of risk are time varying.

The expected depreciation can be obtained straightforwardly as the difference in the expected log SDFs:

$$E_t[s_{t+1}] - s_t = E_t[m_{t+1}^*] - E_t[m_{t+1}]$$  (16)

Under the assumption of lognormality mentioned above, the foreign exchange risk premium is equal to half the difference in the variances of the foreign and domestic SDFs, conditional on information available at time $t$:

$$\mu_{t,t+1} = \frac{\text{var}_t[m_{t+1}^*] - \text{var}_t[m_{t+1}]}{2}$$  (17)

Using the assumed functional form for the SDF given above (equation 5), the expected depreciation is therefore:

$$E_t[s_{t+1}] - s_t = (r_t - r_t^*) - \left(\frac{\Lambda_t'^*\Lambda_t^*}{2} - \frac{\Lambda_t'\Lambda_t}{2}\right)$$  (18)

and the one-period ahead foreign exchange risk premium is:

$$\mu_{t,t+1} = \frac{\Lambda_t'^*\Lambda_t^* - \Lambda_t'\Lambda_t}{2}$$

Clearly, (18) implies that UIP does not hold due to time varying foreign exchange risk premium, which is determined by the same factors as interest rate risks. Nonetheless, UIP should hold when accounted for the risk premium:

$$E_t[\Delta s_{t+1}] - \left(\frac{\Lambda_t\Lambda_t^*}{2} - \frac{\Lambda_t'^*\Lambda_t^*}{2}\right) = r_t - r_t^*$$

3 Estimation

3.1 Data

3.1.1 Nominal forward rates

The bond market data used to estimate the affine term structure models in this paper are estimates of zero-coupon forward rates for the period October 1992 - May 2008 derived from UK, German and US government bonds using the smoothed cubic spline method proposed by Anderson and Sleath (2001). The forward rates we use relate to the periods one month in length starting one, three, five and ten years ahead. Additionally, we use monetary policy rates as proxies for risk-free rates.
While data on UK, US and German rates are available further back, we have limited our sample period to be from October 1992 to May 2008 to avoid obvious structural breaks in the series. For instance, the adoption of inflation targeting in October 1992 represented a substantial change in the United Kingdom’s monetary policy framework. This change is likely to have affected the term structure of interest rates, as perceptions about how monetary policy will react to a variety of factors will affect expectations of future short rates and term premia. Although another obvious break in the series is the creation of the euro in 1999, model estimation over a large number of parameters prevents further reduction of the sample period.

Time series of forward rates in different countries displayed in Figure 1. Forward rates fell at all maturities in all three economies over the sample. The larger range of falls across different maturities in the UK reflected the large falls in ten-year forward rates in the period between October 1992 and late 1999. There is evidence of substantial co-variation across forward rates from different countries, at all maturities, but particularly those longer than one year.

Forward rates of different maturity within an economy are positively correlated; unsurprisingly, this correlation is highest between yields of similar maturities, although even the one- and ten-year forward rates have correlation coefficients between 0.5 and 0.7 (See Figures 3a-3c). Forward rates are also positively correlated across economies (see Figure 2). UK one-year forward rates are more strongly correlated with those in the US than in Germany, whereas the opposite is true for ten-year forward rates. The preliminary principle component analysis suggests that 3 factors required to explain a singular country yield curve, and probably additional factors are needed to explain all three country yield curves jointly (See Tables 1 and 2).
3.1.2 Exchange rates

The exchange rate data we use in this paper are end-of-month data on the sterling:dollar and sterling:euro rates. We use a synthetic series for the sterling:euro exchange for the period before the creation of the euro in 1999. This was obtained by geometrically weighting the bilateral exchange rates of the eleven original euro-area countries using weights based on the country shares of trade with countries outside the euro area. Exchange rate data are much volatile than interest rates and less persistent (see Figure 3d).

The data are consistent with a well-known Fama puzzle: differently from a unity estimate as UIP suggests, the regression of

$$\Delta s_{t+1} = \alpha + \beta (r_t - r^*_{t}) + \text{residual}$$

produces a negative estimates of $\beta$ for Euro-Sterling (although not significant, see Table 3).

