Stochastic Dominance Option Bounds and Nth Order Arbitrage Opportunities

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Abstract

In this paper we first derive optimal Nth order stochastic dominance option bounds from concurrently expiring options. These bounds are given by pricing kernels that have piecewise constant $(N-2)$th derivatives. Using numerical examples we show that option bounds are improved significantly when we raise the order of stochastic dominance rules. When these option bounds are violated there are Nth order arbitrage opportunities interpreted as the comparison of (weighted average) conditional expected returns. We then establish the way to make profits from these arbitrage opportunities in option markets.

Keywords: Option bounds, Option pricing, Stochastic Dominance, Arbitrage opportunities.

JEL Classification Numbers: G13.

Introduction

It has been recognized that meaningful option bounds can be obtained under less strong assumptions than exact option prices. Perrakis and Ryan (1984), Ritchken (1985), and Levy (1985) derive option bounds under the assumption of risk aversion or second order stochastic dominance.[1] Ritchken and Kuo (1989) derive option bounds under the assumption of higher order stochastic dominance rules. Basso and Pianca (1997) and Mathur and Ritchken (2000) obtain option pricing bounds by assuming decreasing absolute (relative) risk aversion (hereafter DARA (DRRA)).

Ryan (2003) tightens the second order stochastic dominance option bounds by using the observed price of one concurrently expiring option at a time. Huang (2004b) uses a new methodology to further improve second order stochastic dominance option bounds and discusses the second order arbitrage opportunities in the markets of concurrently expiring options. The methodology is presented by Huang (2004a) which takes the advantage of options's distinctive features. Using the same methodology, Huang (2004c) improves the DARA and DRRA bounds by using the observed prices of concurrently expiring options. Huang (2004d) derives option bounds from concurrently expiring options when the pricing representative investor's relative risk aversion is bounded.

[1] We use SSD, TSD, and NSD to denote the second, third, and Nth order stochastic dominance.

In this paper we tighten the Nth ($N \geq 3$) order stochastic dominance (hereafter NSD) option bounds by using the observed prices of concurrently expiring options. We show that given the prices of a unit bond, underlying stock, and n option prices, the kth order stochastic dominance option bounds are given by a pricing kernel whose $(N-2)$th derivative is $(n/2)$-segmented and piecewise constant if n is even or $((n+1)/2)$-segmented and piecewise constant if n is odd.

Using numerical examples we show that option bounds from concurrently expiring options are improved significantly when we raise the order of stochastic dominance rules. We find that higher order stochastic dominance rules give satisfactory bounds on deep in-the-money options even using only one observed option.

When the Nth stochastic dominance option bounds are violated then there are Nth order arbitrage opportunities. An Nth order arbitrage can be interpreted as the comparison of conditional expected returns taken weighted average $N-2$ times. We present the Nth order arbitrage portfolios which can be used to make profits when the NSD option bounds are violated.

This paper is also related to the important works by Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000). Cochrane and Saa-Requejo derive option bounds using restrictions on the volatility of the pricing kernel, while Bernardo and Ledoit derive option pricing bounds using restrictions on the deviation of the pricing kernel from a benchmark pricing kernel.

Other related works include Lo (1987), Grundy (1991), and Constantinides and Zariphopoulou (1999, 2001) who all derive option bounds under different conditions.

The structure of the remaining paper is as follows: In Section 1 we derive option bounds from concurrently expiring options when the third order stochastic dominance rule applies. In Section 2 we generalize the results in Section 1 to the case where the Nth ($N \geq 3$) order stochastic dominance rule applies. In Section 3 we present the arbitrage portfolios. In Section 4 we present numerical examples and give comparisons of option bounds under different orders of stochastic dominance rules. The final section concludes the paper.

1 TSD Option Bounds

In this section we derive option bounds from concurrently expiring options assuming third order stochastic dominance. According to Ritchken and Kuo (1989), applying third order stochastic dominance we have a pricing kernel that is decreasing and convex in the underlying stock price.

1.1 The Problem and Third Order Arbitrage

Given the prices of a unit bond, a stock and some European options written on the stock with the same maturity, we want to know the bounds on the price of another option which has the same maturity when Nth ($N \geq 3$) order stochastic dominance rule applies. That is,

$$\max \text{ (or min) } E(c^X(S_T)\phi(S_T))B_0$$

subject to

$$\phi(x) \geq 0; \quad \phi'(x) \leq 0; \quad \phi'(x) \text{ is increasing in x.}$$

$$E(\phi(S_T) = 1$$

$$E(S_T\phi(S_T))B_0 = S_0$$

$$E(c_i(S_T)\phi(S_T))B_0 = c_0^i, \quad i = 1, ..., n.$$

The dual problem is[2]

$$\min \ (\text{or} \ \max) \ \alpha_1 B_0 + \alpha_2 S_0 + \sum_{i=1}^{n} \alpha_i c_0^i$$

subject to

$$\frac{\int_0^s E(\alpha_1 + \alpha_2 x + \sum_{i=1}^n \alpha_i c_i(x) | x < x_0) Pr(x < x_0) dx_0}{\int_0^{S_T} Pr(x < x_0) dx_0}$$
$$\geq (\leq) \frac{\int_0^{S_T} E(c^X(x) | x < x_0) Pr(x < x_0) dx_0}{\int_0^{S_T} Pr(x < x_0) dx_0}, \quad \text{for all } s. \tag{1}$$

The above dual problem suggests a third order arbitrage opportunity if the option bound is violated. Suppose, for example, the upper bound is violated. Then by selling the over-priced option c^X and buying the arbitrage portfolio $L \equiv (\alpha_1, ..., \alpha_{n+2})$, we make a profit at time 0. Now consider the net payoff $(L(S_t) - c^X(S_t))$ at the maturity of the option, where $c^X(S_t)$ and $L(S_t)$ are the payoffs of the option and portfolio respectively. According to (1), the following expression

$$\frac{\int_0^s E(L(S_t) - C_0^X(S_t) | S_t < x_0) Pr(S_t < x_0) dx_0}{\int_0^s Pr(S_t < x_0) dx_0} \tag{2}$$

[2]The proof of the duality can be given by using the Ritchken and Kuo's (1989) results in the discrete case. For brevity, it is omitted.

will be non-negative for all s.

Note $E(L(S_t) - C_0^X(S_t)|S_t < x_0)$ is the expected net payoff conditional on the stock price being lower than x_0. It is clear that (2) is the weighted average of this conditional expectation for all $x_0 < s$. Thus (2) is non-negative for all s implies that the weighted average expectation of the net payoff conditional on the stock price being lower than s is always non-negative for all s. This statement is strong; it simply means that it is very safe to expect that the net payoff at time 1 to be non-negative. This is the essential point of third order arbitrage.