3.2 Estimation Method

Our estimation is based on the state-space form of the model. Indeed, the state equation in this context is the first-order VAR:
\( z_{t+1} = \Phi z_t + \Omega^{1/2} \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim NID(0, I_3) \)

The space equation (otherwise known as the measurement or observation equation) comes from 11:

\[
f^n_t = A_n - A_{n+1} + (B_n' - B_{n+1}') z_t,
\]

which predicts the exact relation between the factors and the forward rates. However, when using more maturities than factors, this exact relation cannot be satisfied by yields of all maturities. Hence, some kind of measurement error is required. We assume that the measurement errors have zero mean and are uncorrelated:

\[
f^n_{t*} = A_n - A_{n+1} + (B_n' - B_{n+1}') z_t + v_t^n, \quad v_t^n \sim NID(0, h)
\]

The full state-space form of the model with measurement errors is then

\[
\begin{align*}
z_{t+1} &= \Phi z_t + \Omega^{1/2} \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim NID(0, I_3) \\
F_t &= A + B' z_t + V_t, \quad V_t \sim NID(0, H), \quad H = \text{diag}(h),
\end{align*}
\]

where \( F_t \) is a stacked vector of observable forward rates in both home and foreign countries, and \( A, B' \) are stacked vector and matrix of corresponding factor loadings. Given this state-space setup, the most convenient way to estimate the parameters is by maximum likelihood based on the Kalman filter.

According to no-arbitrage, it is not required to incorporate the exchange rate into the estimation, given that it is endogenously determined by the dynamics of two stochastic discount factors. However, the estimation procedure is subject to several complicated issues. Primarily, the likelihood function has an extremely complicated form and has to be optimized over 40-50 parameters, which makes it difficult to find a global maximum. Thus, including exchange rate data in the estimation might provide us more precise estimates of unobservable market prices of risk and hence risk premia. The difficulty with incorporating the depreciation rate into the vector of observed variables is that the measurement equation becomes nonlinear. Indeed, the depreciation rate is given by

\[
\Delta s_t = m_t^* - m_t \\
= r_{t-1} + \frac{1}{2} \Lambda_{t-1} \Lambda_{t-1} + \Lambda_{t-1} \varepsilon_t - r^*_t - \frac{1}{2} \Lambda_{t-1} \Lambda_{t-1} - \Lambda_{t-1} \varepsilon_t \\
= \delta - \delta^* + \frac{1}{2} (\lambda' \lambda - \lambda'^* \lambda^*) + (\lambda' - \lambda'^*) z_t \\
+ (\gamma - \gamma^* + \lambda' \beta - \lambda'^* \beta^* + (\lambda'^* - \lambda') \Phi) z_{t-1} + \frac{1}{2} z_{t-1} \beta' \beta z_{t-1} \\
- \frac{1}{2} z_{t-1} \beta'^* \beta^* z_{t-1} + z_{t-1} (\beta'^* - \beta') \Phi z_{t-1} + z_{t-1} (\beta' - \beta'^*) z_t,
\]
which is a nonlinear function of \((z_{t-1}, z_t)\). In this case we can still use maximum likelihood method, based on extended Kalman filter, linearizing the observation equation for the depreciation rate. (See Appendix B for the details).

As we already discussed in the theory section, the model implies that exchange rate risk is explained by the same factors that drive interest rate risk. This assumption is rather strong and is usually statistically rejected (see Bekaert, Wei and Xing (2007), Wu (2007)). In particular, Bekaert, Wei and Xing (2007) suggest that different risk factors may be present in the foreign exchange markets than those in the fixed income markets. To account for this, we introduce an additional variable \(f_t\), assuming it to be orthogonal to interest rate risk, such that

\[
\Delta s_t = r_{t-1} + \frac{1}{2} \Lambda_{t-1}' \Lambda_{t-1} + \Lambda_{t-1}' \xi_t - r_{t-1}' - \frac{1}{2} \Lambda_{t-1}' \Lambda_{t-1} - \Lambda_{t-1}' \xi_t + f_t,
\]

where we assume \(f_t \sim NID(0, F)\). In practice, this assumption allows us to treat \(f_t\) as a measurement error variable in the observation equation, such that the model can be written in the standard state-space form (see (19) in Appendix B).