We now compare third order arbitrage with second order arbitrage. In the case of second order arbitrage, the conditional expectation of the net payoff $E(L(S_t) - C_0^X(S_t)|S_t < s)$ will be always non-negative for all s.[3] However, in the case of third order arbitrage, only a weighted average of this conditional expectation is guaranteed to be non-negative. It is obvious that if for all s the conditional expected net payoff $E(L(S_t) - C_0^X(S_t)|S_t < s)$ is positive, then the weighted average conditional expected net payoff is also non-negative for all s; but the reverse is not true. Hence a second order arbitrage opportunity is always a third order arbitrage opportunity while the reverse is not true.

To derive the TSD option bounds, we first solve a similar but more general problem in which we assume that not only the third order stochastic dominance rule applies but also the absolute value of the pricing kernel's first derivative is bounded from above.

[3] See Ryan (2003) or Huang (2004b).

We will show in this paper that under this condition, the option bounds are given by a pricing kernel that has piecewise constant first derivative, where the number of segments of the risk aversion depends on the number of observed option prices.

Moreover, we will see that for an even number of observed option prices the pricing kernel that gives the option bounds has a certain pattern while for an odd number of observed option prices the pricing kernel that gives the option bounds has a different pattern. Thus in order to explain the solutions more clearly we start with the case where we have only one observed option price then continue with the case where we have two observed options. Building on the above two cases we explore the general case where we have n observed options.

1.2 The Case with One Observed Option

In this subsection we deal with the case where we observe the price of one concurrently expiring option. Before we proceed, we introduce two lemmas.

Lemma 1 (FSS (1999)) *Assume two pricing kernels give the same stock price. If they intersect twice, then the pricing kernel with fatter tails gives higher prices of convex-payoff contingent claims writhen on the stock.*

Proof: See the proof of Theorem 1 in FSS (1999).

Lemma 2 *Assume two pricing kernels give the same prices of the underlying* *stock and an option with strike price K. If they intersect three times, then the pricing kernel with fatter left tail will give higher [lower] prices for all options with strike prices below [above] K than the other.*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's first derivative is bounded from above.

Lemma 3 *Assume the pricing kernel is decreasing and convex in S_t and its first derivative is bounded below by $-\beta$. Assume the current price of a unit bond is B_0, the current price of the underlying stock is S_0, and the current price of an option with strike price K is c_{K0}.*

- *Then the upper bound for an option with strike price below K is given by a pricing kernel, $\phi_1^{**}(x)$, which has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\beta$ for $x < s_1^{**}$, $-\beta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$. That is,*

$$\phi_1^{**}(x) = \begin{array}{ll} \beta^{**}(s_2^{**} - s_1^{**}) + \beta(s_1^{**} - x), & x < s_1^{**} \\ \beta^{**}(s_2^{**} - x), & x \in (s_1^{**}, s_2^{**}) \\ 0, & x \geq s_2^{**}, \end{array}$$

  where s_1^{**}, s^{**}, and β^{**} are to be determined such that $\beta > \beta^{**} > 0$, $E(\phi_1^{**}(S_t)) = 1$, $E(S_t\phi_1^{**}(S_t)) = S_0/B_0$, and $E(c_K(S_t)\phi_1^{**}(S_t)) = c_{K0}/B_0$.

- *The lower bound for an option with strike price below K is given by a pricing kernel, $\phi_0^{*}(x)$, which has a two-segmented and piecewise constant* *first derivative. More precisely its first derivative is equal to* $-\beta^*$ *for* $x < s^*$*, and zero for* $x > s^*$*, and its value is zero at its right tail. That is,*

$$\phi_1^*(x) = \{ \begin{array}{ll} b + \beta^*(s^* - x), & x < s^* \\ b, & x \geq s^*, \end{array}$$

where b*,* β^**, and* s^* *are to be determined such that* $-\beta^* > 0$*,* $E(\phi_1^*(S_t)) = 1$*,* $E(S_t \phi_0^*(S_t)) = S_0/B_0$*, and* $E(c_K(S_t)\phi_1^*(S_t)) = c_{K0}/B_0$*.*

- *The upper (lower) bound for an option with strike price above* K *is given by pricing kernel* $\phi_0^*(x)$ $(\phi_0^{**}(x))$*.*

Proof: From Lemma 2 we need only prove that the pricing kernels described in the lemma intersect all admissible pricing kernels exactly three times and then examine the fatness of their left tails.

We first examine ϕ_1^{**}. Note it has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\beta$ for $x < s_1^{**}$, $-\beta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is zero at its right tail.

From Lemma 2 we need only prove that the pricing kernels described in the lemma intersect all admissible pricing kernels exactly three times and then examine the fatness of their left tails.

We first examine ϕ_1^{**}. Note it is three-segmented and piecewise constant. More precisely $\phi_1^{**} = \overline{\phi}$, $S < s_l$; $\phi_1^{**}(S) = a_1$, $s_l < S < s_u$; $\phi_1^{**}(S) = \underline{\phi}$, $S > s_u$, $\overline{\phi} \geq a_1 \geq \underline{\phi}$. Obviously this pricing kernel intersects any admissible pricing kernel at most three times. However from Lemma 1, it must intersect all admissible pricing kernels at least three times; otherwise they cannot give the same observed option price. Hence ϕ_1^{**} intersects all admissible pricing kernel exactly three times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. For ϕ_1^{*} the proof is similar. Q.E.D.

Proposition 1 *Assume the pricing kernel is decreasing and convex in S_t. Assume the current price of a unit bond is B_0, the current price of the underlying stock is S_0, and the current price of an option with strike price K is c_{K0}. Let $\underline{s}$ be the lowest possible value of S_t.*

- *Then the upper bound for an option with strike price below K is given by the pricing kernel $\varphi_1^{**}(S_t) = a_0 \frac{\delta(S_t - \underline{s})}{p(S_t)} + f_1^{**}(S_t)$, where $p(S_t)$ is the true probability density function and $\delta(x)$ is the Dirac function and*

$$f_1^{**}(S_t) = \{ \begin{array}{cc} \beta^{**}(s^{**} - x), & x < s^{**} \\ 0, & x \geq s^{**}, \end{array}$$

  where a_0, s^{**}, and β^{**} are to be determined such that $E(\varphi_1^{**}(S_t)) = 1$, $E(S_t \varphi_1^{**}\ (S_t)) = S_0/B_0$, and $E(c_K(S_t)\varphi_1^{**}(S_t)) = c_{K0}/B_0$.

- *The lower bound for an option with strike price below K is given by the pricing kernel $\varphi_1^{*}(x) = \phi_1^{*}(x)$.*

- *The upper (lower) bound for an option with strike price above K is given by pricing kernel $\phi_0^{*}(x)$ $(\phi_0^{**}(x))$.*

Proof: Let $\beta \to +\infty$; we immediately obtain the result from Lemma 3.

1.3 The Case with Two Observed Options

In this subsection we deal with the case where we have two observed concurrently expiring options. We first introduce a lemma.

Lemma 4 *Assume two pricing kernels give the same prices of the underlying stock and two options with strike prices* K_1 *and* K_2*, where* $K_1 < K_2$*. If they intersect four times, then the pricing kernel with fatter left tail will give higher (lower) prices for options with strike prices outside (inside)* (K_1, K_2)*.*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's first derivative is bounded from above.

Lemma 5 *Assume the pricing kernel is decreasing and convex in* S_t *and its first derivative is bounded below by* $-\beta$*. Assume the current price of a unit bond is* B_0*, the current price of the underlying stock is* S_0*, and the current prices of two options with strike prices* K_1 *and* K_2 *are* c_0^1 *and* c_0^2 *respectively.*

- *Then the upper bound for an option with a strike price below* K_1 *or above* K_2 *is given by a pricing kernel,* $\phi_2^{**}(x)$*, which has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to* $-\beta$ *for* $x < s_1^{**}$*,* $-\beta^{**}$ *for* $x \in (s_1^{**}, s_2^{**})$*, and zero for* $x > s_2^{**}$*,* *and its value is a positive constant b at its right tail. That is,*

$$\phi_2^{**}(x) = \{ \begin{array}{ll} b + \beta^{**}(s_2^{**} - s_1^{**}) + \beta(s_1^{**} - x), & x < s_1^{**} \\ b + \beta^{**}(s_2^{**} - x), & x \in (s_1^{**}, s_2^{**}) \\ b, & x \geq s_2^{**}, \end{array}$$

where b, s_1^{**}, s_2^{**}, and β^{**} are to be determined such that $\beta > \beta^{**} > 0$, $E(\phi_1^{**}(S_t)) = 1$, $E(S_t\phi_1^{**}(S_t)) = S_0/B_0$, and $E(c^i(S_t)\phi_1^{**}(S_t)) = c_0^i/B_0$, $i = 1, 2$.

- *The lower bound for an option with a strike price below K_1 or above K_2 is given by a pricing kernel,$\phi_2^*(x)$, which has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\beta_1^*$ for $x < s_1^*$, $-\beta_2^*$ for $x \in (s_1^*, s_2^*)$, and zero for $x > s_2^*$. That is,*

$$\phi_2^*(x) = \{ \begin{array}{ll} \beta_1^*(s_2^* - s_1^*) + \beta_2^*(s_1^* - x), & x < s_1^* \\ \beta_1^*(s_2^* - x), & x \in (s_1^*, s_2^*) \\ 0, & x \geq s_2^*, \end{array}$$

where s_1^*, s_2^*, β_1^*, and β_2^* are to be determined such that $\beta_1^* > \beta_2^* > 0$, $E(\phi_1^*(S_t)) = 1$, $E(S_t\phi_1^*(S_t)) = S_0/B_0$, and $E(c^i(S_t)\phi_1^*(S_t)) = c_0^i/B_0$, $i = 1, 2$.

Proof: From Lemma 4 we need only prove that the pricing kernels described in the lemma intersect all admissible pricing kernels exactly four times and then examine the fatness of their left tails.

We first examine ϕ_2^{**}. Note it has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\beta$ for $x < s_1^{**}$,

$-\beta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is a positive constant b at its right tail. We can verify that this pricing kernel intersects any admissible pricing kernel at most four times.

However from Lemma 2, it must intersect all admissible pricing kernels at least four times; otherwise they cannot give the same observed option prices. Hence ϕ_1^{**} intersects all admissible pricing kernel exactly four times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. For ϕ_1^* the proof is similar. Q.E.D.

Proposition 2 *Assume the pricing kernel is decreasing and convex in S_t. Assume the current price of a unit bond is B_0, the current price of the underlying stock is S_0, and the current prices of two options with strike prices K_1 and K_2 are c_0^1 and c_0^2 respectively. Let $\underline{s}$ be the lowest possible value of S_t.*

- *Then the upper bound for an option with a strike price below K_1 or above K_2 is given by the pricing kernel $\varphi_2^{**}(S_t) = a_0 \frac{\delta(S_t - \underline{s})}{p(S_t)} + f_2^{**}(S_t)$, where $p(S_t)$ is the true probability density function and $\delta(x)$ is the Dirac function and*

$$f_2^{**}(S_t) = \left\{ \begin{array}{ll} b + \beta^{**}(s^{**} - x), & x < s^{**} \\ b, & x \geq s^{**}, \end{array} \right.$$

  where a_0, b, s^{**}, and β^{**} are to be determined such that $E(\varphi_2^{**}(S_t)) = 1$, $E(S_t \varphi_2^{**}(S_t)) = S_0/B_0$, and $E(c^i(S_t)\varphi_2^{**}(S_t)) = c_0^i/B_0$, $i = 1, 2$.

- *The lower bound for an option with a strike price below K_1 or above K_2 is given by the pricing kernel $\varphi_2^*(x) = \phi_2^*(x)$.*

Proof: Let $\beta \to +\infty$; we immediately obtain the result from Lemma 5.

1.4 The General Case

In this subsection we deal with the case where we have n observed concurrently expiring options. We first introduce a lemma.

Lemma 6 *Assume two pricing kernels give the same prices of the underlying stock and options with strike prices K_1, K_2, ..., K_n, where $K_1 < K_2 < ... < K_n$. Let $K_0 = 0$ and $K_{n+1} = +\infty$. If the two pricing kernels intersect $n+2$ times then the one with fatter left tail will give higher (lower) prices for all options with strike prices between (K_{2i-2}, K_{2i-1}) $((K_{2i-1}, K_{2i}))$, $i = 1, 2, ...$*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's first derivative is bounded from above.

Lemma 7 *Assume the pricing kernel is decreasing and convex in S_t and its first derivative is bounded below by $-\beta$. Assume the current price of a unit bond is B_0, the current price of the underlying stock is S_0, and the current prices of n options with strike prices K_1, ..., and K_n are c_0^1, ..., and c_0^n respectively. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

- *Assume n is odd. Let $m = (n+1)/2$.*
  - *Then the upper bound for options with strike prices between $(K_{2i-2}$, $K_{2i-1})$, $i = 1, 2, ...$, is given by a pricing kernel, $\phi_n^{**}(x)$, which has a (m+2)-segmented and piecewise constant first derivative. More*

precisely its first derivative is equal to $-\beta$ *for* $x < s_1^{**}$, $-\beta_1^{**}$ *for* $x \in (s_1^{**}, s_2^{**})$, ..., $-\beta_m^{**}$ *for* $x \in (s_m^{**}, s_{m+1}^{**})$*, and zero for* $x > s_{m+1}^{**}$*. That is,* $\phi_n^{**}(x) =$

$$
\begin{array}{ll}
\beta_m^{**}(s_{m+1}^{**} - s_m^{**}) + ... + \beta_1^{**}(s_2^{**} - s_1^{**}) + \beta(s_1^{**} - x), & x < s_1^{**} \\
\text{......} & \\
\beta_m^{**}(s_{m+1}^{**} - x), & x \in (s_m^{**}, s_{m+1}^{**}) \\
0, & x \geq s_{m+1}^{**},
\end{array}
$$

where s_1^{**}, ..., s_{m+1}^{**}, β_1^{**}, ..., *and* β_m^{**} *are to be determined such that* $\beta > \beta_1^{**} > ... > \beta_m^{**}$, $E(\phi_n^{**}(S_t)) = 1$, $E(S_t\phi_n^{**}(S_t)) = S_0/B_0$*, and* $E(c^i(S_t)\phi_n^{**}(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