4 Preliminary results

We have estimated the models with forward rate data restricting our sample period to be from October 1992 to May 2008. In each case, forward rates for each economy are described by three factors. Specifically, we consider: 1) UK, US, and Euro three-factor models estimated separately, 2a) a three-country global-factor model, and 2b) a three-country model, where single country bond prices are explained by two global and 1 local factors; 3a) a set of UK-Euro and 3b) a set of UK-US two-country three factor models. Here we report our preliminary results.

\[\text{NOTE: Tables with parameter estimates to be added.}\]

4.1 Commonality in factors

First of all, we find striking commonality in country factors. In Figures 4a-4c we plot separately estimated factors from singular country models (we number them according to their order in the state equation (1)). For all sets of factors, there is a clear evidence of commonality in factor dynamics, which suggests the existence of global factors. However, third factors from single-country models appear to be less correlated across countries.

As a next step, we extended single-country frameworks to a joint yield curve setting, allowing interest rates to depend on global as well as country-specific factors. Although, global-only factor models perform well for all country combinations (decompositions on risk free rates and risk premia are similar to those of single country models, the fitting errors are small), the best performing models according to MLkH criterion are the models allowing for two global and 1 local factors. The factors implied by this model are shown in Figure 5. The global factors appear to be highly correlated with first two factors from single country models, while local factors have
the same pattern as third factors from those models (compare Figures 4 and 5).

**Figure 4a.** 1st factors from singular country models.

**Figure 4b.** 2nd factors from singular country models.

**Figure 4c.** Third factors from singular country models.
As Figure 5 shows, the local factors are highly correlated with country-specific monetary policy rates, which confirms that the global factor only model would be misspecified in the environment of independent monetary policies across different economies. Additional evidence of the close link between local factors and policy rates comes from the factor loading analysis (see Figure 6a). Loadings on local factors are decreasing with maturity, which confirms that local factors have a close link with local monetary policy and a weak explanatory power for longer maturities. Long term maturity rates are mostly explained by the global factors (Figure 6b), which have an important economic interpretation too. Indeed, following the previous literature studying the relationship between yield factors and macroeconomic factors (see Ang and Piazzesi (2003), Diebold et al (2008)), we found that our extracted global factors are linked to global inflation and global economic activity (see Figures 7a-7b).
4.2 Implied decompositions into risk-free rates and term premia

Figures 8a-c show implied decompositions of 10-year forward rates from our best international model (with 2 global and 1 local factors) and benchmark singular models. The main conclusion about interest rate decompositions from all models is similar: expected risk-free rates and term premia have fallen in both economies since the late 90s. UK term premia became negative in 1998 and has remained close to zero thereafter, which explains a downward sloping UK curve. Instead, US and euro area term premia have been positive.

The residuals from the international model lie almost everywhere in a narrow band of [-50bp, 50bp], which indicate that moving from isolated single country models to a global model we still fit forward rates reasonably well.
4.3 Exchange rate implications

The advantage of the international no-arbitrage model is that we are able to study the decomposition of exchange rate movements into risk free and risk premia related. As Figures 9a-b with the model implied depreciation rates and the risk-free component (UIP path) show, changes in expected exchange rates differ substantially from the differences in risk free rates. Almost everywhere through the sample period, UIP path has been indicating the depreciation of sterling. However, nor the data neither our model confirm this persistent pessimism over sterling, showing the periods of expected sterling weakening that alternate with those of sterling strength. As the model suggests, the dynamics of exchange rates has been determined to a large extent by time.
Even if our model captures broad movements in exchange rates, it fails to match exchange rate volatility. As Graveline (2006) shows, to successfully capture both exchange rate volatility and the term structure of interest rates options data must be included to estimate the model. Thus, a natural extension of this work would be to estimate the model with the joint time series of interest rates, exchange rates, and prices of exchange rate options.