- *The lower bound for options with strike prices between* (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...$, *is given by a pricing kernel* $\phi_n^*(x)$*, which has a (m+1)-segmented and piecewise constant first derivative. More precisely its first derivative is equal to* $-\beta_1^*$ *for* $x < s_1^*$, ..., $-\beta_m^*$ *for* $x \in (s_{m-1}^*, s_m^*)$*, and zero for* $x > s_m^*$*. That is,* $\phi_n^*(x) =$

$$
\begin{array}{ll}
b + \beta_m^*(s_m^* - s_{m-1}^*) + ... + \beta_2^*(s_2^* - s_1^*) + \beta_1^*(s_1^* - x), & x < s_1^* \\
\text{......} & \\
b + \beta_m^*(s_m^* - x), & x \in (s_{m-1}^*, s_m^*) \\
b, & x \geq s_m^*,
\end{array}
$$

where b, s_1^*, ..., s_m^*, β_1^*, ..., β_m^* *are to be determined such that* $\beta_1^* > ... > \beta_m^*$, $E(\phi_n^*(S_t)) = 1$, $E(S_t\phi_n^*(S_t)) = S_0/B_0$*, and* $E(c^i(S_t)\phi_n^*(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

- *Assume* n *is even. Let* $m = n/2$.

– *Then the upper bound for options with strike prices between* (K_{2i-2}, K_{2i-1})*,* $i = 1, 2, ...,$ *is given by a pricing kernel,* $\phi_n^{**}(x)$*, which has a (m+2)-segmented and piecewise constant first derivative. More precisely its first derivative is equal to* $-\beta$ *for* $x < s_1^{**}$*,* $-\beta_1^{**}$ *for* $x \in (s_1^{**}, s_2^{**})$*, ...,* $-\beta_m^{**}$ *for* $x \in (s_m^{**}, s_{m+1}^{**})$*, and zero for* $x > s_{m+1}^{**}$*, and its value at its right tail is a positive constant b. That is,* $\phi_n^{**}(x) =$

$$
\begin{array}{ll}
b + \beta_m^{**}(s_{m+1}^{**} - s_m^{**}) + ... + \beta_1^{**}(s_2^{**} - s_1^{**}) + \beta(s_1^{**} - x), & x < s_1^{**} \\
...... & \\
b + \beta_m^{**}(s_{m+1}^{**} - x), & x \in (s_m^{**}, s_{m+1}^{**}) \\
b, & x \geq s_{m+1}^{**},
\end{array}
$$

where b, s_1^{**}*, ...,* s_{m+1}^{**}*,* β_1^{**}*, ..., and* β_m^{**} *are to be determined such that* $\beta > \beta_1^{**} > ... > \beta_m^{**}$*,* $E(\phi_n^{**}(S_t)) = 1$*,* $E(S_t\phi_n^{**}(S_t)) = S_0/B_0$*, and* $E(c^i(S_t)\phi_n^{**}(S_t)) = c_0^i/B_0$*,* $i = 1, 2, ..., n$*.*

– *The lower bound for options with strike prices between* (K_{2i-2}, K_{2i-1})*,* $i = 1, 2, ...,$ *is given by a pricing kernel,* $\phi_n^*(x)$*, which has a (m+2)-segmented and piecewise constant first derivative. More precisely its first derivative is equal to* $-\beta_1^*$ *for* $x < s_1^*$*, ...,* $-\beta_{m+1}^*$ *for* $x \in (s_m^*, s_{m+1}^*)$*, and zero for* $x > s_{m+1}^*$*. That is,* $\phi_n^*(x) =$

$$
\begin{array}{ll}
\beta_{m+1}^*(s_{m+1}^* - s_m^*) + ... + \beta_2^*(s_2^* - s_1^*) + \beta_1^*(s_1^* - x), & x < s_1^* \\
...... & \\
\beta_{m+1}^*(s_{m+1}^* - x), & x \in (s_{m-1}^*, s_m^*) \\
0, & x \geq s_{m+1}^*,
\end{array}
$$

where s_1^**, ...,* s_{m+1}^**,* β_1^**, ..., and* β_{m+1}^* *are to be determined such* *that* $\beta_1^* > ... > \beta_{m+1}^*$, $E(\phi_n^*(S_t)) = 1$, $E(S_t\phi_n^*(S_t)) = S_0/B_0$, *and* $E(c^i(S_t)\phi_n^*(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

Proof: From Lemma 6 we need only prove that the pricing kernels described in the lemma intersect all admissible pricing kernels exactly $(n+2)$ times and then examine the fatness of their left tails.

We first examine ϕ_n^{**}. Assume n is odd. Note it has a (m+2)-segmented and piecewise constant first derivative, where $m = (n+1)/2$. More precisely, its first derivative is equal to $-\beta$ for $x < s_1^{**}$, $-\beta_1^{**}$ for $x \in (s_1^{**}, s_2^{**})$, ..., $-\beta_m^{**}$ for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$, and its value at its right tail is 0.

We can verify that this pricing kernel intersects any decreasing pricing kernel at most $(n+2)$ times. However from Lemma 6, it must intersect all admissible pricing kernels at least $(n+2)$ times; otherwise they cannot give the same observed option prices. Hence ϕ_n^{**} intersects all admissible pricing kernel exactly $(n+2)$ times. It is not difficult to verify that ϕ_n^{**} has fatter left tail. This proves the first result. Other results can similarly proved. Q.E.D.