5 Conclusion

We have extended the popular no-arbitrage affine term structure model of Duffee (2002) to model jointly bond markets and exchange rates across the UK and its two main trading partners. Using a monthly data set of forward rates for UK, US, and Germany (Euro) from 1992.10 to 2008.5, we have shown two global factors account for a significant variation in bond yields of all countries. Also we have shown that to explain country specific movements in yield curves local factors are required, which are closely linked to monetary policy rates. Although the results are broadly similar to the conclusion of Diebold et al (2007), the advantage of our joint international model is that we are able to decompose interest rates in risk-free rates and risk premia. Additionally, we are able to study the implications for foreign exchange rates. In particular, we show that while the differences in risk-free rate matters, to a large extent, changes in the exchange rate are determined by exchange rate risk premia.
6 Appendix A

In this section, we show the derivation of the recursive equations [7] and [8] shown in the text above. For convenience we re-state the three basic equations of the model. The state variables driving yields are assumed to follow a first-order VAR process:

\[ z_{t+1} = \Phi z_t + \Omega^{1/2} \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim NID(0, I_3) \]

The logarithm of the nominal SDF is:

\[ m_{t+1} = -r - \gamma' z_t - \frac{\Lambda'_1 \Lambda_1}{2} - \Lambda'_1 \varepsilon_{t+1} \]

Finally, we assume that log bond prices are affine in the state vector, so that the log price of a zero-coupon bond with \( n \) periods to maturity at time \( t \) is given by:

\[ \log P^n_t = p^n_t = A_n + B'_n z_t \]

where:

\[ B'_n = \begin{bmatrix} B_{n,1} & B_{n,2} \\ B_{n,1} & B_{n,2} \end{bmatrix} \]

To derive the recursive equations, we begin with the fundamental asset pricing equation, which states that:

\[ P^n_t = E_t [M_{t+1} P^{n-1}_{t+1}] \]

If we assume that the expression within the expectations term on the right-hand side of this equation is jointly lognormally distributed, we can then write:

\[ p^n_t = E_t [m_{t+1} + p^{n-1}_{t+1}] + \frac{1}{2} \text{var}_t [m_{t+1} + p^{n-1}_{t+1}] \]

where lower case letters denote that we have taken the natural logarithm. The proof now proceeds by substituting in for log bond prices and the log SDF and manipulating this expression. First, if we substitute for \( m_{t+1} \) and \( p^{n-1}_{t+1} \) in the expectations term from the previous expression:

\[ E_t [m_{t+1}] + E_t [p^{n-1}_{t+1}] = E_t \left[ -r - \gamma' z_t - \frac{\Lambda'_1 \Lambda_1}{2} - \Lambda'_1 \varepsilon_{t+1} + A_{n-1} + B'_{n-1} z_{t+1} \right] \]

If we now substitute in for \( z_{t+1} \) take through the expectations operator and use the fact that \( E_t \varepsilon_{t+1} = 0 \), we get:
where the last line uses the fact that for the log bond price:

\[ E_t [m_{t+1}] + E_t \left[ p_{t+1}^{n-1} \right] = E_t \left[ -r - \gamma' z_t - \frac{\Lambda'_{t}}{2} - \Lambda_{t}' \varepsilon_{t+1} + A_{n-1} + B_{n-1}' \left( \Phi z_t + \Omega^{1/2} \varepsilon_{t+1} \right) \right] \]

\[ = -r - \gamma' z_t - \frac{\Lambda'_{t}}{2} + A_{n-1} + B_{n-1}' \Phi z_t \]

Second,

\[ \text{var}_t [m_{t+1} + p_{t+1}^{n-1}] = \text{var}_t \left[ -r - \gamma' z_t - \frac{\Lambda'_{t}}{2} - \Lambda_{t}' \varepsilon_{t+1} + A_{n-1} + B_{n-1}' \varepsilon_{t+1} \right] \]

If we now substitute in for \( z_{t+1} \) as before and drop the constant terms (since these have zero variance and can be ignored), we obtain:

\[ \text{var}_t [m_{t+1} + p_{t+1}^{n-1}] = \text{var}_t \left[ -r - \gamma' z_t - \frac{\Lambda'_{t}}{2} - \Lambda_{t}' \varepsilon_{t+1} + A_{n-1} + B_{n-1}' \varepsilon_{t+1} \right] \]

Expanding out the right-hand side of this expression, we obtain:

\[ \text{var}_t [m_{t+1} + p_{t+1}^{n-1}] = \text{var}_t \left[ -r - \gamma' z_t - \frac{\Lambda'_{t}}{2} - \Lambda_{t}' \varepsilon_{t+1} + A_{n-1} + B_{n-1}' \varepsilon_{t+1} \right] \]

where the last line uses the fact that \( B_{n-1}' \Omega^{1/2} \) is a scalar. Now we can obtain an expression for the log bond price:

\[ p_{t}^{n} = \left( -r - \gamma' z_t - \frac{\Lambda'_{t}}{2} + A_{n-1} + B_{n-1}' \Phi z_t \right) + \frac{1}{2} \left( \Lambda'_{t} A_t - 2B_{n-1}' \Omega^{1/2} A_t + B_{n-1}' \Omega B_{n-1} \right) \]

\[ = -r - \gamma' z_t + A_{n-1} + B_{n-1}' \Phi z_t - \frac{B_{n-1}' \Omega B_{n-1}}{2} \]

We now substitute in for \( A_t \) and collect terms:

\[ p_{t}^{n} = -r - \gamma' z_t + A_{n-1} + B_{n-1}' \Phi z_t - \frac{B_{n-1}' \Omega B_{n-1}}{2} \]

\[ = \left( -r + A_{n-1} - B_{n-1}' \Omega^{1/2} \frac{\lambda + \beta z_t}{2} \right) + \left( -\gamma' + B_{n-1}' \left( \Phi - \Omega^{1/2} \beta \right) \right) z_t \]

Finally, substituting in for \( p_{t}^{n} \) on the left-hand side, we get the following expression:
\[ A_n + B'_n z_t = \left( -r + A_{n-1} - B'_{n-1} \Omega^{1/2} \lambda + \frac{B'_{n-1} \Omega B_{n-1}}{2} \right) + \left( -\gamma' + B'_{n-1} \left( \Phi - \Omega^{1/2} \beta \right) \right) z_t \]

From this equation, we get the two recursive equations in the text:

\[
\begin{align*}
A_n &= -r + A_{n-1} - B'_{n-1} \Omega^{1/2} \lambda + \frac{B'_{n-1} \Omega B_{n-1}}{2} \\
B'_n &= -\gamma' + B'_{n-1} \left( \Phi - \Omega^{1/2} \beta \right)
\end{align*}
\]

Since we know that \( R_0^0 = 1 \), we can start up these recursions with:

\[
A_0 = 0, \quad B'_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}
\]
Theorem 1 ATSM with depreciation rate included as observable variable can be presented in the state-space form:

\[ x_t = \Gamma x_{t-1} + \Theta \epsilon_t \]

\[ Y_t = f(x_t) + \rho_t, \]

where \( f \) is continuous and differentiable.

**Proof.** Let’s denote the old state vector by \( z_t \). Since depreciation rate is a function of both, current and past \( z_t \), we have to change the system and introduce \( z_{t-1} \) into a new state vector

\[ x_t = \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix} \]

Then the state equation becomes

\[ x_t = \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ z_{t-2} \end{bmatrix} + \begin{bmatrix} \Omega^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix} \]

In the extended ATSM the space equation is a system of observation equations for yields and a depreciation rate. Yields are linear functions of \( z_t \) and hence of \( x_t \)

\[ y_t = A + Bz_t + v_t \]

\[ = A + \begin{pmatrix} b & 0 \end{pmatrix} x_t + v_t \]