Proposition 3 *Assume the pricing kernel is decreasing and convex in S_t. Assume the current price of a unit bond is B_0, the current price of the underlying stock is S_0, and the current prices of n options with strike prices K_1, ..., and K_n are c_0^1, ..., and c_0^n respectively. Let $\underline{s}$ be the lowest possible value of S_t. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

- *Assume n is odd. Let $m = (n+1)/2$.*
  - *Then the upper bound for options with strike prices between* $(K_{2i-2}$,

$K_{2i-1})$, $i = 1, 2, ...,$ *is given by the pricing kernel* $\varphi_n^{**}(S_t) = a_0 \frac{\delta(S_t - \underline{s})}{p(S_t)} +$ $f_n^{**}(S_t)$, *where* $p(S_t)$ *is the true probability density function,* $\delta(x)$ *is the Dirac function, and* $f_n^{**}(x) =$

$$\begin{array}{ll} \beta_m^{**}(s_m^{**} - s_{m-1}^{**}) + ... + \beta_2^{**}(s_2^{**} - s_1^{**}) + \beta_1^{**}(s_1^{**} - x), & x < s_1^{**} \\ \multicolumn{2}{c}{......} \\ \beta_m^{**}(s_m^{**} - x), & x \in (s_{m-1}^{**}, s_m^{**}) \\ 0, & x \geq s_m^{**}, \end{array}$$

where a_0, s_1^{**}, ..., s_m^{**}, β_1^{**}, ..., *and* β_m^{**} *are to be determined such that* $\beta_1^{**} > ... > \beta_m^{**}$, $E(\phi_n^{**}(S_t)) = 1$, $E(S_t\phi_n^{**}(S_t)) = S_0/B_0$, *and* $E(c^i(S_t)\phi_n^{**}(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

- *The lower bound for options with strike prices between* (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ *is given by the pricing kernel* $\varphi_n^*(x) =$

$$\begin{array}{ll} b + \beta_m^*(s_m^* - s_{m-1}^*) + ... + \beta_2^*(s_2^* - s_1^*) + \beta_1^*(s_1^* - x), & x < s_1^* \\ \multicolumn{2}{c}{......} \\ b + \beta_m^*(s_m^* - x), & x \in (s_{m-1}^*, s_m^*) \\ b, & x \geq s_m^*, \end{array}$$

where b, s_1^*, ..., s_m^*, β_1^*, ..., β_m^* *are to be determined such that* $\beta_1^* > ... > \beta_m^*$, $E(\phi_n^*(S_t)) = 1$, $E(S_t\phi_n^*(S_t)) = S_0/B_0$, $E(c^i(S_t)\phi_n^*(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

- *Assume* n *is even. Let* $m = n/2$.

  - *Then the upper bound for options with strike prices between* $(K_{2i-2}$, $K_{2i-1})$, $i = 1, 2, ...,$ *is given by a pricing kernel* $\varphi_n^{**}(S_t) = a_0 \frac{\delta(S_t - \underline{s})}{p(S_t)} +$

$b + f_n^{**}(S_t)$, *where* $p(S_t)$ *is the true probability density function,* $\delta(x)$ *is the Dirac function, and* $f_n^{**}(x) =$

$$
\begin{cases}
\beta_m^{**}(s_m^{**} - s_{m-1}^{**}) + ... + \beta_2^{**}(s_2^{**} - s_1^{**}) + \beta_1^{**}(s_1^{**} - x), & x < s_1^{**} \\
...... & \\
\beta_m^{**}(s_m^{**} - x), & x \in (s_{m-1}^{*}, s_m^{*}) \\
0, & x \geq s_m^{**},
\end{cases}
$$

where a_0, b, s_1^{**}, ..., s_m^{**}, β_1^{**}, ..., *and* β_m^{**} *are to be determined such that* $\beta_1^{**} > ... > \beta_m^{**}$, $E(\phi_n^{**}(S_t)) = 1$, $E(S_t\phi_n^{**}(S_t)) = S_0/B_0$, *and* $E(c^i(S_t)\phi_n^{**}(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

– *The lower bound for options with strike prices between* (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...$, *is given by a pricing kernel* $\varphi_n^*(x) =$

$$
\begin{cases}
\beta_{m+1}^{*}(s_{m+1}^{*} - s_m^{*}) + ... + \beta_2^{*}(s_2^{*} - s_1^{*}) + \beta_1^{*}(s_1^{*} - x), & x < s_1^{*} \\
...... & \\
\beta_{m+1}^{*}(s_{m+1}^{*} - x), & x \in (s_m^{**}, s_{m+1}^{**}) \\
0, & x \geq s_{m+1}^{*},
\end{cases}
$$

where s_1^*, ..., s_{m+1}^*, β_1^*, ..., *and* β_{m+1}^* *are to be determined such that* $\beta_1^* > ... > \beta_{m+1}^*$, $E(\phi_n^*(S_t)) = 1$, $E(S_t\phi_n^*(S_t)) = S_0/B_0$, *and* $E(c^i(S_t)\phi_n^*(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

Proof: Let $\beta \to +\infty$; we immediately obtain the result from Lemma 5.

2 NSD Option Bounds

In this section we apply the Nth order stochastic dominance rule to derive option bounds from concurrently expiring options. According to Ritchken and Kuo (1989), applying Nth order stochastic dominance we have such a pricing kernel that its derivatives alternate to be negative and positive up to the $(N-1)$th order.

2.1 The Problem and Nth Order Arbitrage

Given the prices of a unit bond, a stock and some European options written on the stock with the same maturity, we want to know the bounds on the price of another option which has the same maturity when Nth ($N \geq 3$) order stochastic dominance rule applies. That is,

$$\max \ (\text{or} \ \min) \ E(c^X(S_t)\phi(S_t))B_0$$

subject to

$$\phi^{(i)}(x) \geq 0, \quad \text{for even } i \leq N; \quad \phi^{(i)}(x) \leq 0, \quad \text{for odd } i \leq N$$

$$E(\phi(S_t) = 1$$

$$E(S_t\phi(S_t))B_0 = S_0$$

$$E(c^i(S_t)\phi(S_t))B_0 = c_0^i, \quad i = 1, ..., n.$$

Before we present the dual problem we first explain the notation. We define $E^{(N-1)}(.|S_t < s)$ as follows: given a function $f(x)$,

$$E^{(i)}(f(S_t)|S_t < s) = \frac{\int_0^s E^{(i-1)}(f(S_t)|S_t < y)Pr^{(i-1)}(S_t < y)dy}{Pr^{(i)}(S_t < y)},$$

$$E^{(1)}(f(S_t)|S_t < s) = E(f(S_t)|S_t < s),$$

$$Pr^{(i)} = \int_0^s Pr^{(i-1)}(S_t < y)dy,$$

$$Pr(1)(S_t < s) = Pr(S_t < s).$$

From the above definition, it is clear that $E^{(i)}(c^X(x)|x < s)$ is the weighted average of $E^{(i-1)}(c^X(x)|x < s)$ with weight equal to $Pr^{(i-1)}(x < s)$. Noting $E^{(1)}(f(S_t)|S_t < s) = E(f(S_t)|S_t < s)$, by induction $E^{(i)}(c^X(x)|x < s)$ is the conditional expectation of $f(S_t)$ (conditional on the stock price being lower than s) taken weighted average $i-1$ times. We now present the dual problem.