Depreciation rate is given by

\[ \Delta s_t = m_t^* - m_t = r_{t-1} + \frac{1}{2} \Lambda_{t-1}^\prime \Lambda_{t-1} + \Lambda_{t-1}^\prime \epsilon_t - r_{t-1} - \frac{1}{2} \Lambda_{t-1}^\prime \Lambda_{t-1} - \Lambda_{t-1}^\prime \epsilon_t \]

\[ = \delta + \gamma z_{t-1} + \frac{1}{2} (\lambda + \beta z_{t-1}) \gamma (\lambda + \beta z_{t-1}) + (\lambda + \beta z_{t-1}) \gamma \Omega^{-\frac{1}{2}} (z_t - \Phi z_{t-1}) \]

\[ - \delta^* - \gamma^* z_{t-1} - \frac{1}{2} (\lambda^* + \beta^* z_{t-1}) \gamma (\lambda^* + \beta^* z_{t-1}) - (\lambda^* + \beta^* z_{t-1}) \gamma \Omega^{-\frac{1}{2}} (z_t - \Phi z_{t-1}) \]

\[ = \delta - \delta^* + \frac{1}{2} (\lambda - \lambda^*) \gamma \]

\[ + (\lambda - \lambda^*) \gamma (\gamma + \gamma^* + \lambda^* \beta - \lambda^* \beta^* + (\lambda^* - \lambda^*) \Omega^{-\frac{1}{2}} \Phi) z_{t-1} \]

\[ + \frac{1}{2} z_{t-1}^\prime \beta^* \Omega^{-\frac{1}{2}} \Phi z_{t-1} + z_{t-1}^\prime (\beta^* - \beta^*) \Omega^{-\frac{1}{2}} z_{t-1} \]

\[ = a + b_{1(1x6)} x_t + x_t^\prime C x_t, \]
where

\[
\begin{align*}
a &= \delta - \delta^* + \frac{1}{2} (\lambda' \Omega \lambda - \lambda'^* \Omega^* \\
b &= \begin{pmatrix}
(\lambda' - \lambda'^*)^{-1/2} \\
(\gamma' - \gamma'^* + \lambda' \beta - \lambda'^* \beta^* + (\lambda'^* - \lambda') \Phi \Omega^{-1/2} \\
\beta' \beta^* - \beta'^* \beta^* + (\beta'^* - \beta) \Omega^{-1/2} \Phi
\end{pmatrix}^T \\
c &= \begin{pmatrix}
0 \\
(\beta' - \beta'^*) \Omega^{-1/2} \Phi \\
\frac{1}{2} (\beta' \beta^* - \beta'^* \beta^*) + (\beta'^* - \beta) \Omega^{-1/2} \Phi
\end{pmatrix}
\end{align*}
\]

The observation equations for yields and depreciation rate can be combined:

\[
Y_{t(m+1)x1} = \begin{pmatrix}
y_t(m_{x1}) \\
\Delta s_t
\end{pmatrix} = \begin{pmatrix}
A_{(m_{x1})} \\
a
\end{pmatrix} + \begin{pmatrix}
B_{i(m_{x6})} x_t \\
b_{i(m_{x6})} x_t
\end{pmatrix} + \begin{pmatrix}
0_{(m_{x1})} \\
\varepsilon_t
\end{pmatrix}
\]

so that the state-space system is given by:

\[
\begin{align*}
x_t &= \Gamma x_{t-1} + \Theta^{1/2} \epsilon_t \\
y_t &= f(x_t) + \rho_t,
\end{align*}
\]

where \( \rho_t \sim N(0, R) \). Although the state equation is linear in \( x_t \), the observation equation is nonlinear: \( f(x_t) \) is quadratic, and hence continuous and differentiable.