The dual problem[4]

$$\min \ (\text{or } \max) \ \alpha_1 B_0 + \alpha_2 S_0 + \sum_{i=1}^{n} \alpha_{n+2} c_0^i$$

subject to

$$E^{(N-1)}(\alpha_1 + \alpha_2 S_t + \sum_{i=1}^{n} \alpha_{n+2} c^i(S_t)|S_t < s) \geq (\leq) E^{(N-1)}(c^(S_t)|S_t < s),$$

The above dual problem suggests an Nth ($N > 3$) order arbitrage opportunity if the option bound is violated. Nth order arbitrage can be understood in the same way as third order arbitrage.[5] Nth order arbitrage can be interpreted as the comparison of conditional expected payoffs taken weighted average $N-2$ times.

Obviously, if $E^{(N-2)}(f(S_t)|S_t < s) \geq 0$ for all s then its weighted average $E^{(N-1)}(f(S_t)|S_t < s) \geq 0$ for all s; but the reverse is not true. Thus an $(N-1)$th

[4] The proof of the duality is similar to the case of TSD. For brevity, it is omitted.

[5] See the argument for the case of TSD.

order arbitrage opportunity is always an Nth order arbitrage opportunity while the reverse is not true.

In the next subsection we will present the NSD option bounds. We will show that under the NSD assumption, the option bounds are given by pricing kernels which have piecewise constant $(N-2)$th derivative, where the number of segments of the $(N-2)$th derivative depends on the number of observed options.

2.2 The Solution

As in the TSD case, to derive the NSD option bounds we can first solve a similar but more general problem in which we assume that not only the Nth order stochastic dominance rule applies but also the absolute value of the pricing kernel's Nth derivative is bounded above. However, since the derivation is similar to the TSD case, it is omitted entirely for brevity. We present our results in the general case straightaway.

Proposition 4 *Assume the pricing kernel $\phi(x)$ satisfies the Nth order stochastic dominance rule, i.e., $\phi^{(i)}(x) \geq 0$, for even $i < N$, $\phi^{(i)}(x) \leq 0$, for odd $i < N$, and $|\phi^{(N-1)}(x)|$ is increasing in x. Assume $|\phi^{(N-2)}(x)|$ is bounded above by β. Assume the current price of a unit bond is B_0, the current price of the underlying stock is S_0, and the current prices of n options with strike prices K_1, ..., and K_n are c_0^1, ..., and c_0^n respectively. Let $\underline{s}$ be the lowest possible value of S_t. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

1. *Assume n is odd. Let $m = (n+1)/2$.*

(a) *Then the upper bound for options with strike prices between* (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ *is given by the pricing kernel* $\varphi_n^{**}(S_t) = a_0 \frac{\delta(S_t - \underline{s})}{p(S_t)} + f_n^{**}(S_t)$, *where* $p(S_t)$ *is the true probability density function,* $\delta(x)$ *is the Dirac function, and* $f_n^{**}(x)$ *has a (m+1)-segmented and piecewise constant (N-2)th derivative, where the absolute of value of its (N-2)th derivative is equal to* β_1^{**} *for* $x < s_1^{**}$, ..., β_m^{**} *for* $x \in (s_{m-1}^{**}, s_m^{**})$, *and zero for* $x > s_m^{**}$. *More precisely,* $f_n^{**}(x) = f_{ni}^{**}(x)$, *for* $x \in (s_{i-1}^{**}, s_i^{**})$, $i = 1, ..., m+1$, *where* $s_0^{**} = 0$, $s_{m+1}^{**} = +\infty$, $f_{n(m+1)}^{**}(x) = 0$, $f_{nm}^{**}(x) = \frac{\beta_m^{**}}{(N-2)!}(s_m^{**} - x)^{N-2}$, *and for* $i = 1, ..., m-1$,

$$\begin{aligned} f_{ni}^{**}(x) &= f_{n(i+2)}^{**}(s_{i+1}^{**}) + \beta_i^{**} \frac{(s_i^{**} - x)^{N-2}}{(N-2)!} \\ &+ \beta_{i+1}^{**} \sum_{1}^{N-2} \frac{(s_{i+1}^{**} - s_i^{**})^j}{j!} \frac{(s_i^{**} - x)^{(N-2-j)}}{(N-2-j)!}, \end{aligned}$$

where a_0, s_1^{**}, ..., s_m^{**}, β_1^{**}, ..., *and* β_m^{**} *are to be determined such that* $E(\phi_n^{**}(S_t)) = 1$, $E(S_t \phi_n^{**}(S_t)) = S_0/B_0$, *and* $E(c^i(S_t)\phi_n^{**}(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

(b) *The lower bound for options with strike prices between* (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ *is given by the pricing kernel* $\varphi_n^*(x) = \phi_{on}^*(x)$, *which is derived in the above proposition.*

2. *Assume* n *is even. Let* $m = n/2$.

(a) *Then the upper bound for options with strike prices between* (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ *is given by a pricing kernel* $\varphi_n^{**}(S_t) = a_0 \frac{\delta(S_t - \underline{s})}{p(S_t)} + f_n^{**}(S_t)$, *where* $p(S_t)$ *is the true probability density function,* $\delta(x)$ *is the Dirac function, and* $f_n^{**}(x)$ *has a (m+1)-segmented and piecewise* *constant (N-2)th derivative, where the absolute of value of its (N-2)th derivative is equal to* β_1^{**} *for* $x < s_1^{**}$, ..., β_m^{**} *for* $x \in (s_{m-1}^{**}, s_m^{**})$, *and zero for* $x > s_m^{**}$. *More precisely,* $f_n^{**}(x) = f_{ni}^{**}(x)$, *for* $x \in (s_{i-1}^{**}, s_i^{**})$, $i = 1, ..., m+1$, *where* $s_0^{**} = 0$, $s_{m+1}^{**} = +\infty$, $f_{n(m+1)}^{**}(x) = b$, $f_{nm}^{**}(x) = b + \frac{\beta_m^{**}}{(N-2)!}(s_m^{**} - x)^{N-2}$, *and for* $i = 1, ..., m-1$,

$$\begin{aligned} f_{ni}^{**}(x) &= f_{n(i+2)}^{**}(s_{i+1}^{**}) + \beta_i^{**}\frac{(s_i^{**} - x)^{N-2}}{(N-2)!} \\ &\quad + \beta_{i+1}^{**}\sum_{1}^{N-2}\frac{(s_{i+1}^{**} - s_i^{**})^j}{j!}\frac{(s_i^{**} - x)^{(N-2-j)}}{(N-2-j)!}, \end{aligned}$$

where a_0, b, s_1^{**}, ..., s_m^{**}, β_1^{**}, ..., *and* β_m^{**} *are to be determined such that* $E(\phi_n^{**}(S_t)) = 1$, $E(S_t\phi_n^{**}(S_t)) = S_0/B_0$, *and* $E(c^i(S_t)\phi_n^{**}(S_t)) = c_0^i/B_0$, $i = 1, 2, ..., n$.

(b) The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...$, *is given by a pricing kernel* $\varphi_n^*(x) = \phi_{en}^*$, *which is derived in the above proposition.*

3 The Arbitrage Portfolios

Using a method similar to the one use by Huang (2000b) we can derive the arbitrage portfolios when the Nth order stochastic dominance option bounds are violated. We first introduce the following notation:

$$\bar{s}_j^{(N)} \equiv E^{(N-1)}(S_t | S_t < s_j)$$

$$\bar{f}_{l_j}^{(N)} \equiv E^{(N-1)}(f(S_t) | S_t < s_j),$$

where $f(x)$ is any given function, and

$$(\bar{c}_{l_j}^{(N)X})' \equiv \frac{d}{dy} E^{(N-1)}(c^X(S_t)|S_t < y)|_{y=s_{l_j}}.$$

We have the following result.

Proposition 5 *Assume the pricing kernel is decreasing in S_t. Assume the price of a unit bond is B_0, the underlying stock price is S_0, and the prices of n options with strike prices K_1, K_2, ..., K_n are c_0^1, c_0^2, ..., and c_0^n respectively. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

- *Assume n is odd. Let $m = (n+1)/2$.*
  - *For options with strike prices X between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...$, when its lower bound is violated the arbitrage portfolio is given by*

    $$(\alpha_1, \alpha_2, ..., \alpha_{n+2}) = (|A_1|, |A_2|, ..., |A_{n+2}|)/|A|, \tag{3}$$

    where A is given by

    $$\left\{ \begin{array}{ccccc} 1 & \bar{s}_{l_1}^{(N)} & \bar{c}_{l_1}^{1} & ... & \bar{c}_{l_1}^{n(N)} \\ 0 & (\bar{s}_{l_1}^{(N)})' & (\bar{c}_{l_1}^{1(N)})' & ... & (\bar{c}_{l_1}^{n(N)})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m}^{(N)} & \bar{c}_{l_m}^{1(N)} & ... & \bar{c}_{l_m}^{n(N)} \\ 0 & (\bar{s}_{l_m}^{(N)})' & (\bar{c}_{l_m}^{1(N)})' & ... & (\bar{c}_{l_m}^{n(N)})' \\ 1 & E(S_t) & E(c^1(S_t)) & ... & E(c^n(S_t) \end{array} \right\} \tag{4}$$

    $s_{l_1}, ..., s_{l_m}$ are determined by 1(b) in Proposition 3, and A_i, $i = 1, 2, ..., n+2$, is obtained from A by replacing its ith column by

    $$(\bar{c}_{l_1}^{X(N)}, (\bar{c}_{l_1}^{X(N)})', ..., \bar{c}_{l_m}^{X(N)}, (\bar{c}_{l_m}^{X(N)})', E(c^X(S_t)))^T.$$

– *For options with strike prices X between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...,$ when its upper bound is violated the arbitrage portfolio is given by*

$$(\alpha_1, \alpha_2, ..., \alpha_{n+2}) = (|B_1|, |B_2|, ..., |B_{n+2}|)/|B|, \tag{5}$$

where B is given by

$$\left\{\begin{array}{ccccc} 1 & \underline{s} & c^1(\underline{s}) & ... & c^n(\underline{s}) \\ 1 & \bar{s}_{l_1}^{(N)} & \bar{c}_{l_1}^{1(N)} & ... & \bar{c}_{l_1}^{n(N)} \\ 0 & (\bar{s}_{l_1}^{(N)})' & (\bar{c}_{l_1}^{1(N)})' & ... & (\bar{c}_{l_1}^{n(N)})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m}^{(N)} & \bar{c}_{l_m}^{1(N)} & ... & \bar{c}_{l_m}^{n(N)} \\ 0 & (\bar{s}_{l_m}^{(N)})' & (\bar{c}_{l_m}^{1(N)})' & ... & (\bar{c}_{l_m}^{n(N)})' \end{array}\right\} \tag{6}$$

$s_{l_1}, ..., s_{l_m}$ are determined by 1(a) in Proposition 3, and B_i, $i = 1, 2, ..., n+2$, is obtained from B by replacing its ith column by

$$(c^X(\underline{s}), \bar{c}_{l_1}^{X(N)}, (\bar{c}_{l_1}^{X(N)})', ..., \bar{c}_{l_m}^{X(N)}, (\bar{c}_{l_m}^{X(N)})')^T.$$

– *For options with strike prices X between $(K_{2i-1},\ K_{2i})$, $i = 1, 2, ...,$ when its lower bound is violated the arbitrage portfolio is given by (5); when its upper bound is violated the arbitrage portfolio is given by (3)*

- *Assume n is even. Let $m = n/2$.*

  – *For options with strike prices X between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...,$ when its lower bound is violated the arbitrage portfolio is given by*

$$(\alpha_1, \alpha_2, ..., \alpha_{n+2}) = (|U_1|, |U_2|, ..., |U_{n+2}|)/|U|, \tag{7}$$

where U is given by

$$\left\{\begin{array}{ccccc} 1 & \bar{s}_{l_1}^{(N)} & \bar{c}_{l_1}^{1(N)} & \dots & \bar{c}_{l_1}^{n(N)} \\ 0 & (\bar{s}_{l_1}^{(N)})' & (\bar{c}_{l_1}^{1(N)})' & \dots & (\bar{c}_{l_1}^{n(N)})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_{m+1}}^{(N)} & \bar{c}_{l_{m+1}}^{1(N)} & \dots & \bar{c}_{l_{m+1}}^{n(N)} \\ 0 & (\bar{s}_{l_{m+1}}^{(N)})' & (\bar{c}_{l_{m+1}}^{1(N)})' & \dots & (\bar{c}_{l_{m+1}}^{n(N)})' \end{array}\right\} \tag{8}$$

$s_{l_1}, ..., s_{l_m}$ *are determined by 2(b) in Proposition 3, and* U_i, $i = 1, 2, ..., n+2$, *is obtained from* U *by replacing its* i*th column by*

$$(\bar{c}_{l_1}^{X(N)}, (\bar{c}_{l_1}^{X(N)})', ..., \bar{c}_{l_{m+1}}^{X(N)}, (\bar{c}_{l_{m+1}}^{X(N)})')^T.$$