In the case of non-linearities in the state or space equations, the extended Kalman filter must be used. The extended Kalman filter, when only the measurement equation is nonlinear, is obtained by linearizing \( f(x_t) \) around the conditional mean \( \hat{x}_{t|t-1}^* \):

\[
f(x_t) = f(\hat{x}_{t|t-1}^*) + H_t \cdot (x_t - \hat{x}_{t|t-1}^*),
\]

where \( H_t = \frac{\partial f(x_t)}{\partial x_t} |_{x_t = \hat{x}_{t|t-1}^*} \). The prediction step equations are the same as before:

\[
\begin{align*}
\hat{x}_{t|t-1}^* &= \Gamma x_{t-1} \\
P_{t|t-1}^* &= \Gamma P_{t-1} \Gamma^T + \Theta
\end{align*}
\]

The update step equation under the extended Kalman Filter is modified to account for the linearization:

\[
\begin{align*}
\hat{x}_t &= \hat{x}_{t|t-1}^* + P_{t|t-1} H_t^T F_t^{-1} v_t, \\
P_t &= P_{t|t-1} - P_{t|t-1} H_t^T F_t^{-1} H_t P_{t|t-1}
\end{align*}
\]
where

\[ v_t = Y_t - f(\hat{x}_{t|t-1}), \]
\[ F_t = H_t P_{t|t-1} H_t' + R, \]

**Proposition 2** In the case of our model, the measurement equation for the extended Kalman filter takes the form

\[ Y_t = f(\hat{x}_{t|t-1}) + H_t (x_t - \Gamma x_{t-1}) + \rho_t, \]

where

\[
H_t = \begin{pmatrix}
B_{(m\times 6)} \\
(b' + (C + C')\Gamma x_{t-1})'
\end{pmatrix}
\]

\[
f(\hat{x}_{t|t-1}) = \begin{pmatrix}
A_{(m\times 1)} + B_{(m\times 6)} \Gamma x_{t-1} \\
a + b_{(1\times 6)} \Gamma x_{t-1} + x_{t-1}' \Gamma C_{(6\times 6)} \Gamma x_{t-1}
\end{pmatrix}
\]

**Proof.**

\[
\frac{\partial}{\partial x_t} f_s|_{x_t=\hat{x}_{t|t-1}} = (b' + (C + C')x_t|_{x_t=\hat{x}_{t|t-1}})' = (b' + (C + C')\Gamma x_{t-1})'
\]

\[
\frac{\partial}{\partial x_t} f_p|_{x_t=\hat{x}_{t|t-1}} = B
\]

hence

\[
f(x_t) = \begin{pmatrix}
A_{(m\times 1)} + B_{(m\times 6)} \Gamma x_{t-1} \\
(a + b_{(1\times 6)} \Gamma x_{t-1} + x_{t-1}' \Gamma C_{(6\times 6)} \Gamma x_{t-1})
\end{pmatrix}
+ \begin{pmatrix}
B_{(m\times 6)} \\
(b' + (C + C') \Gamma x_{t-1})'
\end{pmatrix}
(x_t - \Gamma x_{t-1})
\]

\[ \blacksquare \]
Table 1. Principle component analysis of singular country yields

<table>
<thead>
<tr>
<th>Principal component</th>
<th>Yield loadings</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$y_{12}^t$</td>
</tr>
<tr>
<td>1</td>
<td>0.9215</td>
</tr>
<tr>
<td>2</td>
<td>0.9941</td>
</tr>
<tr>
<td>3</td>
<td>0.9998</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2. Principle component analysis of international yields.

<table>
<thead>
<tr>
<th>Principal component</th>
<th>Proportion of total variance explained</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>UK and US</td>
</tr>
<tr>
<td>1</td>
<td>0.8473</td>
</tr>
<tr>
<td>2</td>
<td>0.9570</td>
</tr>
<tr>
<td>3</td>
<td>0.9877</td>
</tr>
<tr>
<td>4</td>
<td>0.9972</td>
</tr>
<tr>
<td>5</td>
<td>0.9993</td>
</tr>
<tr>
<td>6</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

Table 3. Results from Fama regression

<table>
<thead>
<tr>
<th>UK and US</th>
<th>UK and Germany</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td></td>
</tr>
</tbody>
</table>
References


[26] **Wu, S (2007)**, 'Interest rate risk and the forward premium anomaly in foreign exchange markets', *Journal of Money, Credit & Banking*