– *For options with strike prices* X *between* $(K_{2i-2},\, K_{2i-1})$, $i = 1, 2, ...$, *when its upper bound is violated the arbitrage portfolio is given by*

$$(\alpha_1, \alpha_2, ..., \alpha_{n+2}) = (|V_1|, |V_2|, ..., |V_{n+2}|)/|V|, \tag{9}$$

where V is given by

$$\left\{\begin{array}{ccccc} 1 & \underline{s} & c^1(\underline{s}) & \dots & c^n(\underline{s}) \\ 1 & \bar{s}_{l_1}^{(N)} & \bar{c}_{l_1}^{1(N)} & \dots & \bar{c}_{l_1}^{n(N)} \\ 0 & (\bar{s}_{l_1}^{(N)})' & (\bar{c}_{l_1}^{1(N)})' & \dots & (\bar{c}_{l_1}^{n(N)})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m}^{(N)} & \bar{c}_{l_m}^{1(N)} & \dots & \bar{c}_{l_m}^{n(N)} \\ 0 & (\bar{s}_{l_m}^{(N)})' & (\bar{c}_{l_m}^{1(N)})' & \dots & (\bar{c}_{l_m}^{n(N)})' \\ 1 & E(S_t) & E(c^1(S_t)) & \dots & E(c^n(S_t)) \end{array}\right\} \tag{10}$$

$s_{l_1}, ..., s_{l_m}$ *are determined by 2(a) in Proposition 3, and* V_i, $i =$

$1, 2, ..., n+2$, is obtained from V by replacing its ith column by

$$(c^X(\underline{s}), \bar{c}_{l_1}^{X(N)}, (\bar{c}_{l_1}^{X(N)})', ..., \bar{c}_{l_m}^{X(N)}, (\bar{c}_{l_m}^{X(N)})', E(c^X(S_t)))^T.$$

- *For options with strike prices X between $(K_{2i-1},\ K_{2i})$, $i = 1, 2, ...,$ when its lower bound is violated the arbitrage portfolio is given by (9); when its upper bound is violated the arbitrage portfolio is given by (7).*

Proof: The proof is very similar to the case of second order stochastic dominance. For brevity, it is omitted. For details, see Huang (2004b).

4 Nuermical Examples

In this section we simulate the option bounds assuming the underlying stock price follows a log-normal distribution. In order to compare our results with the second order stochastic dominance option bounds given by Huang (2004b), we use the same distribution to derive the option bounds. The strike prices of the observed options are presented in Table 1 while the parameters of the log-normal distribution are presented in Table 2.

Table 1 Table 2

Applying the results derived in this paper we calculate the 2nd, 3rd, ..., and 6th stochastic dominance option bounds for nine options with strike prices between 35 and 75 using one observed options. The simulated option bounds are presented in Table 3 and Table 4. We also present the gaps between the upper and lower option bounds in Table 5.

Table 3

From these results, we can see clearly that the option bounds become tighter when we assume higher order stochastic dominance rules. In the case of the at-the-money option the gap between the upper and lower bounds is reduced from 1.665 to 0.409 by three quarters when we raise the order of stochastic dominance rule from the second to the sixth. We can also see that the incremental benefit from raising the order of stochastic dominance rules is generally diminishing.

Table 4

Moreover, for deep in-the-money options high order stochastic dominance rules give very tight option bounds. Taking the option with strike price 35 for an example, the gap between the upper and lower option bounds under 6SD is less than 3% of the Black-Scholes option price. However, for out-of-the-money options the option bounds are not satisfactory even we raise the order of stochastic dominance rules. Taking the option with strike price 75 for an example, the gap between the upper and lower option bounds under 6SD is almost than 47% of the Black-Scholes option price.

Table 5

5 Conclusions

In this paper we derive Nth order stochastic dominance option bounds from concurrently expiring options. We show that given the prices of a unit bond, underlying stock, and n option prices, the kth order stochastic dominance option bounds are given by a pricing kernel the $(N-2)$th derivative of which is $(n/2)$-segmented and piecewise constant if n is even or ((n+1)/2)-segmented and piecewise constant if n is odd. Since stochastic dominance rules are generally accepted, the derived option bounds in this paper are practically meaningful.

Using numerical examples we find that the option bounds become significantly tighter when we raise the order of stochastic dominance rules. We also find that the incremental benefit from raising the order of stochastic dominance rules is generally diminishing. Moreover, we find that for deep in-the-money options high order stochastic dominance rules give very tight option bounds while for out-of-the-money options the option bounds are not satisfactory even we raise the order of stochastic dominance rules.

The results have important implications for arbitrage opportunities in the markets of concurrently expiring options. When the option bounds are violated we can construct arbitrage portfolios to take the advantage.

It may also be worth mentioning that the pricing kernels that give the lower bounds on option prices also give lower bounds on the variance of stock price under the risk neutral probability measure.[6]

[6]This is because the square of stock price is the sum of a series of call options. Also see Lemma 2 in Huang (2004a).

Table 1. Target and Observed options

	Strike Price
First Observed Option	55
Second Observed Option	45
Third Observed Option	60

Table 2. Parameters of the price process

Stock Price	50
Drift	0.2
Standard Deviation	0.2
Risk Free Rate	0.1
Time to Maturity	1

Table 3. NSD lower option bounds using one observed option

	B-S Price	L^{SSD}	L^{TSD}	L^{4SD}	L^{5SD}	L^{6SD}
35	18.3612	18.34	18.347	18.35	18.352	18.353
40	13.9963	13.89	13.933	13.951	13.96	13.965
45	9.9943	9.697	9.843	9.897	9.919	9.93
50	6.6348	6.132	6.423	6.51	6.542	6.558
55	4.0915	3.642	3.93	4.004	4.029	4.041
60	2.354	2.354	2.354	2.354	2.354	2.354
65	1.273	0.868	1.115	1.173	1.194	1.204
70	0.6524	0.147	0.45	0.528	0.555	0.569
75	0.3196	0	0.146	0.214	0.238	0.25

Table 4. NSD upper option bounds using one observed option

Strike		U^{SSD}	U^{TSD}	U^{4SD}	U^{5SD}	U^{6SD}
35	18.3612	19.901	19.267	19.049	18.953	18.9
40	13.9963	15.644	14.946	14.709	14.606	14.549
45	9.9943	11.534	10.844	10.619	10.522	10.47
50	6.6348	7.787	7.231	7.06	6.989	6.951
55	4.0915	4.661	4.366	4.279	4.245	4.228
60	2.354	2.354	2.354	2.354	2.354	2.354
65	1.273	1.481	1.435	1.358	1.331	1.319
70	0.6524	0.885	0.855	0.779	0.739	0.721
75	0.3196	0.505	0.487	0.441	0.405	0.388

Table 5. Gaps between NSD option bounds using one observed option

Strike	*B-S Price*	SSD	TSD	4SD	5SD	6SD
35	18.3612	1.561	0.92	0.699	0.603	0.548
40	13.9963	1.754	1.013	0.758	0.655	0.589
45	9.9943	1.837	1.001	0.722	0.625	0.551
50	6.6348	1.655	0.808	0.55	0.479	0.409
55	4.0915	1.019	0.436	0.275	0.241	0.199
60	2.354	0	0	0	0	0
65	1.273	0.613	0.32	0.185	0.158	0.125
70	0.6524	0.738	0.405	0.251	0.211	0.166
75	0.3196	0.505	0.341	0.227	0.191	0.15

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